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APPLICATION OF THE ALGEBRAIC ABERRATION EQUATIONS TO OPTICAL DESIGN

By I. C. Gardner

ABSTRACT

The phase of optical engineering which deals with lens design and the measurement of the aberrations of a lens system is worthy of a more comprehensive treatment than is available in English. Even in our technical schools and universities it is seldom that one finds a course dealing adequately with optical imagery which goes beyond the first order or Gaussian equations. In geometrical optics there are two applications of the aberration equations. In the first, one has what may be termed the direct problem. The specifications of the lens system are given and the aberrations are to be determined. In the inverse problem one is to determine the specifications of the lens system which will have the desired aberration characteristics. Although the second problem is much the more important the literature dealing with it is relatively meager. There is no treatise entirely satisfactory, either in German or English, which gives the third order aberrations in a convenient form for the inverse solution with a simple and consistent notation and sign convention.

And yet the control of the aberrations of a projected system is the central problem of optical design. To understand the aberrations, one must have had experience in computing and in measuring them. The need of a treatment of the aberrations covering a different field than that of the existing treatises has often been felt and has been recently well expressed by Mr. Emley [1] at a meeting of the Optical Society of London, in which he says, "It is when the student attempts to get hold of expressions from which he can calculate these quantities (aberrations), however, that his difficulties begin. There is no standard English work leading him from his elementary geometrical and physical optics to problems of this kind. It has always seemed to me that there is a distinct gap in the subject which, looked at from any point of view, should be filled. There are many advanced works on instrument design and other specialized problems, but the ordinary type of student I have in mind can not follow them, partly because of the gap alluded to and partly because of the variety of symbols and constants used."

It is hoped that the following treatment will partially fill this gap and at the same time serve as a reference book for the solution of problems in lens design. The laboratory measurement of the aberrations is not dealt with. The aberration equations are presented in such a manner as to permit their direct application to problems of lens design. Only the assumptions of geometrical optics in the restricted sense are applied, and all references to diffraction effects, resolving power, and kindred subjects are omitted. The derivations of the equations are omitted except for a brief reference in Appendix 2, and the physical interpretation and manner of application of the equations to problems is stressed. It is the intention thus to produce a grammar or handbook for reference which will contain the information necessary for the algebraic third order design of optical systems composed of thin lenses.

[1] H. H. Emley, Trans. Opt. Soc., **27**, p. 233; 1925–26.

For the third order equations there are two systems differing in the choice of parameters. The one termed the continental system (see p. 79) will be found treated, among others, by Schwarzschild,[2] von Rohr,[3] and Southall.[4]

The second system, termed the Taylor system, was originated by Coddington [5] and much extended by Taylor.[6] Modifications have been introduced into the Taylor-Coddington equations which simplify them in appearance and which enable the two systems of equations to be carried along in parallel throughout the treatment. The reader can, accordingly, make his own choice as to which group of equations is to be used.

The notation employed is substantially that of Schwarzschild and von Rohr, and its choice is justified by the wealth of material already published in which this notation is employed. Some slight changes have been made to avoid confusion when the notation shall be extended to projected publications dealing with the stage of optical design in which trigonometric ray tracing is employed. To avoid the attachment of two significations to the same symbol, it has been necessary to depart from the notation of Taylor in many instances. The sign convention adopted is the one believed to be most nearly universal in treatises on applied optics.

The first order equations of imagery are dealt with only in such detail as is necessary to provide the basis of notation and sign convention to be used in the third order equations.[7] The equations and a general description of each third order aberration for a single lens are given, after which the equations are extended to a system of thin lenses. A general discussion of the method of controlling the aberrations of an optical system follows with two numerical examples. In the first example the aberrations of a Ramsden eyepiece are determined. In the second example a Kellner eyepiece is designed to have given aberration characteristics. This last illustration is worked out in considerable detail in order to illustrate fully the different applications of the third-order equations. Following this there are given the equations for the third-order aberrations of thick plates or reflecting prisms and the method of their application to the design of optical systems which contain thin lenses and reflecting prisms or plane parallel plates.

There are four appendixes, which are as follows:

Appendix 1.—The notation and sign conventions used, with equivalent symbols as applied by Taylor.

Appendix 2.—The Seidel equations as given by Schwarzschild, with the method of derivation of the thin lens equations.

Appendix 3.—A series of plates giving dimensional drawings of the principal prisms used in optical systems. These plates were prepared by Otto Kaspereit, of Frankford Arsenal, and were originally included in Elementary Optics and Applications to Fire Control Instruments, revision of January, 1924, Ordnance Department Publication No. 1065. Thanks are due the Ordnance Department, United States Army, for its courtesy in permitting the inclusion of these drawings.

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[2] Schwarzschild, K., Untersuchungen zur geometrischen Optik. I. Einleitung in die Fehlertheorie optischer Instrumente auf Grund des Eikonalbegriffs. Abh. der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Math-Phys. Kl. Neue Folge, 4, No. 1; 1905.

[3] von Rohr, M., Die Bilderzeugung in optischen Instrumenten. Julius Springer, Berlin; 1904. English translation by Kanthack. H. M. Stationery Shop, London; 1920.

[4] Southall, J. P. C., Principles and Methods of Geometrical Optics. The Macmillan Co., New York; 1910.

[5] Coddington, H., A Treatise on the Reflexion and Refraction of Light. Simkin & Marshall, London; 1829.

[6] Taylor, H. D., A System of Applied Optics. Macmillan & Co. (Ltd.), London; 1906.

[7] If an aberration is measured by the angle subtended by the aberration disk at the pupil point all the monochromatic aberrations considered are of the third order. In a symmetrical system the aberrations of even order vanish. Hence, after the first-order equations the third-order equations offer the next approximation.

Appendix 4.—A table giving values of the functions of n used in the third-order equations for values of n from 1.4 to 1.75. This table has been computed by H. U. Graham and will be found very useful in connection with the third-order equations.

In conclusion, the writer wishes to express his gratitude to C. D. Hillman, of Keuffel & Esser Co., and to Mr. Kaspereit for the care which they have taken in reading the manuscript and for their many constructive suggestions which have been adopted. Mr. Kaspereit has not only read the manuscript but has checked practically all the computations and compiled a list of errata which were used in revising the numerical parts.

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I. INTRODUCTION

In the equations for the design of an optical system the imagery is considered as consisting of a set of third order aberrations superposed upon a first order imagery. The first order, or Gaussian, equations give, in general, an approximate description of the imagery which is not sufficiently complete to serve as a basis for the final design of the system. In the first order equations, the value of o, the radius of the entrance pupil of lens, and β, the angular distance of object point from the axis, do not enter. These equations are, in fact, the limiting forms which one obtains as o and β are allowed to approach zero in the more exact equations. They are, therefore, rigorously true in the limit as aperture of incident bundle of rays and field of view vanish. Such imagery is sometimes termed paraxial imagery for, as aperture and field of view approach zero, the rays which participate in the image formation approach parallelism with the axis.

The imagery actually realized departs from first order imagery because a useful aperture for the incident bundle of rays or a useful extent of field entails sensible departures from parallelism on the part of the rays participating in the imagery. These departures from first order imagery are said to be due to aberrations which are the systematic small deviations considered as superposed upon the first order imagery. If, in the development of the system of equations, terms of the third order in $\frac{o}{f}$ (f = focal length and o/f is the half aperture ratio) and β are retained, one obtains the expressions for the aberrations corresponding to the terms in $\frac{o^3}{f^3}, \frac{o^2}{f^2}\beta, \frac{o}{f}\beta^2$ and β^3. In addition, there are two chromatic aberrations arising from the failure of the glass or other optical medium to refract equally light of different wave lengths. Whereas first order imagery is an imagery in which straight lines go into straight lines and points are imaged as points, the presence of aberrations implies a departure from these desirable characteristics and with aberrations of sufficient magnitude the image is poorly defined and distorted. By using third order equations, however, direct solutions may be obtained which indicate the design for optical systems such that certain aberrations vanish to the degree consistent with the approximations of the equations.

The number of variables and the interrelations between them are so numerous and complicated that if a direct solution of the third order equations is to be obtained simplifying assumptions must be introduced. All lens components are assumed to be thin lenses; that is, to have thicknesses so small that they may be neglected in comparison with the distances from lens to object and image points. Unless otherwise noted, in the drawings which follow, the lenses represented are to be treated as thin lenses, although represented as having considerable thickness, and distances measured from the lens actually originate at the assumed common vertex of the two spherical surfaces which bound the lens of zero axial thickness.

II. PARAMETERS EMPLOYED IN THE THIRD ORDER EQUATIONS OF IMAGERY

Two sets of equations for the determination of the third order aberrations are in common use. The one is based upon the work of Coddington [8] and much extended by Taylor.[9] This will be referred to as the Taylor system of equations. The second set is commonly employed in German literature and is used by Koenig [10] in his dis-

[8] Coddington, Henry, A Treatise on the Reflection and Refraction of Light. Simpkins & Marshall, London; 1829.

[9] Taylor, H. Dennis, A System of Applied Optics. Macmillan & Co. (Ltd.).

[10] Koenig, A., Chapter 7 of "The formation of images in optical instruments," edited by V. Rohr, English translation by Kanthack. H. M. Stationery Shop; London.

cussion of optical computation, and will be referred to as the continental system. That these two systems are identical was, perhaps, first clearly set forth by Nakamura,[11] who proves the identity of the equations of the two systems for freedom from spherical and comatic aberration and who designates the two systems as English and continental, respectively. Either of the two systems of equations can be derived conveniently from the equations of Schwarzschild,[12] which were derived by use of the Eikonal [13] function. In the following discussion both sets of equations will be given. The Taylor system will be modified from the form given by Taylor by the introduction of mathematical devices used by Schwarzschild [12] which simplify the equations, at least in appearance, and which make the parallelism between the two systems more apparent.

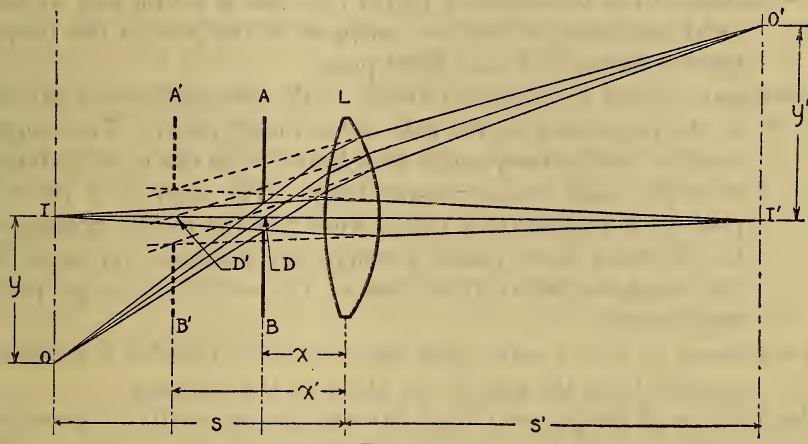

FIG. 1.—*First order imagery*

IO and *I'O'* are the object and image planes. *AB* and *A'B'* are the entrance and exit pupils.

1. FIRST ORDER EQUATIONS OF IMAGERY FOR A SINGLE LENS

The parameters for the third order equations of imagery are based upon the constants of the imagery obtained by the application of the first order equations. Figure 1 shows the relationship between object and image for a thin lens. Points *I* and *O* are axial and oblique object points, respectively, with conjugate points at *I'* and *O'*. For the first order equations of imagery the optical system is completely specified when the focal length *f*, or power ϕ, of the lens is given as defined by the equation

$$\phi=\frac{1}{f}=(n-1)\left(\frac{1}{r}-\frac{1}{r'}\right) \tag{1}$$

[11] Nakamura, On the calculation of a thin aplanatic objective, Japanese J. Phys., **2**, p. 85; 1923.

[12] Schwarzschild, K.: Untersuchungen zur geometrischen Optik. I. Einleitung in die Fehlertheorie optischer Instrumente auf Grund des Eikonalbegriffs. Abh. der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Math-Phys. Kl. Neue Folge, **4**, No. 1; 1905.

[13] See Appendix 1.

n = index of refraction of the lens relative to air which is assumed to be the surrounding medium.

r = radius of curvature of first surface of lens; that is. the surface which receives the incident light.

r' = radius of curvature of second surface of lens.

r and r' are positive if the respective surfaces are convex toward the incident light, negative if concave toward it.

The equations of first order imagery are

$$\frac{1}{s'}=\frac{1}{s}+\frac{1}{f} \tag{2}$$

and

$$y'=\frac{s'}{s}y \tag{3}$$

s = distance from the common vertex (the lens is a thin lens without axial thickness) of the two surfaces of the lens to the projection on the axis of the object point.

s' = distance from the common vertex of the two surfaces of the lens to the projection on the axis of the image point. The lengths s and s' may subsequently be referred to as the axial distances of object and image, respectively. The sign of s or s' is positive if a generating point, when moving in the direction of the incident light, passes through the common vertex of the two surfaces before it arrives at the object or image point, respectively.

y = distance of object point from the axis and is positive if measured upward from the axis in the plane of the diagram.

y' = distance of image point from the axis and is positive if measured upward from the axis in the plane of the diagram. The lengths y and y' may be referred to as the lateral distances of object and image, respectively.

In Figure 1, r, s', and y' are positive, r', s, and y are negative. The object and image lie in the same meridional plane and, without sacrifice of generality, this can be assumed to be the s, y, and s', y' plane. Consequently, a third equation for the z and z' coordinates is not necessary.

To the degree of approximation involved in the first order equations of imagery, homocentric bundles of rays in the object space go into homocentric bundles in the image space, and the higher order departures from this simple imagery, arising from lack of homogeneity as regards wave length in the incident bundle and the entry into the image formation of rays other than the paraxial, are suppressed. The first order equations, as has been mentioned, give, in general, a good first approximate description of the imagery which one actually obtains, and a more accurate description is given by the addition of the third order equations which indicate the differences between first order imagery and third order imagery.

In addition to the Cartesian coordinates of the object point (s and y), the first order equations require only the value of f in order to effect a solution for the determination of the image point. Before the aberration equations are to be solved it is necessary or desirable to have additional parameters by which the variation of index of lens with color, the shape of the lens, the convergence of the incident bundle, the convergence of the chief rays, the height of incidence of marginal ray, and the angular distance of the object from the center of the field are specified. As in some cases these parameters used in the two systems of equations (the Taylor and continental) are different, there will be given below a parallel treatment of the two sets of parameters.

2. VARIATION OF THE INDEX OF GLASS WITH COLOR

For the chromatic aberration it is customary to select three wave lengths in the spectrum, λ', λ, and λ'', which are assumed to be here designated in the order of decreasing wave length. The ends of the spectral interval treated are defined by λ' and λ'' and the primary chromatic aberrations are said to be eliminated when the images formed by these two wave lengths register accurately over the entire field. The intermediate wave length λ is the one for which the monochromatic aberrations are corrected and any constants of the lens, when not definitely specified otherwise, are referred to λ. To apply the third order and chromatic aberration equations it is necessary to have given

n_λ = the index of refraction for the intermediate wave length, relative to the surrounding medium, of the glass or other optical material employed

and

$$\Delta n = n_{\lambda''} - n_{\lambda'} \tag{4}$$

or

$$\nu = \frac{n_\lambda - 1}{n_{\lambda''} - n_{\lambda'}} \tag{5}$$

For instruments in which the image formed is viewed directly by the eye it is customary to select for λ', λ, and λ'' the wave lengths corresponding to the C, D, and F lines of the spectrum.[14] In such a case the constants of the lens are referred to the D spectrum line and

[14] The selection of the C, D, and F spectrum lines to characterize the chromatic aberrations of an optical system for visual use originated at a time when the number of convenient laboratory sources giving bright line spectra was much more limited than now. At present there is a tendency to make other selections of lines which are more favorably located in the spectrum. For a discussion of this see Weidert, Zeits. f. Tech. Phy., 7, pp. 304–309, 1926. In particular, the selection of the D line (λ=5,893) as the position in the spectrum for the elimination of the monochromatic aberrations seems unfortunate. Some manufacturers of optical glass regularly give the index for the 5,461 mercury line. It lies much nearer the maximum of the luminosity curve for white light than the D line, and it would seem preferable to correct the monochromatic aberrations for this wave length.

$$\Delta n = n_F - n_C \tag{6}$$

$$\nu = \frac{n_D - 1}{n_F - n_C} \tag{7}$$

When the symbol ν is used without special qualification, particularly in the catalogues of glass manufacturers, it may be understood to refer to the ratio formed with the choice of wave lengths as indicated in equation (7). For photographic instruments, in which the focussing is done by the eye and the final image recorded photographically, it is customary to select the wave lengths corresponding to D, F, and G' for λ', λ, and λ''. The notation for variation of index with color, as described above, is employed both in the Taylor and continental systems of equations.

3. SHAPE OF THE LENS

In the first order equations of imagery the position of the image point or character of imagery is independent of any variation in r and r', provided that the two are varied together in such a manner that f remains constant. In the third order equations, however, the final image is a function of the manner in which the total bending of the ray is divided between the first and second surfaces. Different pairs of radii which give lenses of the same focal length yield lenses of different shape (see fig. 2), and in the Taylor system a shape factor σ[15] is introduced, defined by any of the following equations:

$$\left.\begin{aligned} \sigma &= \frac{r'+r}{r'-r} \\ \sigma &= -1 + \frac{2\,(n-1)}{\phi\, r} \\ \sigma &= +1 + \frac{2\,(n-1)}{\phi\, r'} \end{aligned}\right\} \tag{8}$$

or

$$\left.\begin{aligned} \frac{1}{r} &= \frac{(1+\sigma)\,\phi}{2\,(n-1)} \\ \frac{1}{r'} &= -\frac{(1-\sigma)\,\phi}{2\,(n-1)} \end{aligned}\right\} \tag{9}$$

and the lens is completely determined by the values of σ, n, and ϕ.

The shape factor σ is a dimensionless parameter which assumes all values from $-\infty$ to $+\infty$. The shapes both for positive and negative lenses corresponding to typical values of σ are shown diagrammatically in Figure 2 and are given on facing page.

[15] Taylor designates the shape factor by x instead of σ.

$\sigma < -1$ Meniscus,
$\sigma = -1$ First surface of lens plane,
$-1 < \sigma < 0$ The two surfaces are oppositely curved with the curvature of first surface less in absolute value than that of second,
$\sigma = 0$ Lens is equiconvex or equiconcave,
$0 < \sigma < +1$ The two surfaces are oppositely curved with the curvature of the second surface less in absolute value,
$\sigma = +1$ Second surface of lens plane,
$+1 < \sigma$ Meniscus.

As σ increases in absolute value the lens becomes more and more deeply curved, and it is seldom in practice that σ falls without the

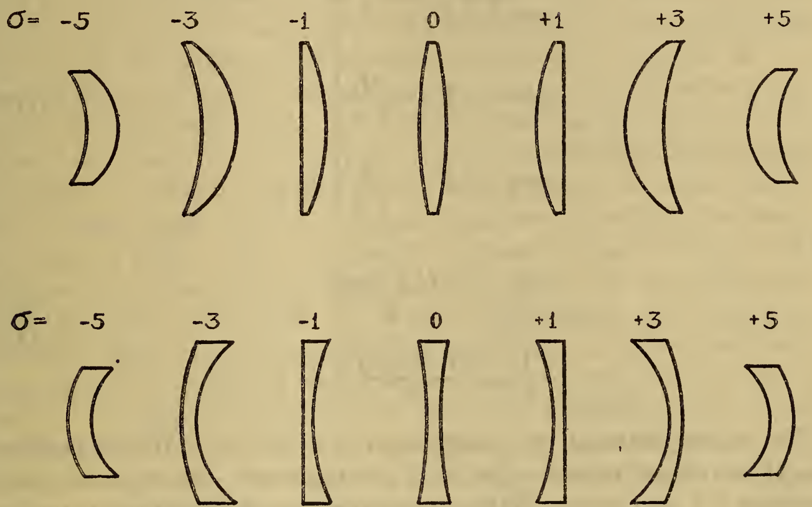

FIG. 2.—*Variations in a lens with change of shape factor*

The upper series shows converging lenses of the same power, but with shape factors varying from −5 through zero to +5. The lower row shows a similar series of diverging lenses.

range of values lying between +3 and −3 except for components in which ratio of diameter to focal length is small. As σ is dimensionless it is the same for all lenses which are geometrically similar, a feature which at times is a great convenience. If the signs of the curvatures of the two faces of a lens are changed, the value of σ is unaltered. If the lens is reversed in position so that the face which first received the incident light becomes the face by which light leaves the lens, the sign of σ is changed, the absolute value remaining constant.

In the continental system the terms in $\frac{1}{r}$, in the aberration equations are eliminated by equation (1). This leaves terms in $\frac{1}{r}$ and the parameters which determine the lens are

$$\frac{1}{r}, \ n, \text{ and } \phi$$

4. CONVERGENCE OF THE INCIDENT BUNDLE OF RAYS

The third order equations of imagery contain both the first and second powers of $\frac{1}{s}$ and $\frac{1}{s'}$, and some simplification is necessary before a direct solution can be obtained. In the Taylor system the position of object and image are specified by the values of

$$\pi \text{ and } \phi$$

where π is a dimensionless axial distance factor defined by the equivalent equations [16]

$$\left.\begin{aligned} \pi &= \frac{s'+s}{s'-s} \\ \pi &= -1 - \frac{2}{s\,\varphi} \\ \pi &= +1 - \frac{2}{s'\,\varphi} \end{aligned}\right\} \tag{10}$$

or

$$\left.\begin{aligned} \frac{1}{s} &= -\frac{(1+\pi)\varphi}{2} \\ \frac{1}{s'} &= \frac{(1-\pi)\varphi}{2} \end{aligned}\right\} \tag{11}$$

The values assumed by π range from $-\infty$ to $+\infty$. With a positive lens if the object is real—that is, if s is negative—the value of π lies between -1 and $+\infty$. With a negative lens, if the object is real, π lies between -1 and $-\infty$. In either case when the object and image are coincident in the plane of the lens $\pi = \infty$.

In the continental system $\frac{1}{s'}$ is eliminated from the aberration equations by equation (2), and the axial distances of the object and image are specified, the one explicitly, the other implicitly by the values of

$$\frac{1}{s} \text{ and } \varphi$$

5. CONVERGENCE OF THE CHIEF RAYS

(*a*) Entrance and Exit Pupils and Iris.—In Figure 1 there is a diaphragm at $A\ B$ which limits the bundle of rays incident upon the first surface from any point in the object space. At $A'\ B'$ the area conjugate, with respect to the lens, to the aperture at D is represented by the opening in the dotted diaphragm. Since, by construction, the

[16] Taylor designates the position factor by α instead of π and terms it the vergency factor.

peripheries of the apertures in A B and $A'B'$ are conjugate, it follows that every incident ray which grazes the edge of the diaphragm at A B will, after refraction (when produced), graze the edge of the diaphragm at A' B'. It further follows that all bundles in the image space are limited by the image diaphragm at A' B' just as the incident bundle is limited by the diaphragm at A B. From each point in the object space, therefore, there proceeds a conical bundle of rays with vertex at object point, and all such bundles have aperture A B as a common section. Similarly, to each point of the image there proceeds a bundle of emergent rays with vertex at the image point and with the aperture A' B' common to all such bundles. Even if several diaphragms are present there will, in general, be one either in the object or image space and its conjugate image diaphragm which limits the entering and emergent bundles as do A B and A' B'. The opening A B in the object space is termed the entrance pupil, and A' B' is the exit pupil. It is evident that the entrance and exit pupils will always be conjugate. It may happen that the physical diaphragm is back of the lens in the image space, in which case it serves as exit pupil and the conjugate aperture in front of the lens is the entrance pupil. If no diaphragm is present, other than the clear aperture of the lens, the entrance and exit pupils are coincident and lie in the plane of the lens. If several diaphragms are present, to determine which is the actual limiting diaphragm all apertures in the image space are projected backward through the lens into the object space. The physical diaphragm or conjugate aperture in the object space, which subtends the smallest angle at the axial object point, is the entrance pupil. The physical diaphragm which limits the rays, whether it is in the object or image space, is termed the iris to distinguish it from the other diaphragms.

(*b*) PUPIL POINTS AND CHIEF RAYS.—The axial points of the entrance and exit pupils are termed the entrance and exit pupil points and are conjugate. A ray from an object point which passes through the entrance pupil point, or a ray from an exit pupil point to an image point, is known as a chief ray. In the object space there exists a bundle of chief rays with vertex at entrance pupil point, each ray of which corresponds to a different object point. Similarly, in the image space there is a conjugate bundle of chief rays with vertex at exit pupil point and a ray ending at each image point. The two conjugate bundles image the entrance pupil point at the exit pupil point. Such a bundle of rays [17] is shown in the lower drawing of Figure 3.

[17] A bundle of chief rays formed in this manner differs from a bundle of rays proceeding from an object point in that the chief rays are not coherent; that is, there is no fixed phase relation between the different rays of the bundle, whereas the rays proceeding from an object point are coherent. This is a real difference from the standpoint of physical optics, but insignificant so long as only the geometrical relations are under consideration.

(c) Entrance and Exit Windows.—After the entrance pupil point has been located one may locate the entrance window. The entrance window is the diaphragm in the object space which subtends the smallest angle at the entrance pupil point. In Figure 1 the only diaphragm present in the object space, other than the entrance pupil, is the cell limiting the clear aperture of the lens. It, therefore, is the entrance window and it limits the aperture of the bundle of incident chief rays just as the entrance pupil limits the bundle of incident rays proceeding from any object point. The assumed diaphragm conjugate to the entrance window is the exit window and limits the aperture of the emergent bundle of chief rays. In Figure 3 an

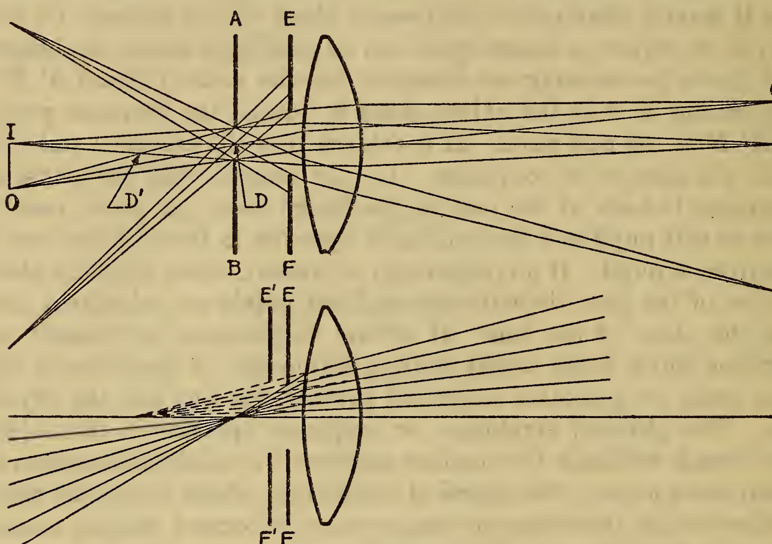

Fig. 3.—*First order imagery*

IO and *I'O'* are the object and image planes. The entrance pupil is at *AB*, the entrance window at *EF*. The lower diagram shows a bundle of chief rays with entrance window at *EF*, exit window at *E'F'*.

additional diaphragm has been added to the system of Figure 1 in order that the entrance window may be indicated in a more general manner than in Figure 1. At *D* and *D'* are the entrance and exit pupil points and at *E F* and *E' F'* are the entrance and exit windows. The pupils limit the bundle of rays from any object point, the windows limit the field of view. As a useful mnemonic, one may picture an observer in a room standing back from a window and looking out. The pupil of the eye limits the bundle of rays from any point outside and is the entrance pupil. The window of the room limits the field of view of external objects visible and is the entrance window.

(d) Parameters Giving the Convergence of the Chief Rays.—The positions of the entrance and exit pupils affect the character of third order imagery and, therefore, there must be parameters introduced to define their location. In the Taylor system the term ϵ is used in a manner analogous to that in which π is employed for defining the convergence of an incident bundle of rays. The eccentricity or lateral distance factor, designated by ϵ, is so called because it is a measure of the eccentricity of the point of incidence of a chief ray from a marginal point.[18] It bears the same relation to the bundle of chief rays as does π to the bundle of rays proceeding from an object point. The value of ϵ is defined by the following equivalent equations:

$$\left.\begin{aligned}\epsilon&=\frac{x'+x}{x'-x}\\ \epsilon&=-1-\frac{2}{\varphi x}\\ \epsilon&=+1-\frac{2f}{\varphi x'}\end{aligned}\right\}\tag{12}$$

$$\left.\begin{aligned}\frac{1}{x}&=-\frac{(1+\epsilon)\varphi}{2}\\ \frac{1}{x'}&=\frac{(1-\epsilon)\varphi}{2}\end{aligned}\right\}\tag{13}$$

where (see fig. 1)

x = distance from the common vertex of the two surfaces of the lens to the entrance pupil point,

x' = distance from the vertex of the two surfaces of the lens to the exit pupil point.

x or x' is positive if a generating point, when moving in the direction of the incident light, passes through the common vertex of the two surfaces before it arrives at the object or image point, respectively.

As the two pupil points are conjugate

$$\frac{1}{x'}=\frac{1}{x}+\frac{1}{f}\tag{14}$$

Like π and σ, ϵ ranges in value from $-\infty$ to $+\infty$. With a positive lens, if the iris is in front of the lens, the value of ϵ lies between -1 and $+\infty$; if back of the lens, between $-\infty$ and -1. With a negative lens, if the iris is in front, ϵ is between $-\infty$ and -1; if back of the lens, between -1 and $-\infty$. If no diaphragm other than the lens cell is present $\epsilon=\infty$ and the chief rays pass through the center of the lens. Therefore, although ϵ is a measure of eccentricity it is of such a nature that ϵ becomes infinite as the eccentricity vanishes. In the Taylor system of equations the parameters employed to denote the convergence of the chief rays are

$$\epsilon \text{ and } \varphi$$

[18] Taylor uses β instead of ϵ to denote eccentricity factor and terms it the vergency of the chief rays.

30906°—27——2

In the continental system $\frac{1}{x'}$ is eliminated by equation (14) and one uses

$$\frac{1}{x} \text{ and } \varphi$$

6. AUXILIARY RAY BY WHICH HEIGHT OF INCIDENCE OF MARGINAL RAY IS DETERMINED

The system of equations as given by Taylor[19] have been modified by the introduction of the coefficients g and h which correspond to the y and h of Schwarzschild. These two coefficients are also used in the same manner in the continental system. This modification in the Taylor system of equations makes the parallelism between the two systems more evident, results in a neater form for the equations, and in some cases simplifies their interpretation.

From the axial object point an auxiliary ray is traced (see figs. 1 and 4) which passes through the entrance pupil at the distance p[20] from the axis. Then by definition the ray is incident on the lens at

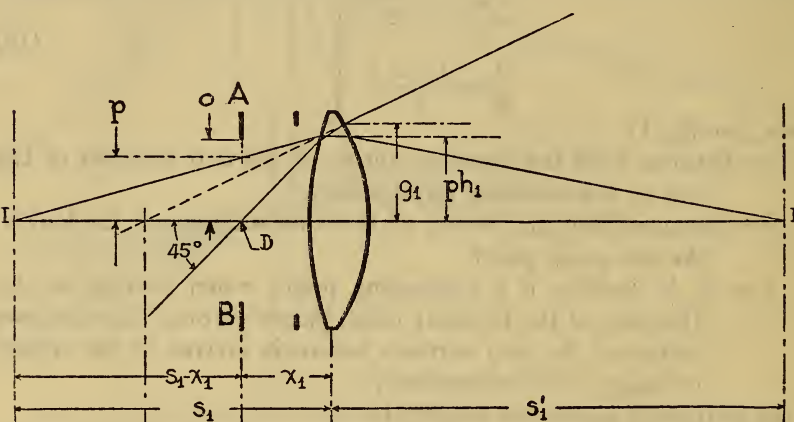

FIG. 4.—*First order imagery*

The entrance pupil is at *AB*, the entrance window at *EF*. The auxiliary rays by which g^1 and h^1 are determined are indicated.

the distance ph_1 from the axis and h_1 is the ratio of height of incidence of ray on lens to the height of incidence of same ray on the plane of the entrance pupil. From Figure 4 it is evident that

$$h_1 = \frac{s_1}{s_1 - x_1} \tag{15}$$

The subscript 1 in equations (15) and (16) indicates that the equation only applies to a single lens or to the first lens of an optical system. The definitions of g and h for the other lenses of a system will be given later.

[19] See footnote 6, p. 74.

[20] p will be used as a running coordinate to give the height of incidence on the plane of the entrance window of any ray transmitted by the lens. The maximum value of p is 0, the radius of the entrance pupil.

It should be noted that h as here defined does not correspond to the y as used by Taylor. With the present definition h is a dimensionless constant, with Taylor y is the actual height of incidence. The denominator $s-x$ is used by Schwarzschild and makes the expressions for the angular values of the aberrations particularly simple and compact.

The distance ph_1 serves as a measure of the divergence of any ray from the chief ray. The extreme ray of an incident pencil is incident on the lens at the distance oh_1 from the corresponding chief ray, where o is the radius of the entrance pupil.

7. AUXILIARY RAY BY WHICH ANGULAR DISTANCE OF OBJECT FROM CENTER OF FIELD IS MEASURED

For specifying the distance of any object point from the axis in the formulas it is convenient to use, not y_1, but the distance of point of incidence of chief ray from axis of lens. An auxiliary ray is traced (see fig. 4) which passes through the entrance pupil point and makes an angle of β_0 with the axis. It is incident upon the lens at the distance $g \tan \beta_0$ from the axis where g is defined by the equation

$$g_1 = x_1 \tag{16}$$

The distance of point of incidence of any chief ray from the axis is

$$g_1 \tan \beta_1$$

where β_1 = angle between chief ray and axis.

8. SUMMARY OF DIFFERENT PARAMETERS

Following there is given in résumé the parameters employed in the third order equations of imagery:

	Taylor	Continental
Variation of index with color	Δn or ν	Δn or ν
Shape of lens	n, σ, φ	$n, \frac{1}{r}, \varphi$
Convergence of incident bundle	π, φ	$\frac{1}{s}, \varphi$
Convergence of chief rays	ϵ, φ	$\frac{1}{x}, \varphi$
Height of incidence of marginal ray	o, h_1	o, h_1
Angular distance from center of field	$g_1, \tan \beta_1$	$g_1, \tan \beta_1$

III. CHROMATIC ABERRATIONS OF A SINGLE THIN LENS

The aberrations of a lens system may be divided, according to their origin, into two main groups—the chromatic and the monochromatic. Of the first group the two principal aberrations are longitudinal and lateral chromatic. In their derivation it is assumed that the first order equations of imagery apply rigorously and the aberrations arise only from the variation of the index of refraction of the lens with wave length. These two aberrations are of the order $o\varphi\Delta n$ and $\Delta n \tan\beta_1$. In the optical media commonly employed, Δn is of such value that the chromatic and the third order monochromatic aberrations are comparable in their effect upon the quality of imagery.

1. LONGITUDINAL CHROMATIC ABERRATION OF A SINGLE LENS

Reference to equation (1) shows that the focal length of a lens is a function of the index of refraction and the value of s' for a heterochromatic object point, therefore, will be different for the different wave lengths present in the incident bundle. For each wave length there is formed a separate image of the object point, and these images may be considered for the present as distributed along a short segment of the chief ray.[21] This is shown in Figure 5, where the sepa-

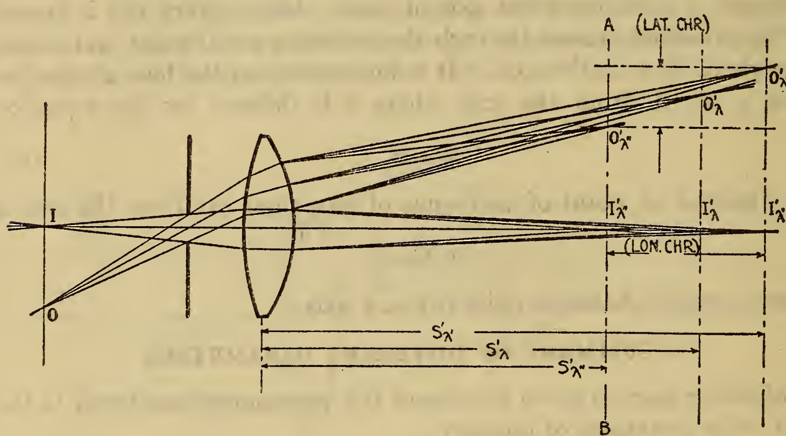

FIG. 5.—*Chromatic aberrations of a lens*

The axial image point I' illustrates longitudinal chromatic aberration. The oblique point O' has both longitudinal and lateral chromatic aberration.

ration of the different images is much exaggerated to lend greater clearness to the illustration. If an image plane is selected which cuts this segment at some point and is perpendicular to the axis, for some one wave length the image will be correctly focussed, and the images formed by light of other wave lengths will be slightly out of focus and appear as small circular disks concentric and superposed. To make the explanation more concrete, let it be assumed that λ' and λ'' (see p. 81) are the wave lengths corresponding to the C and F spectrum lines as is customary for instruments to be used visually. Let an image plane normal to the axis be passed through the focus of the

[21] In general, except for an axial point, the images formed by different wave lengths will not lie on a common chief ray. This is clearly shown by the location of the images $O'_{\lambda'}$, O'_{λ}, and $O'_{\lambda''}$ in Figure 5. Longitudinal chromatic aberration is the component of the separation of the chromatic images measured parallel to the axis. The lateral component will be ignored until the discussion of lateral chromatic aberration. (V, infra.)

F rays as at $A\ B$ (fig. 5). Since the focal length of the lens for the C rays is greater than for the F rays, the image formed by them will be farther from the lens than that formed by the F rays. In such a case one has longitudinal chromatic aberration which may be measured in two ways. In the λ'' plane the image formed by the λ' rays will be a circular disk. This circular disk may be projected backward through the lens by the first order equations and the conjugate disk in the object space determined. The angle subtended by this disk at the entrance pupil point will be designated as the angular value of the longitudinal chromatic aberration (Ang. Lon. Chr.). The linear distance, measured parallel to the optic axis from the image formed by the λ'' rays to the λ' rays, and termed the longitudinal chromatic aberration (Lon. Chr.), may be taken as a measure of the aberration. The values of the aberration, measured by the two methods are

$$(\text{Ang. Lon. Chr.}) = 2\frac{oh^2\varphi}{\nu} \tag{17}$$

$$(\text{Lon. Chr.}) = s'^2\frac{\varphi}{\nu} \tag{18}$$

In each case a positive sign indicates that a line originating at the image formed by the shorter wave length—that is, λ'' or in the illustrative case at the F image—and extending to the image formed by the longer wave length—that is, λ', or the C image—lies in the direction traveled by the incident light. The Ang. Lon. Chr. is the necessary angular separation of two points in the object space if the two images, as enlarged by longitudinal chromatic aberration, are to touch but not overlap.

If the trivial cases are excluded in which o, φ or $s' = 0$, it is evident that for a single lens the longitudinal chromatic aberration can not be caused to vanish unless $\nu = \infty$. Unfortunately no media for the construction of a dioptric system are known in which $\nu = \infty$, as this implies an index which does not vary with the wave length. As ν is always positive, for a single lens the chromatic aberration always has the same sign as φ. In the discussion of optical systems it will be shown that the chromatic aberration as defined above may be eliminated by the use of two or more lenses constructed from different types of glass.

2. LATERAL CHROMATIC ABERRATION OF A SINGLE LENS

In Figure 5 there is illustrated the imagery of an axial and an oblique point by a lens with the diaphragm placed some distance in front of it. With the oblique bundle the lens may be considered as playing two rôles. Because of the curvature of its two surfaces it alters the convergence of the rays of the bundle and causes it to be brought to a focus. But, also, as the section of the lens traversed by the bundle is thicker at one side than at the other, because of the eccentric refraction, it is evident that the lens acts as a prism and, in the example illustrated, bends the bundle down toward the axis. The alteration in convergence effected by the lens on the three wave lengths shown causes them to focus at different distances from the lens and gives rise to the longitudinal chromatic aberration of the same magnitude (to the present order of approximation) as that of the axial ray. But the prismatic action, too, will be different for the different colors with the image formed by the shorter wave lengths bent the more and therefore nearer the axis. If all wave lengths are present in the incident bundle, the image of an oblique point will be a short spectrum lying in the image plane on a line passing through the axial point of field and, in the case illustrated, with the blue end toward the center. If the length of this spectrum, extending from the λ' to λ'' wave lengths, is projected by the first order equations into the object plane, the angle subtended by it at the entrance pupil point is the angular value of the lateral chromatic aberration (Ang. Lat. Chr.) and its length measured normal to the axis is the lateral chromatic aberration (Lat. Chr.). For a single lens the values of the two aberrations, measured in the manner given above, are

$$\text{(Ang. Lat. Chr.)} = -gh\frac{\varphi}{\nu}\tan\beta \tag{19}$$

$$\text{(Lat. Chr.)} = -s'g\frac{\varphi}{\nu}\tan\beta \tag{20}$$

If the coefficient of $\tan\beta$ as yielded by either of the above equations is positive, it indicates that the image formed by the longer wave length is farther from the axis than that formed by the shorter wave length, as in the case illustrated in Figure 5.

As more oblique object points are considered with images farther from the center of the field, it is evident that the prismatic action of the lens is greater, and consequently the difference in the bending of the two pencils becomes more marked. As a point recedes from the center of the field, therefore, the lateral chromatic aberration increases and this is evidenced in the equations by the factor $\tan\beta$ in the right-hand member. If the diaphragm is in the plane of the lens, all the bundles of rays, oblique as well as axial, pass through the center of the lens and the convergence is modified, but the lens does not act as a prism. In such a case the longitudinal chromatic aberration remains, but there is no lateral chromatic. This condition is obtained from the equation by setting $g=0$. The sign of lateral chromatic aberration changes as the sign of g changes; that is, as the diaphragm is placed before or behind the lens. If the diaphragm is not in the plane of the lens, the lateral chromatic aberration can not be eliminated for a single lens.

One may approach lateral chromatic aberration from a second viewpoint. The scale to which an object is reproduced in the image space is a function of the focal length. As the focal length differs for the different colors one can expect the different images to vary in size. Consequently, one has a series of superposed images, one for each color, which register perfectly only at the axial point. As one passes from the center to the edge of the field this difference in scale becomes more apparent and the distance between corresponding points of blue and red images becomes greater and greater.

3. RELATIVE IMPORTANCE OF LONGITUDINAL AND LATERAL CHROMATIC ABERRATION

Longitudinal chromatic aberration is a function of o but not of $\tan \beta$. It is the same, therefore, in magnitude over all parts of the field and increases as the aperture is increased. On the other hand, lateral chromatic aberration is independent of size of aperture. For an optical system having a large aperture but small field of view it is important to correct for the longitudinal chromatic aberration, while with a large field of view and small aperture the elimination of the lateral chromatic is the more important.

Lateral chromatic aberration is ordinarily easily detected in a binocular or opera glass. A sharply defined dark object against a bright background, a chimney or smokestack against the sky is a suitable test object. Turn the binocular to bring this object wholly to one side of the center of the field. The vertical borders will be slightly tinged with fringes of color which increase as the binoculars are turned to bring the object nearer the border of the field of view. If the edge nearer the center is blue, the lateral chromatic aberration is positive as defined by equations (19) and (20).

IV. MONOCHROMATIC ABERRATIONS OF A SINGLE THIN LENS

The two preceding aberrations arise because of the heterochromatic character of the incident bundle and the variation of the index of refraction of the lens with wave length. The five succeeding aberrations have their origin in the fact that the different parts of a monochromatic bundle of rays pass through different parts of the lens and, except in special cases, are brought to a focus at different points on the image plane. They are, therefore, grouped as the monochromatic aberrations. The monochromatic aberrations are commonly adjusted to give the most desirable compromise for the intermediate wave length; that is, for λ (see p. 81). In this way a satisfactory compensation may generally be secured over the entire interval extending form λ' to λ'' and the chromatic aberrations are adjusted, as has already been noted, to bring the images formed by the two wave lengths, at the end of the interval, together.

The five monochromatic aberrations of the third order in $\frac{o}{f}$ and $\tan \beta$ are:

Spherical aberration of order, $\frac{o^3}{f^3}$

Coma of order, $\frac{o^2}{f^2} \tan \beta$

$\left.\begin{array}{l}\text{Curvature of field}\\ \text{Astigmatism}\end{array}\right\}$ of order, $\frac{o}{f} \tan^2 \beta$

and

Distortion of order, $\tan^3 \beta$

where o is the radius of the entrance pupil and β the angular distance of the object from the center of the field.

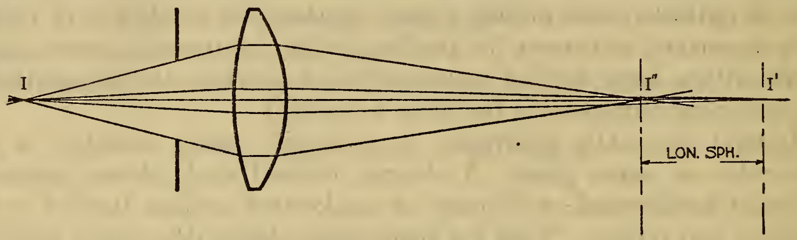

FIG. 6.—*Longitudinal spherical aberration of the axial image point I'*

1. SPHERICAL ABERRATION OF A SINGLE THIN LENS

Spherical aberration for an axial point is illustrated in Figure 6. The paraxial rays, represented by the rays drawn close to the axis, are brought to a focus at I', which is the image point as located by the first order equations. Rays which are incident on the lens at points farther from the center of lens are brought to a focus nearer the lens and in particular the rays at the edge of the clear aperture focus on the axis at I''. If, now, an image plane is selected perpendicular to the axis at I', it is evident that the hollow cone of rays which focuses at I'' will have opened out and the image produced by it on the image plane will be a circular annulus. Intermediate zones of the lens will produce smaller annuli, and the image at I', produced by the entire lens, will be a circular disk.[22] This departure

[22] In discussing the character of image as influenced by third order aberrations, only the rules of geometric optics will be applied. Actually the quality of definition as predicted by geometric optics is greatly modified by diffraction effects and other phenomena of physical optics. The condition of best focus does not necessarily lie at the best geometric focus, and often the actual definition realized is much better than is to be expected on the basis of geometric optics. These differences, however, do not need to be taken into account during the stage of the design in which the third order equations are applied.

from first order imagery is termed spherical aberration, and it is the only one of the monochromatic aberrations which results in an entirely symmetrical enlargement of the image of a point.

If this circular disk resulting from spherical aberration is considered as projected backward through the lens, one obtains in the object plane the conjugate circular area. The angle at the entrance pupil point, subtended by the diameter of this conjugate circular area, is the angular value of the spherical aberration (Ang. Sph.). The length measured from I'' to I' (fig. 6), which is positive in the case illustrated (extending in the same sense as the direction traversed by the incident light), is termed the longitudinal spherical aberration (Lon. Sph.). For a single lens the values[23] of the aberration according to the two methods of measurement are

$$(\text{Ang. Sph.})=\frac{o^3}{4}\,h^4\,\varphi^3\,A \tag{21}$$

$$(\text{Lon. Sph.})=\frac{o^2}{8}\,s'^2h^2\,\varphi^3\,A \tag{22}$$

The coefficient of spherical aberration A is defined by either of the two equations:

Taylor system

$$A=\frac{n+2}{n(n-1)^2}\sigma^2+\frac{4(n+1)}{n(n-1)}\sigma\pi+\frac{3n+2}{n}\pi^2+\frac{n^2}{(n-1)^2} \tag{23}$$

Continental system

$$A=4\Big(\frac{n^2}{(n-1)^2}+\frac{3n+1}{n-1}\,\frac{1}{\varphi s}+\frac{3n+2}{n}\,\frac{1}{\varphi^2 s^2}$$
$$-\frac{2n+1}{n-1}\,\frac{1}{\varphi r}+\frac{n+2}{n}\,\frac{1}{\varphi^2 r^2}-\frac{4(n+1)}{n}\,\frac{1}{\varphi^2 rs}\Big) \tag{24}$$

(*a*) General Characteristics of Spherical Aberration.—In equations (21) and (22) a positive value indicates that the rays passing through the edge of the lens cut the axis to the left of the paraxial image when the light is traveling from left to right. The spherical aberration indicated in Figure 6 is positive. Reference to either of the equations shows that for a positive lens the aberration will be positive unless A vanishes or is negative. In equation (23) all terms are necessarily positive except the second, which will be negative if σ and π have opposite signs. If object and image are each real, the absolute value of π will be less than 1. It can be easily shown that for any available index, for values of π less than 1 in absolute value, there is no real value of σ for which $A=0$. In other words, if object and image are to be real, the spherical aberration can not be made to

[23] Equation (21) gives (Ang. Sph.) in radians. To obtain the value in minutes, one multiplies by 3,438. Logarithm 3,438=3.5363.

vanish, but will have the same sign as φ. With a positive lens and a real object and image, this condition may be described by stating that the edge rays are always bent too much toward the axis, as in Figure 6. A lens acts on the marginal rays like a prism having its surfaces tangent to the surface of lens at points of incidence and emergence. A prism bends the rays least when in the position of minimum deviation; that is, when the entrant and emergent rays make equal angles with the normals to the two faces. From this analogy it is apparent that the edge rays will be bent least; that is, conditions will be most favorable for the elimination of spherical aberration if the lens is so turned that the entrant and emergent rays near the edge make equal or nearly equal angles with the normals to the two surfaces. This is equivalent to saying that the total bending of the edge rays by the lens should be equally divided between the two surfaces. Reference to the equations which define σ and π and to the example illustrated in Figure 7 will show that this, interpreted mathematically, signifies that σ and π have opposite signs, which is a necessary condition if A is to be a minimum.

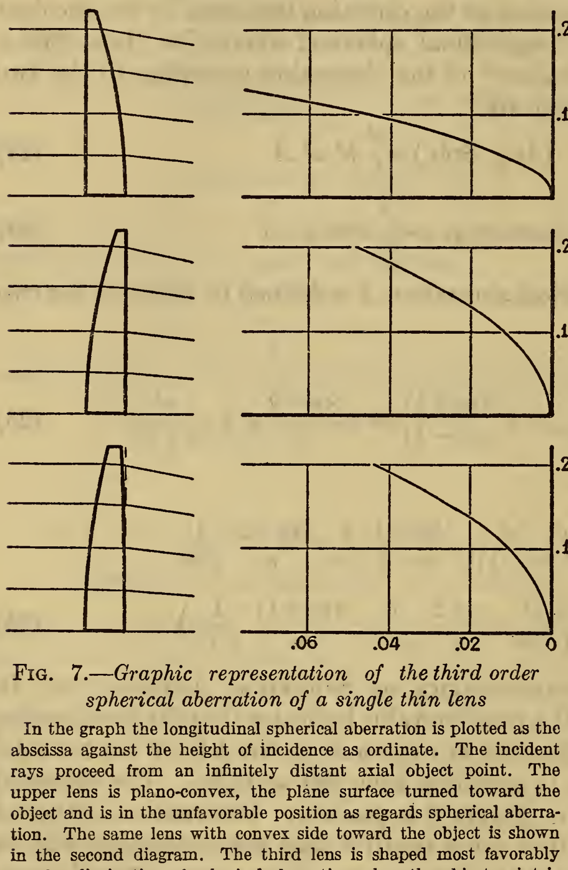

Fig. 7.—*Graphic representation of the third order spherical aberration of a single thin lens*

In the graph the longitudinal spherical aberration is plotted as the abscissa against the height of incidence as ordinate. The incident rays proceed from an infinitely distant axial object point. The upper lens is plano-convex, the plane surface turned toward the object and is in the unfavorable position as regards spherical aberration. The same lens with convex side toward the object is shown in the second diagram. The third lens is shaped most favorably for the elimination of spherical aberration when the object point is infinitely distant. The focal length of the lens in each instance is 1.

(*b*) Shape of a Single Thin Lens for Minimum Spherical Aberration.—In Figure 7 the first drawing illustrates the ray paths through a plano-convex lens from an infinitely distant object. The lens is turned with the plane face toward the distant object. The total bending of the rays is at the second surface, and this is the unfavorable position for a small amount of spherical aberration. If, as in the second drawing, the lens is reversed in position, the rays are bent almost equally at the surfaces, and the spherical aberration, although positive, is much decreased; but neither of these cases is the most favorable for the elimination of spherical aberration in the image of an infinitely distant object. To determine the most favorable shape, one sets π equal to -1 in the expression for A and determines, in the usual maner by differentiation, the value of σ which gives the minimum value. If $n=1.5$, for the minimum value, $\sigma=0.71$, and the lens is double convex with the first radius almost exactly six times the second. Such a lens is shown in the third drawing of Figure 7. To the right of each of the illustrations of Figure 7 there is a graph in which the spherical aberration as computed by equation (22) is plotted in the usual manner. As ordinates one plots *ph*, the "height" as it is usually termed, or distance from center of lens to point of incidence of ray. As abscissas the corresponding Lon. Sph. is plotted to a scale five times as open as the scale of ordinates. The diagram is plotted for a lens of unit focal length and the index of refraction selected is 1.5. For each of the above three cases there is given below the Ang. Sph. for an aperture of *f*/16.[24]

		Seconds
(Ang. Sph.)	Plano-convex lens	+56. 7
	Convexo-plane lens	+14. 7
	Minimum spherical	+13. 5

As a general rule when object and image are each real, a given double convex lens will give the greater freedom from spherical aberration when turned with the surface of greater curvature toward the more remote of the two conjugate points. If object and image are equidistant from the lens, the lens should be equiconvex, in which case all terms drop out of A (Taylor system) except the last.

(c) Values of π for Which a Component Free from Aberration Can Be Designed.—It has already been stated that it is impossible to design a single lens (with spherical surfaces) to give freedom from spherical aberration when object and image are real. It is not difficult to determine the values of π for which the spherical aberration may be made to vanish. The expression for A (equation (23)) may be written

$$A=a\sigma^2+b\sigma\pi+c\pi^2+d \tag{25}$$

where the letters a, b, c, and d represent the coefficients of equation (23). Writing $A=0$ and solving for σ

$$\sigma=-\frac{1}{2}\ \frac{b\pi}{a}\pm\sqrt{\frac{b^2\pi^2}{4a^2}-\frac{c}{a}\pi^2-\frac{d}{a}} \tag{26}$$

[24] A trigonometric computation of the spherical aberration for these three lenses gives the following values:

		Seconds
(Ang. Sph.)	Plano-convex lens	+57. 0
	Convexo-plane lens	+14. 8
	Minimum spherical	+13. 6

Setting the discriminant equal to zero and solving for π

$$\pi = \pm 2\sqrt{\frac{ad}{b^2-4ac}} \tag{27}$$

and substituting the values of a, b, c, and d

$$\pi = \frac{\pm 1}{n-1}\sqrt{(n+2)n} \tag{28}$$

The corresponding values of σ by substitutions in equation (26) are

$$\sigma = 2\,\frac{n+1}{n+2}\sqrt{(n+2)n} \tag{29}$$

These are the boundary values for which A can be made to vanish. For larger absolute values of π, A can be made negative; that is, the lens can be overcorrected. Using the two formulas above, the following table has been computed. Real solutions giving a lens free from spherical aberration exist when π has an absolute value greater than that tabulated. If π has the value tabulated, spherical aberration will vanish (to the third order) when σ has the tabulated value. The value of s for unit focal length corresponding to the positive value of π as tabulated, is given in the fourth column.

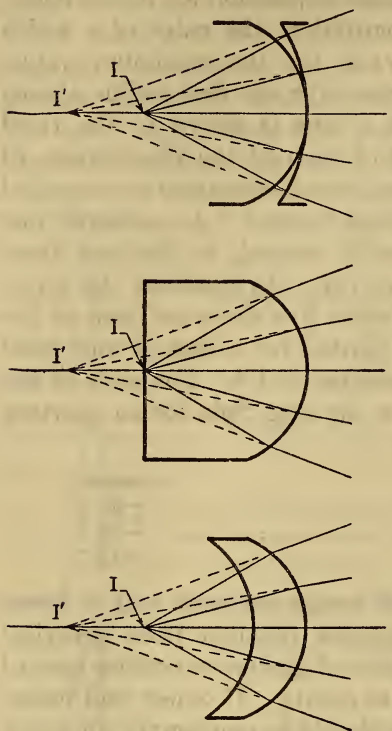

Fig. 8.—*The single lens and the aplanatic points*

The upper diagram shows a lens of zero axial thickness (negative thickness at edge) which images I at I' with complete freedom from spherical aberration and coma for points near the axis. The second diagram shows the realization of similar freedom from aberration by homogeneous immersion as in a microscope objective. The third lens is a meniscus with center of curvature of first surface at the object point. It gives similar freedom from aberration.

Table 1.—*Values of π for which the third order spherical aberration of a single lens may vanish*

n	π	σ	s
1.4	5.46	3.08	−0.31
1.45	4.97	3.18	−.34
1.50	4.58	3.27	−.36
1.55	4.26	3.37	−.38
1.60	4.00	3.47	−.40
1.65	3.78	3.56	−.42
1.70	3.58	3.66	−.44

(*d*) Aplanatic Points.—The thin lens illustrated in Figure 8 is an interesting special case. Both radii are intrinsically negative and

$$r=\frac{n+1}{n}r' \tag{30}$$

In the illustration the lens is actually drawn of zero axial thickness and the edge thickness is *negative* as the less curved surface is actually the first surface of the lens. If

$$s=\frac{n+1}{n}r' \tag{31}$$

a solution of the first order equations shows that

$$\frac{1}{f}=\frac{1-n}{1+n}\frac{1}{r'} \tag{32}$$

and

$$s'=(n+1)\ r' \tag{33}$$

Applying the equations for spherical aberration

$$\sigma=-(2n+1) \tag{34}$$

$$\pi=\frac{n+1}{n-1} \tag{35}$$

and

$$A=0. \tag{36}$$

For these two points, therefore, the third order spherical aberration vanishes. But it will be noticed that s and r are identical; that is, the object is situated at the center of curvature of the first surface. It is impossible to realize the lens of zero thickness as shown. But if the first surface of the lens is replaced by a plane surface passing through the object point, as shown in the second illustration, the lens becomes a thick lens in which the deviation of the rays occurs in exactly the same manner as in the original thin lens. This lens, therefore, is free from third order aberration. But it can also be shown that this pair of object and image points is free from all higher order spherical aberrations and free from coma as well.[25] A bundle of monochromatic rays proceeding from I gives rise to a rigorously homocentric refracted bundle with vertex at I', no matter how large the aperture. A pair of conjugate points free from spherical aberration and coma (*v.* infra) is termed aplanatic, and such points are referred to as the aplanatic points of the spherical surface. The first component of a microscope objective is generally designed to utilize this aplanatism. In the homogeneous immersion objective it

[25] Southall, Principles and Methods of Geometrical Optics, pp. 291–292.

is completely realized; in the water immersion or dry objectives, the aplanatism is only approximately obtained. The third illustration of Figure 8 shows a second form of lens for utilizing the aplanatic points. The first surface may be any surface concentric with the object point, and it is evident that the path of the rays is identical with that in the two lenses already described.

2. COMA OF A SINGLE THIN LENS

Spherical aberration results, as already stated, in a symmetrical enlargement of the image and as it is a function of o^3, but not of tan β, it is the same in amount (to a third order approximation) over all parts of the field. Coma, however, results in an asymmetric enlarge-

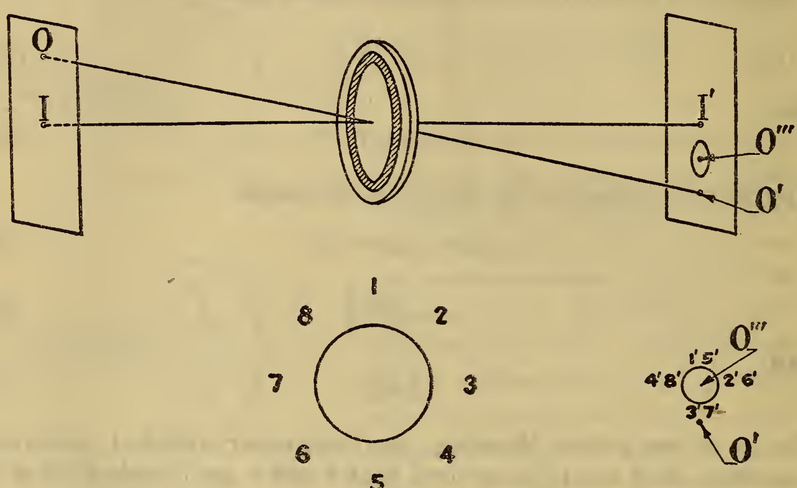

FIG. 9.—*Coma of the third order*

The oblique point O is imaged at O' by the chief ray and as a circle with center at O''' by the rays which pass through the shaded annulus on the lens.

ment of the image, is of the order o^2 tan β and, therefore, it is evidently zero on the axis and increases in magnitude as one proceeds from the center of the field outward. Reference to Figure 6 and the description of spherical aberration shows that if a lens is considered as divided into concentric circular zones each zone will focus a given axial object point at a different axial image point, and the composite image, in any selected image plane, is a circular disk made up of superposed circular concentric annuli. In the absence of coma this will also be true for the image of a point removed from the axis. In general, however, the asymmetry in the incident bundle produced by the displacement of the object point from the axis introduces an asymmetric flare in the image which is superposed on the spherical aberration and which is termed coma.

(*a*) FORMATION OF COMA IN THE ABSENCE OF SPHERICAL ABERRATION.—The formation of the image by a zone of a lens having coma (in the absence of spherical abberration) is shown in Figure 9. The diagram is in perspective, and the annular zone of the lens by which the image is formed is shaded. The chief ray from O passes through O' where the image is formed by rays infinitely near the chief ray. The image of O formed by the shaded area of lens is not concentric with O' but is the circular annulus with center at O'''. The relation between the points of the zone of lens and annular ring is shown in the two circles below the diagram in perspective and is not what one might at first expect. Rays passing through the zone of lens at points *1* and *5* form an image at the point most remote from O', as indicated by the numbers *1'* and *5'*. Rays traversing the lens at points *3* and *7* focus at *3'* and *7'*, the point nearest O'. Intermediate

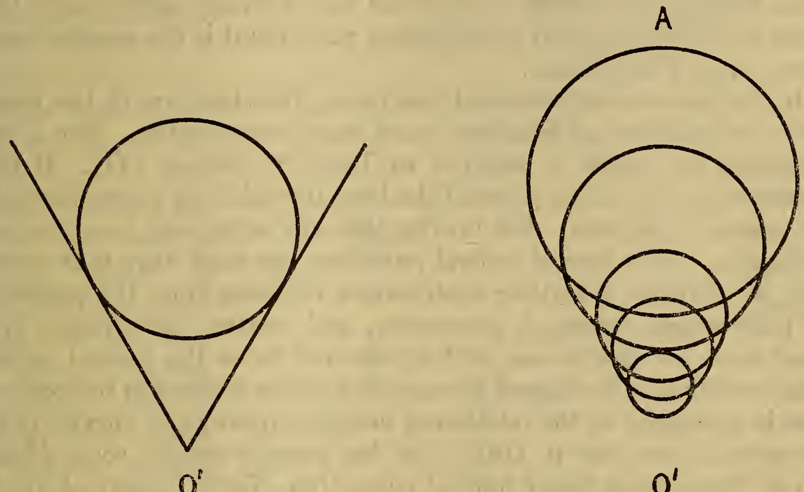

FIG. 10.—*Structure of comatic flare*

The first diagram shows that the tangents to any comatic circle in pure third order coma, drawn through the image point as determined by the chief ray, include an angle of 60°. The second diagram shows the series of comatic circles which together compose the comatic flare.

rays of the zone focus as indicated by the corresponding numbers. Hence, the annulus in the image plane is really a double or duplex circle, each point of which is the focus formed through diametrically opposite points of the zone. In fact, if coma of the type here described is the only aberration present, the image of a point will be a complete circle even though half the zone of the lens is obscured by a diaphragm having a diameter of the lens as one edge. Coma which arises when the diaphragm is in the plane of the lens and when there is no spherical aberration, as in the case here illustrated, is termed pure coma to distinguish it from the coma, to be treated later, which results from spherical aberration in combination with eccentric refraction (diaphragm not in plane of lens).

The metric relations in an image possessing pure coma are interesting. If tangents to the comatic circles are drawn through O', as in Figure 10, the angle included by the tangents is 60°. The image of a point formed by the entire lens is the assemblage of the images formed by the separate zones and is a family of circles having common tangents including an angle of 60°, as shown in the second drawing of Figure 10. The image of a point, therefore, is a figure shaped somewhat like a comet with a bright, well-defined point and an asymmetric flare extending in a radial direction toward or from the center which becomes more faint and diffuse as it recedes from the bright point. The overall length of the comatic flare is $O'A$ and will be referred to as the linear pure coma (Lin. Pure Coma). If, in the usual manner, this length is projected back into the object plane, the angle subtended by it at the entrance pupil point is the angular pure coma (Ang. Pure Coma).

In the presence of spherical aberration the structure of the third order comatic image becomes much more complicated. For a description, the reader is referred to Taylor,[26] Section VIII. If the entrance pupil is in the plane of the lens, the chief ray passes through the center of the lens. For brevity this will be referred to as central refraction. In a case of central refraction one may have pure coma and, in addition, a further modification resulting from the presence of pure coma, spherical aberration, and central refraction. The total coma in such a case will be referred to as the normal coma. This terminology is adopted because its relation to coma in the general case is analogous to the relation of normal curvature to curvature in the general case (see p. 110). One has angular normal coma (Ang. Norm. Coma) and linear normal coma (Lin. Norm. Coma), of which the values are given by the equations

$$(\text{Ang. Norm. Coma}) = -\frac{3}{4}\, o^2 h^2 \varphi^2 C \tan\beta, \tag{37}$$

$$(\text{Lin. Norm. Coma}) = -\frac{3}{4}\, o^2 h s' \varphi^2 C \tan\beta. \tag{38}$$

The coefficient of coma, C, is defined by the equation in the Taylor system

$$C = \frac{n+1}{n\,(n-1)}\sigma + \left(2 + \frac{1}{n}\right)\pi \tag{39}$$

or in the continental system

$$C = \frac{2(n+1)}{n}\,\frac{1}{\varphi r} - \frac{2(2n+1)}{n}\,\frac{1}{\varphi s} - \frac{2n}{n-1} \tag{40}$$

If the coefficient of tan β yielded by the above equations is positive, it indicates that the flare extends outward; that is, away from the optic axis with the bright pointed end turned inward.

[26] See footnote 6, p. 74.

If $A=0$, $C\neq 0$, for central refraction one has only pure coma.[27] If $A\neq 0$, $C=0$, one has pure coma sufficient to neutralize the coma which arises from spherical aberration, and if the entrance pupil is in the plane of the lens the image is free from coma.

(*b*) Flare Produced by Normal Coma and Spherical Aberration.—When the refraction is central—that is, when the diaphragm is in the plane of the lens—one has only normal coma of the type determined by equations (37) and (38). In a lens system, in general, the refraction is eccentric and there will be additional coma which arises from the combined effect of eccentric refraction and spherical aberration. The manner in which coma of this nature arises is shown diagrammatically in Figure 11. In the upper figure the course of the rays from an infinitely distant oblique object point is shown. There is no pure coma, and therefore the rays in the object space, symmetrically placed with respect to the ray passing through the center of the lens, intersect upon the conjugate ray in the image space, and the images produced by the different zones of the lens are centered with respect to each other, but do not lie in one plane, since it is assumed that there is spherical aberration. The section of the refracted cone of rays formed on any plane in the neighborhood of the image, and perpendicular to the axis of the lens, is therefore circular (except for the slight foreshortening arising from the obliquity of the object point), and there is none of the side flare

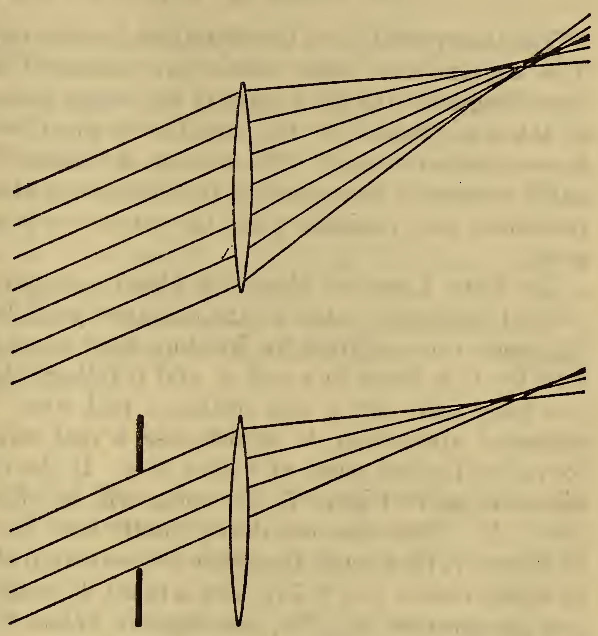

Fig. 11.—*A lens with spherical aberration but no pure coma*

When the bundle of rays passes through the lens centrally as in the upper diagram, there is spherical aberration, but no comatic asymetry. With a diaphragm without the plane of the lens, as in the lower diagram, the eccentric refraction and spherical aberration conspire to produce a comatic flare.

[27] Pure coma, as here defined, is the coma which arises from the failure of the lens to satisfy the sine condition. See Southall, l. c., p. 400–415. Taylor proves that $C=0$ and the satisfying of the sine condition are identical when there is no spherical aberration. In the presence of spherical aberration, however, $C=0$ has the significance given above.

characteristic of coma. In the lower part of Figure 11 the course of the rays is shown when a diaphragm is placed some distance in front of the lens. It is evident from the drawing that the symmetrical character of the image has disappeared as a result of the selective action of the stop and the spherical aberration. A complete formula for the length of comatic flare must, therefore, include coma of this sort and is given in the following equations:

$$\text{(Ang. Coma)} = \frac{3}{8} o^2 (gh^3\varphi^3 A - 2h^2\varphi^2 C) \tan \beta \tag{41}$$

$$\text{(Lin. Coma)} = \frac{3}{8} o^2 s' (gh^2\varphi^3 A - 2h\varphi^2 C) \tan \beta \tag{42}$$

The interpretation of the equations is as for equations (37) and (38). The angular and linear values are measured in a radial direction extending from the axial point of the image plane and if the coefficient of $\tan \beta$ as yielded by the equation is positive it indicates that the flare extends outward. The term in A (defined in equations (23) and (24)) represents coma arising from spherical aberration and eccentric refraction and vanishes when the refraction is central; that is, when $g=0$.

(*c*) Thin Lens of Minimum Coma.—If one has a thin lens and central refraction—that is, the entrance pupil located in the plane of the lens—the condition for freedom from coma is $C=0$. The expression for C is linear in σ and π, and it follows that for any value of π one may solve for σ and obtain a real root. This contrasts with spherical aberration, in which case a real solution for $A=0$ exists for only a limited range of values of π. If the object is at an infinite distance, as in Figure 7, the coma will be eliminated if $\sigma = +0.80$ $(n=1.5)$. This does not differ greatly from the third case illustrated in Figure 7, that most favorable for reduction of spherical aberration, in which case $\sigma = +0.71$. For a point 2° from the center of the field and an aperture of *f*/16, the angular values [28] of the coma for the three cases of Figure 7 are

		Seconds
(Ang. Norm. Coma)	Plano-convex lens	+31.7
	Convexo-plane lens	−3.5
	Lens of Min. Sph	+1.6

If one solves simultaneously the equations

$$A=0 \tag{43}$$

$$C=0 \tag{44}$$

one obtains the values of π and σ for which the spherical aberration and coma both vanish. The roots are

$$\pi = \pm\frac{n+1}{n-1} \tag{45}$$

$$\sigma = \mp (2n+1) \tag{46}$$

[28] A trigonometric computation of the coma for these three lenses gives the following values:

		Seconds
(Ang. Norm. Coma)	Plano-convex lens	+32.8
	Convexo-plane lens	−3.6
	Lens of Min. Sph	+2.0

Reference to page 99, shows that this solution includes the aplanatic points. Abbe was the first to limit the use of the term aplanatic to conjugate points which are free from both spherical aberration and coma.

A solution of the equations

$$\frac{\partial A}{\partial \sigma}=0, \tag{47}$$

$$C=0, \tag{48}$$

gives the values of π and σ for which the lens of minimum spherical aberration is free from coma. If the case $n=0$ is excluded, the two equations can be satisfied simultaneously only when $\sigma=0$, $\pi=0$. In this case the lens is equiconvex or equiconcave and object and image are symmetrically placed. The impossibility, in general, of constructing a lens of minimum spherical aberration free from coma is illustrated in the tabulated values of the coma of the three lenses of Figure 7 given above.

If eccentric refraction is admitted, one has an additional degree of freedom in the placing of the diaphragm. One can first design the component so that the spherical aberration vanishes or is a minimum after which the location of the diaphragm is determined by solving for g in the equation

$$gh^3\varphi^3A-2h^2\varphi^2C=0 \tag{49}$$

If one applies this to the lens of minimum spherical aberration ($n=1.5$, $\pi=-1$, $\sigma=+0.71$), the value of g for zero coma is -0.07. If the lens is provided with a diaphragm serving as entrance pupil and placed 0.07 of the focal length in front of the lens, coma due to spherical aberration will be introduced sufficient in amount and of proper sign to compensate for the normal coma. (See p. 102.)

3. CURVATURE OF IMAGE AND ASTIGMATISM OF A SINGLE THIN LENS.

It has been shown that when an oblique point is imaged by a lens the portions of the image formed by different parts of the lens are shifted with respect to each other in a direction lying in the plane of the image, and that the result is an asymmetric side flare which is known as coma. In addition to this, there is a shifting of the different parts of the image in a direction normal to the image plane which differs from spherical aberration in that it is different for object points at different distances from the axis and which gives rise to astigmatism and curvature of field, two aspects of the aberration of order $\frac{o}{f}\tan{}^2\beta$. The principal features of this phase of the imagery of an oblique point are illustrated in Figure 12.

The plane which contains the chief ray before and after refraction and the optic axis is indicated by the vertical hatching, and is termed the primary plane. In the usual optical diagram, such as Figures 1 and 2, only the primary plane is shown. The secondary plane contains the chief ray, is normal to the primary plane, and is indicated by the oblique hatching in Figure 12. As a result of the obliquity of the object point the lens introduces different amounts of convergence in the two planes with the result that the rays lying in the primary plane come to an approximate focus at O'_p those in the secondary plane at O'_s. Mathematically this is described by stating that the emergent wave front is a surface having two principal curvatures which lie in the primary and secondary planes.

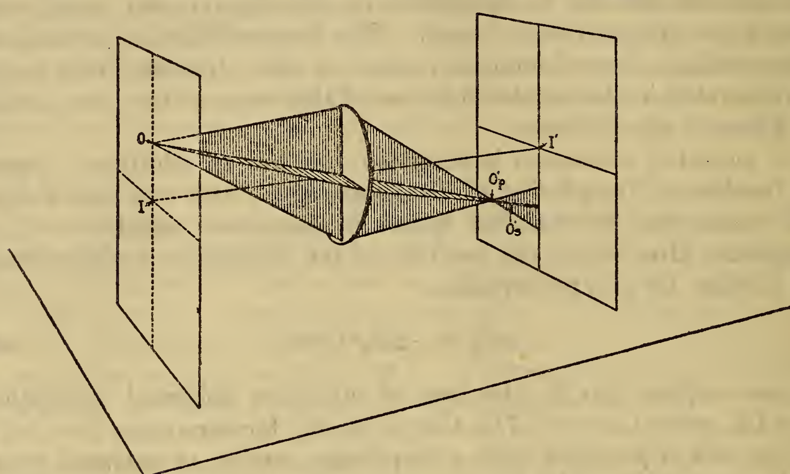

FIG. 12.—*Formation of astigmatic image*

The primary pencil of rays lies in the plane with vertical hatching and produces the image at O'_p. The secondary rays lie in the pencil with oblique hatching and form an image at O'_s.

All rays of the emergent bundle pass through a small line perpendicular to the chief ray and lying in the secondary plane at O'_p, and similarly all the rays pass through a second straight line perpendicular to the optic axis and lying in the primary plane at O'_s. These are lines only in the third order sense; that is, the disk caused by the aberration is of the third order in one dimension and of higher order in the other when measured in the usual manner by the angle subtended at entrance pupil point. These two lines are termed the Sturm focal lines and are the portions of smallest cross sectional area of the emergent bundle, and hence of greatest light intensity on the basis of geometrical optics. Consequently, they are the positions where the best image of the oblique object point lies. But whereas each point has been considered hitherto as having but one image, here the point O has two images, one at O'_p, the second at O'_s, which are referred to, respectively, as the primary and secondary images. The primary image is a short line, or rather the short portion of the arc of a circle concentric about the axial point in the image plane. This is illustrated in Figure 13, which shows a section of the emergent bundle in the vicinity of the image.

Let it be assumed that the object to be imaged is a system of lines passing through the axial point of the object plane and a system of concentric circles, as shown in Figure 14. On the primary image surface each point is imaged as a short circular arc having its center

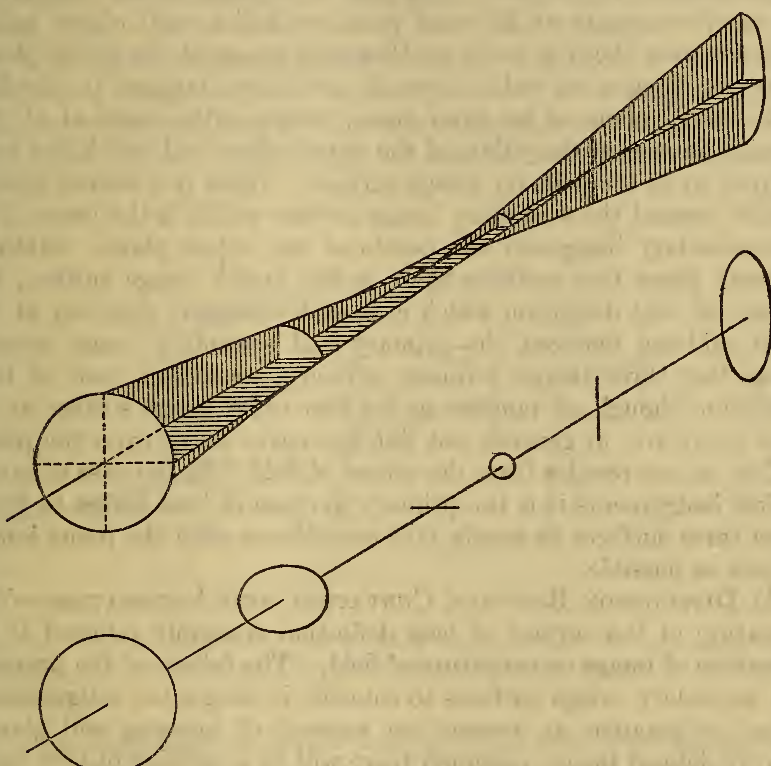

FIG. 13.—*Imagery in the neighborhood of an astigmatic image point*

The lower diagram shows the Sturm's focal lines, the elliptical sections of the bundles on either side, and the so-called circle of least confusion between the focal lines.

of curvature at the axial point. Consequently, the image will be as shown by the second drawing of Figure 14, with the straight lines diffusely imaged and with the circles sharply defined. However, the imagery of the circles is not perfect, as the dotted circle will go over into a continuous circle less bright than the solid ones. This is because the image of each point on the circle is a short arc, and these arcs bridge over the interruptions in the dotted circle, thereby preventing their reproduction in the image. Similarly, in the secondary image surface, one has the straight lines sharply defined and the

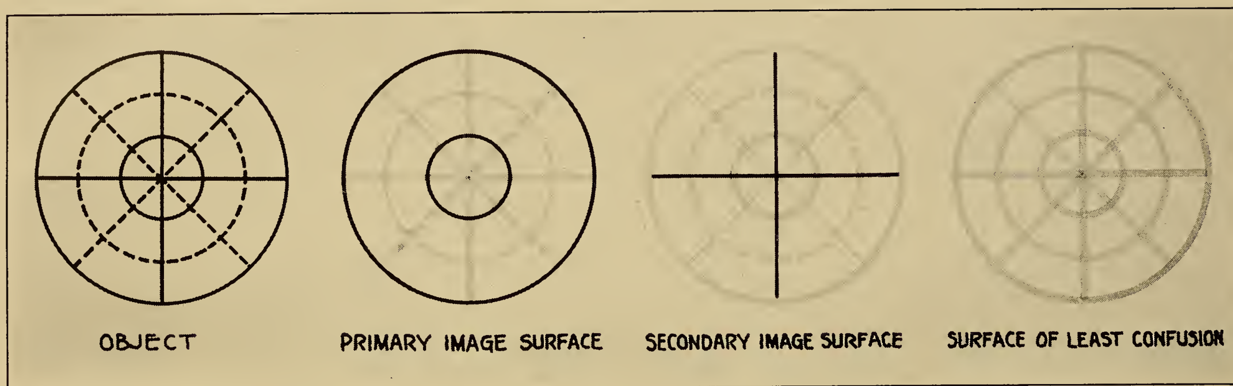

FIG. 14.—*The character of image on the different image surfaces in the presence of astigmatism*

The object, shown at the left, consists of radial full and dotted lines and concentric circles, full and dotted. The center of the object is on the axis of the system. On the primary image surface all concentric circles are strongly defined, but the dotted circles go into full circles of lower brightness. On the secondary image surface the radial lines are sharply defined, the dotted lines going into less bright full lines. On the surface of least confusion all lines are equally blurred.

circles diffused, and as before a dotted straight line is rendered as a continuous one. To get the best average definition for such an object, a surface approximately halfway between the primary and secondary image surface should be selected in which the diffusion is approximately the same in amount in all directions (see fig. 13). The image on this surface is shown farthest to the right in Figure 14.

(*a*) Four Image Surfaces.[29]—It is therefore seen that one has four image surfaces of importance corresponding to an object plane perpendicular to the axis. The first is a plane surface normal to the axis, conjugate at its axial point with the axial object point. This contains the first order or Gaussian image of the entire plane. There is, then, a curved surface of revolution, tangent to the first order image plane at its axial point, which is the locus of all the primary images of the points of the object plane and which has been referred to as the primary image surface. There is a second similar surface termed the secondary image surface which is the locus of all the secondary images of the points of the object plane. Midway between these two surfaces there is the fourth image surface, the surface of best definition which cuts each emergent chief ray at the point midway between the primary and secondary image points. These last three image surfaces, primary, secondary, and of best definition, though all tangent to the first order image surface at its axial point are, in general, not flat but curve away from the plane surface as one recedes from the center of field. In the case of many optical instruments it is the primary purpose of lens design to bring these three surfaces as nearly into coincidence with the plane image surface as possible.

(*b*) Distinction Between Curvature and Astigmatism.—The curvature of the surface of best definition is simply referred to as curvature of image or curvature of field. The failure of the primary and secondary image surfaces to coincide is designated astigmatism. When astigmatism is present, no amount of focusing will give a sharply defined image, although there will be a position of best focus as illustrated in Figure 14. If the primary and secondary image surfaces coincide, the surface of best definition is also coincident with the first two surfaces. If these three coincident surfaces are curved, one has curvature of field without astigmatism. In such a case any circular zone of the field may be made sharply defined by appropriate focusing, but as a result of the presence of curvature all parts of the field can not be sharply focussed simultaneously. If the curvatures of the primary and secondary image surfaces are equal, but in opposite sense, then the surface of best definition, lying midway between them, is flat and coincident with the first order image plane. In such a case one has a flat field; that is, freedom from curvature,

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[29] There is a fifth surface, the Petzval surface, which will be introduced later. See p. 110.

but the astigmatism is not eliminated. The marginal points can never be focussed sharply, but all parts of the field will be best defined on a plane surface when the axial point is sharply focussed.

(c) EQUATIONS OF CURVATURE AND ASTIGMATISM.—In the presence of astigmatism the primary and secondary image surfaces are different distances from the first order image plane, and, hence, the section which it makes of the emergent bundle is an ellipse with its two axes lying in the primary and secondary planes (see fig. 15). The ellipse in the object plane, conjugate to the elliptical image in the image plane, may be determined by first order imagery. The angles at the entrance pupil point, subtended by the axes of this ellipse, which lie in the primary and secondary planes, are defined as angular primary curvature (Ang. Pri. Curv.) and angular secondary curvature (Ang. Sec. Curv.), respectively. A second measure of the curva-

FIG. 15.—*Sturm's focal lines, the primary image surface, the secondary image surface, and the elliptical aberration area on the plane containing the paraxial image point*

ture and astigmatism is the distance from the first order image plane to the primary and secondary image points. These values will be referred to as the longitudinal primary curvature (Lon. Prim. Curv.) and longitudinal secondary curvature (Lon. Sec. Curv.) and are analogous to the longitudinal spherical aberration. A third measure of curvature and astigmatism is the inverse of the radius of curvature at the axial point of the primary and secondary image surfaces. These will be designated as primary and secondary curvature, (Pri. Curv.) and (Sec. Curv.). The values of these different measures of curvature are determined by the equations:

$$(\text{Ang. Pri. Curv.}) = o\left\{\frac{\varphi}{n} + 3\left(\varphi + g^2h^2\varphi^3\,\frac{A}{4} - gh\varphi^2 C\right)\right\}\tan^2\beta \qquad (50)$$

$$(\text{Ang. Sec. Curv.}) = o\left\{\frac{\varphi}{n} + \left(\varphi + g^2h^2\varphi^3\,\frac{A}{4} - gh\varphi^2 C\right)\right\}\tan^2\beta \qquad (51)$$

$$(\text{Lon. Pri. Curv.}) = \frac{s'^2}{2h^2}\left\{\frac{\varphi}{n} + 3\left(\varphi + g^2h^2\varphi^3\frac{A}{4} - gh\varphi^2 C\right)\right\}\tan^2\beta \quad (52)$$

$$(\text{Lon. Sec. Curv.}) = \frac{s'^2}{2h^2}\left\{\frac{\varphi}{n} + \left(\varphi + g^2h^2\varphi^3\frac{A}{4} - gh\varphi^2 C\right)\right\}\tan^2\beta \quad (53)$$

$$(\text{Pri. Curv.}) = \frac{\varphi}{n} + 3\left(\varphi + g^2h^2\varphi^3\frac{A}{4} - gh\varphi^2 C\right) \quad (54)$$

$$(\text{Sec. Curv.}) = \frac{\varphi}{n} + \left(\varphi + g^2h^2\varphi^3\frac{A}{4} - gh\varphi^2 C\right) \quad (55)$$

A positive sign indicates that the image surface is concave toward the incident light.

(*d*) Petzval Curvature.—It will be noted that if the quantity in the round brackets vanishes the primary and secondary surfaces will have the same curvature; that is, there will be no astigmatism. In such a case the value of the curvature will be $\frac{\varphi}{n}$, which is known as the Petzval curvature and which defines the Petzval surface. This is the only case in which the astigmatism can be caused to vanish, and it follows that a flat image free from astigmatism can not be obtained with a single lens except for the trivial case in which $\varphi = 0$. The curvature of the Petzval surface is of fundamental importance. It is independent of shape of lens and position of object or diaphragm in third order imagery. In the presence of astigmatism the primary image surface always curves away from the Petzval surface three times as rapidly as the secondary image surface. This is shown by the identity of the values of the two curvatures except for the factor 3 in front of the rounded parentheses.

(*e*) Normal Curvature.—If the diaphragm is in the plane of the lens, then $g = 0$ and the two curvatures are $\frac{\varphi}{n} + 3\varphi$ and $\frac{\varphi}{n} + \varphi$, respectively. These have been termed the normal curvatures by Taylor, and as they are independent of σ and π, it follows that for a single lens with central refraction the values of the curvature are beyond the control of the designer except as φ and n can be changed. If the image is in the plane of the lens, then $h = 0$, and again one has only the normal curvatures. This condition is approximately realized in the field lens of many eyepieces.

(*f*) Curvature Arising from Eccentric Refraction Plus Spherical Aberration or Coma.—The term in A represents the contribution to the curvature resulting from the combination of spherical aberration and eccentric refraction. The manner in which this arises may be readily understood. In case of a real image and undercorrected spherical, the rays passing through the margin of the lens come to a focus nearer the lens than do the central rays. Consequently if a diaphragm is placed in front of or behind the lens to produce eccentric refraction the images of object points nearer the edge of the field will be produced, because of the selective action of the diaphragm, by the edge rays. These rays are refracted most strongly by the lens, and the positive curvature of the image surface will therefore be increased.

Similarly, the term in C is the contribution to the curvature resulting from the combination of coma and eccentric refraction. But as coma is asymetric, the direction of the shift changes with the sign of g or h. With a single positive lens, A is rarely negative, and it follows that coma plus eccentric refraction is the chief means by which one can secure negative curvature to compensate for the positive curvature of the preceding term. By choosing the proper shape factor, one may so determine A and C that the primary image surface, the secondary image surface, or the surface of best definition (having curvature equal to the mean of the primary and secondary) may be made flat, but this flatness can not be secured without the presence of astigmatism and coma.

(*g*) Control of the Curvature of a Single Lens.—It has already been noted that when the diaphragm is in the plane of the lens ($g=0$), or when the image is in the plane of the lens ($h=0$), it is impossible to exercise any control upon the curvature and astigmatism by changing the shape of the lens. The curvatures will be normal, and the relatively small amount of control through change of index is all that is available. If, however, a diaphragm not in the plane of the lens is assumed, an additional degree of freedom is available and there are several methods of correction. One may set

$$\varphi+g^2h^2\varphi^3\frac{A}{4}-gh\varphi^2C=0 \tag{56}$$

in which case an image free from astigmatism but having the Petzval curvature will be obtained. If one sets

$$\frac{\varphi}{n}+2(\varphi+g^2h^2\varphi^3\frac{A}{4}-gh\varphi^2C=0 \tag{57}$$

(the mean of primary and secondary curvatures) the surface of best definition will be flat, but there will be considerable astigmatism. Also either the primary or secondary curvature may be caused to vanish. To obtain a solution it is necessary to solve a quadratic in g, and it may frequently happen that the roots are imaginary.

If one returns to the lens of minimum spherical aberration of Figure 7, it will be found that imaginary roots are obtained for each solution suggested above. A little consideration will show that this is to be expected. A correction of the normal positive curvature can only be secured by the selective action of the diaphragm plus spherical or comatic aberration. But it has been shown that this lens is substantially free from each of these. If an infinitely distant object is to be imaged by a single lens, and the field must be sharp and flat for a considerable distance from the axis, it is necessary to effect a compromise. Some coma must be introduced with a consequent reduction of sharpness for points near the axis in order that sufficient control may be secured to enable the marginal image points to be brought into the plane of the image. The form usually adopted is a concavo-convex meniscus with the diaphragm in front of the lens or a convex-concave meniscus with the diaphragm following.

Coma is an aberration which causes the definition to fall off as an object recedes from the center of the field. Astigmatism (q. v.) has a similar effect, but whereas astigmatism is a function of $\tan^2\beta$ coma is a function of $\tan \beta$ and is therefore the more important for small values of β. For a lens of large aperture coma may seriously interfere with definition, even for points very near the axis. In the microscope objective the elimination of coma is almost if not quite as important as the compensation of the spherical aberration.

As astigmatism and coma each increases from center of field outward, the effect of coma is frequently attributed to astigmatism and curvature of field. With a point source the two are easily distinguished. If the image is formed near the edge of the field and viewed under relatively low magnification (20 to 30 diameters) in the presence of coma, an unmistakable side flare will be present.

The elimination of curvature and astigmatism is particularly important in the case of photographic objectives. The field of view is relatively large and, if definition is to be good over all the field, there must be reasonable freedom from astigmatism and curvature. In an instrument in which the image is viewed by the eye a moderate amount of curvature is not seriously detrimental, as the eye can compensate for this to a certain extent by varying the accommodation, but on the photographic plate the entire image must be registered simultaneously on a plane surface.

4. DISTORTION OF A SINGLE THIN LENS

The presence of any of the preceding aberrations has a detrimental effect on the definition. Distortion differs from these, as its presence in no wise affects the quality of definition of the image of a point, but rather its location in the image plane. In the absence of distortion a geometric figure in the object plane is imaged as a geometrically similar figure in the image plane. In the presence of distortion this metric relationship between object and image is not preserved.

In Figure 16 cases of positive and negative distortion are illustrated. If the distortion is positive, a rectangular network is imaged as in the upper left-hand figure. This is termed "pincushion" or "hour-glass" distortion. With negative distortion the lines are bent in the opposite sense and one has "barrel" distortion. The distortion of a lens is always important if the metric properties of the image are to be utilized. A binocular may have considerable distortion and still be satisfactory. In fact, it may even be desirable to introduce a small amount of positive distortion as otherwise a straight line near the edge of the field may appear to be bowed in at the center because of the circular boundary of the field of view. A photographic lens used for landscape photography may, without serious detriment, possess a great deal of distortion, as there are generally few straight lines in the field of view by which the distortion may be made evident. But if the lens is to be used as a process lens for copying drawings, or if it is to be used for the production of aerial photographs from which maps are to be constructed, it is obviously of the greatest importance that the distortion be made as small as possible.

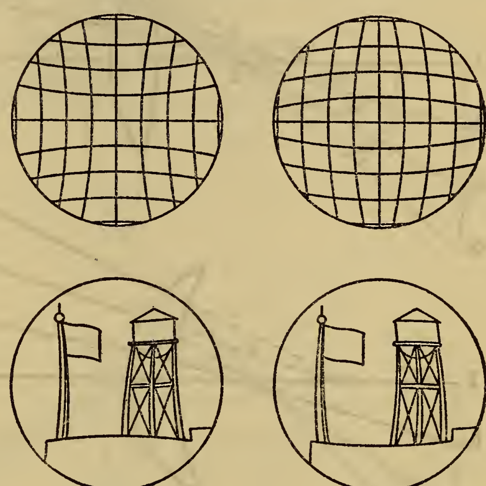

Fig. 16.—*Images as produced by a lens with distortion*

The left-hand images show positive or "pincushion" distortion. In the right-hand images one has negative or "barrel" distortion.

(*a*) Conditions Necessary for Freedom from Distortion for all Positions of Object Plane.—In Figures 17 and 18 the paths of the principal rays with and without distortion are illustrated. In the upper diagram of Figure 17 a bundle of rays is illustrated which passes through the pupil points D and D' and which has the following characteristics: If each incident ray and its conjugate refracted ray are produced to intersect, all the points of intersection lie in a plane normal to the axis of the lens. (The trace of this plane is indicated by the dotted line.) Also the incident and refracted pencils converge homocentrically to the pupil points D and D'. These two characteristics will be subsequently referred to as (*a*) The coplanarity of points of intersection of incident and refracted chief rays. (*b*) The freedom of pupil points from aberration.

In the case, illustrated in the upper diagram, Figure 17, it is evident that

$$\frac{\tan\,\alpha'_a}{\tan\,\alpha_a}=\frac{\tan\,\alpha'_b}{\tan\,\alpha_b}=\frac{\tan\,\alpha'_c}{\tan\,\alpha_c}=\text{etc.} \tag{58}$$

and if any object plane $I\ O$ is selected, homologous lengths in it and the conjugate image plane $I'\ O'$ have the constant ratio expressed by the equation

$$\frac{I'O'_a}{IO_a}=\frac{I'O'_b}{IO_b}=\frac{I'O'_c}{IO_c}=\text{etc.} \tag{59}$$

Geometric figures in the object plane are imaged as geometrically similar figures, and the image is said to be free from distortion.

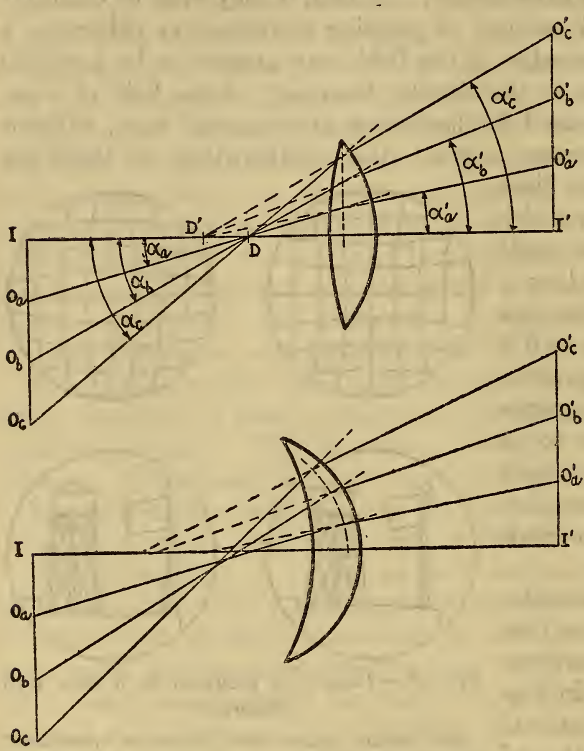

FIG. 17.—*Two lenses which give a distortion-free image*

In the upper diagram the intersections of conjugate chief rays lie in a plane normal to the axis, and the pupil points D and D' are free from aberration. Any pair of conjugate planes are free from distortion. In the lower diagram the intersections of the chief rays do not lie in a plane and the pupil points show considerable aberration. For the particular pair of conjugate planes indicated there is no distortion, but this will not be true for other pairs of planes

In the second diagram of Figure 17 the intersections of incident and refracted chief rays are neither coplanar nor are the pupil points free from aberration. But even so, it is possible for equation (59) to be satisfied for a particular object and image plane as, for example, those indicated in the diagram, in which case the image is free from distortion. It is only necessary that the distortions arising from the two causes compensate each other. But in any given case this compensation can only be effected for a particular pair of planes, and by such means freedom from distortion for all pairs of planes, as in the first case of Figure 17, can not be obtained. The upper diagram of Figure 18 illustrates the case in which one has coplanarity of the intersection points, but the pupil points are not free from aberration. On the other hand, in the lower figure one has freedom from aberration of the pupil points but not coplanarity of the intersection points.

(*b*) EQUATIONS OF DISTORTION.—The displacement of a point from its distortion free position is in a radial direction toward or from the center of the field. One may use this displacement (Dist.) as a measure of distortion, or one may in the usual manner project this displacement backward through the lens into the object plane and use as a measure the angle subtended at the entrance pupil point by this length. This last measure will be referred to as the angular distortion (Ang. Dist.). The values of these two measures of distortion are given by the two equations

$$(\text{Ang. Dist.}) = \frac{1}{4}\left(g^2\varphi^2 T + g^3 h\varphi^3 \frac{B}{2}\right)\tan^3\beta, \tag{60}$$

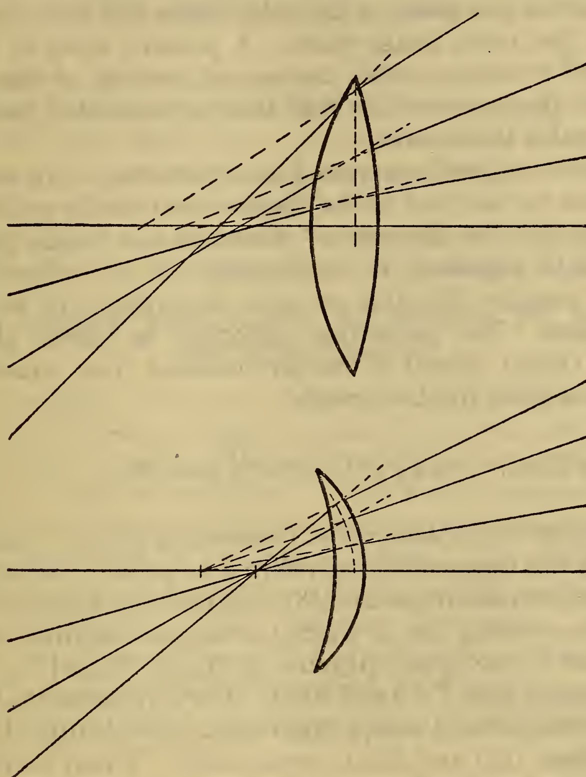

Fig. 18.—*Two lenses which show distortion*

In the first diagram the chief rays intersect in a plane, but the pupil points have aberrations. In the second diagram the pupil points are free from aberration, but the conjugate chief rays do not intersect in a plane.

$$(\text{Dist.}) = \frac{s'}{4h}\left(g^2\varphi^2 T + g^3 h\varphi^3 \frac{B}{2}\right)\tan^3\beta. \tag{61}$$

The coefficients of distortion T and B are defined by the equations

Taylor

$$T = \frac{n+1}{n(n-1)}\sigma + \frac{1}{n}\epsilon \tag{62}$$

Continental

$$T = \frac{1}{\varphi}\left[\frac{2(n+1)}{n}\frac{1}{r} - \frac{2}{n}\frac{1}{x} - \frac{2}{n-1}\varphi\right], \tag{63}$$

Taylor

$$B=\frac{n+2}{n(n-1)^2}\sigma^2+\frac{4(n+1)}{n(n-1)}\sigma\epsilon+\frac{3n+2}{n}\epsilon^2+\frac{n^2}{(n-1)^2} \tag{64}$$

Continental

$$B=\frac{4}{\varphi^3}\left[\frac{n^2}{(n-1)^2}\varphi^3+\frac{3n+1}{n-1}\frac{\varphi^2}{x}+\frac{3n+2}{n}\frac{\varphi}{x^2}-\frac{2n+1}{n-1}\frac{\varphi^2}{r}+\frac{n+2}{n}\frac{\varphi}{r^2}-\frac{4(n+1)}{n}\frac{\varphi}{rx}\right] \tag{65}$$

In the above formulas the distortion is measured from the position of the distortion free point to the point where the chief ray actually pierces the first order image plane. A positive value of the coefficient of $\tan^3 \beta$ indicates that the actual position of the image is farther from the center of the field than its distortion free position, a negative value the reverse.

Often the distortion is expressed as a percentage. In such a case the distortion is the ratio of the displacement of the point from its true position to the distance of distortion-free image point from center of field expressed in hundredths. It is evident that the measure of angular distortion or linear distortion will be identical in this system. The percentage distortion in object plane (Per Cent Dist. Object Plane) is readily obtained from equations (60) or (61) and is given by the formula

$$\text{(Per Cent Dist.)} \;=25\left(g^2\varphi^2 T+g^3 h\varphi^3\frac{B}{2}\right)\tan^2\beta \tag{66}$$

The condition for coplanarity of intersection points of chief rays is $T=0$. This has been sometimes referred to as the condition for constancy of tangent ratio (equation (58)), but this does not seem desirable, as the only interesting case in which the tangent condition is satisfied is that shown in the upper diagram of Figure 17, and for this it is equally necessary that $T=0$ and $B=0$. It will be noted that B is the same as the coefficient A except that ϵ and x are substituted for π and s (see equations (23) and (24)), respectively. B may be referred to as the coefficient of pupil point aberration, since it is a measure of the spherical aberration of the conjugate entrance and exit pupil points. In Figure 18 the upper diagram illustrates the case $T=0$, $B\neq 0$; the lower $T\neq 0$, $B=0$.

If T and B do not equal zero separately, but the quantity within the parentheses vanishes (equations (60) and (61)) one has the type of freedom from distortion illustrated in the lower diagram of Figure 17. As the parentheses contain a term in h, it is evident that it will vanish for only a particular pair of conjugate planes, an interpretation consistent with the discussion of the diagram given above.

(*c*) Thin Lens of Zero Distortion.—Reference to equation (60) shows at once that the distortion vanishes for the case $g=0$; that is, when the entrance and exit pupils are in the plane of the lens. This will be true for a thin lens of any shape and for any position of object. A second interesting case arises when $h=0$; that is, when the image is in the plane of the lens. The distortion is then independent of the value of B, and a distortion-free image can always be obtained by solving the linear equation $T=0$. If one attempts to solve the equations $T=0$, $B=0$, it is found that the roots are imaginary, and, hence, with a single thin lens, it is impossible to obtain the condition illustrated in the upper diagram of Figure 17. However, this case is important, as such a condition can be obtained with a system of lenses.

If the position of the object plane is given, h is determined. Equation (60) can then be equated to zero, yielding an equation which is quadratic in σ and linear in g. Hence, for a lens of any shape, a value of g can be found which gives an image free from distortion, but if the position of the entrance pupil is held fast and one tries to determine a σ for which the distortion vanishes, there will be many cases in which the roots are imaginary and no real solution can be obtained.

V. FIRST ORDER EQUATIONS EXTENDED TO A SYSTEM OF THIN LENSES

In the preceding treatment of first and third order equations of imagery, the thin lens has been considered as a unit, and the control of the aberrations which may be exercised in the case of the single lens has been dealt with in some detail. The limited extent to which the aberrations may be eliminated in a single lens makes necessary the use of several lenses forming an optical system. The introduction of each additional lens adds degrees of freedom which permit the more perfect compensation of the different aberrations. Before the aberration equations can be applied to the system of thin lenses, it is necessary to develop the application of the first order equations in order that the different parameters may be determined for the components of the system.

1. STEP-BY-STEP METHOD OF LOCATING THE IMAGE FORMED BY A SYSTEM OF THIN LENSES

The image of an object point formed by a system of thin lenses may be located by the application of equations (2) and (3) in succession to each lens. The lenses are numbered in order beginning with the one which receives the incident light, and subscripts are used to denote the lens to which any length is referred. For the first lens

$$\frac{1}{s'_1}=\frac{1}{s_1}+\frac{1}{f_1} \tag{67}$$

$$y'_1=\frac{s'_1}{s_1}y_1 \tag{68}$$

Before these formulas are applied to the second lens it is necessary to use the transformation formulas

$$s_2 = s_1' - t'_1 \tag{69}$$

$$y_2 = y'_1 \tag{70}$$

where t'_1 = the distance measured along the axis from the first to the second lens.

Equations (2) and (3) can now be applied to the second lens, after which transformation equations similar to Nos. 69 and 70 are applied before proceeding with the third lens. The significance of the transformation equations is illustrated in Figure 19. This process is

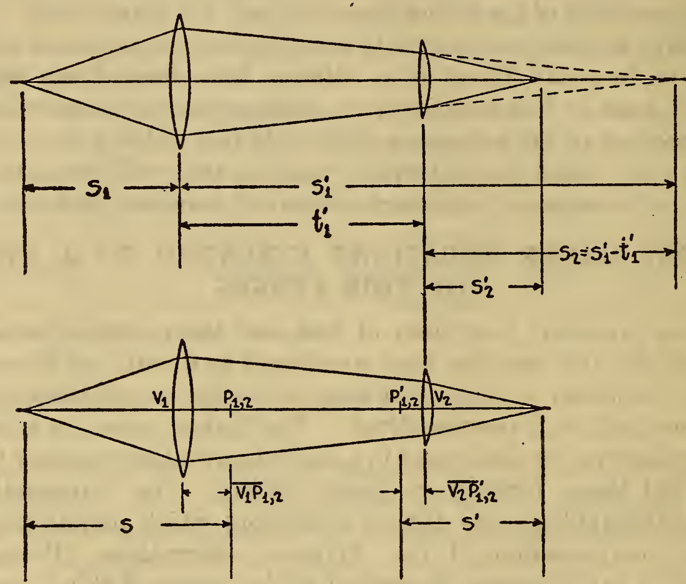

FIG. 19.—*Imagery by a system of thin lenses*

The upper diagram illustrates the derivation of the transformation equations when the final image is located by the step-by-step method. In the lower diagram are shown the lengths which enter when the lenses are combined by equations (71), (72), and (73).

continued until the image formed by the last lens, which is the image of the original object point as produced by the entire system, is located. This step-by-step method of locating the image is usually necessary in the design of an optical system, as the values of the object and image distances for the intermediate lenses are required in the aberration equations.

2. EQUIVALENT FOCAL LENGTH AND PRINCIPAL POINTS OF A SYSTEM OF LENSES

(*a*) A SYSTEM COMPOSED OF TWO LENSES.—If desired, the entire system may be combined and replaced by an equivalent thin lens, in which case the location of an image point can be determined by a single application of equations (2) and (3). The focal length of the thin lens, equivalent to a two-lens system, is determined by the equation

$$f_{1,2}=\frac{f_1 f_2}{f_1+f_2-t'_1} \tag{71}$$

One next determines the location of the principal points $P_{1,2}$ and $P'_{1,2}$. The first principal point is on the axis, in the object space, and its distance from V_1, the vertex of the first lens, is given by the equation

$$\overline{V_1P_{1,2}}=\frac{t'_1 f_1}{f_1+f_2-t'_1} \tag{72}$$

Similarly, the second principal point, which is in the image space, is distant $\overline{V_2P'_{1,2}}$ from the vertex of the second lens, where

$$\overline{V_2P^1_{1,2}}=\frac{-t'_1 f_2}{f_1+f_2-t'_1} \tag{73}$$

If $\overline{V_1P_{1,2}}$ or $\overline{V_2P'_{1,2}}$ is positive, it indicates that the incident light passes through the vertex of the lens before arriving at the respective principal point. If the optical system consists of only two lenses, the point conjugate to any object point can be determined by a single application of equations (2) and (3), provided that s and s' are defined as follows:

s = distance from $P_{1,2}$, the first principal point to the projection on the axis of the object point.

s' = distance from $P'_{1,2}$, the second principal point, to the projection on the axis of the image point.

s or s' is positive, if a generating point, moving in the direction of the incident light, passes through the principal point before it arrives at the object or image point.

y and y' are defined as before (see p. 80).

The second drawing of Figure 19 illustrates the application of equations (2) and (3) after the equivalent focal length of the two-lens combination has been determined.

(*b*) A System of More Than Two Lenses.—If there are more than two lenses in the system, equations (71), (72), and (73) may by applied successively until the equivalent focal length of the entire combination has been determined. It is convenient to rewrite these equations in the following form:

$$f_{1,2}=\frac{f_1 f_2}{f_1+f_2-\overline{P'_1P_2}} \tag{74}$$

$$\overline{P_1P_{1,2}}=\frac{\overline{P'_1P_2}f_1}{f_1+f_2-\overline{P'_1P_2}} \tag{75}$$

$$\overline{P^1_2P_{1,2}}=\frac{-\overline{P'_1P_2}f_2}{f_1+f_2-\overline{P'_1P_2}} \tag{76}$$

The subscripts $_1$ and $_2$ refer to the first and second members which are being combined, and either member may be either a single lens or a combination. The symbols with the double subscripts $f_{1,2}$, $P_{1,2}$, $P'_{1,2}$ refer to the equivalent focal length, first and second principal points of the system formed by the combination of members 1 and 2. The length $\overline{P'_1P_2}$ is the distance from the second principal point of the first lens system to the first principal point of the second system and is positive if, when considered in the sense indicated, it extends in the direction traveled by the incident light. If the first or second member to which the formula is applied is a thin lens instead of a system of lenses, no difficulty will be introduced if it is remembered that for a thin lens the principal points are coincident with the common vertex of the two surfaces. It is evident that by the repeated application of equations (74), (75), and (76) the focal length and principal points of a system made up of any number of lenses can be determined. It should be noted that at each step the location of the principal point is given with respect to the principal point of one of the preceding subordinate members. Hence, to determine the actual location of the principal point for the complete system it is necessary to retrace and sum up all the partial distances.

(*c*) A System of Thin Lenses in Contact.—An important special case is a system composed of any number of lenses of zero thickness in contact. In applying equations (74), (75), and (76) the distance between the two principal points $\overline{P_1P'_2}$ in each case is zero, and as a final result it will be found that for k lenses in contact

$$\frac{1}{f_{1,k}}=\sum_{1}^{k}\frac{1}{f_1} \tag{77}$$

and that the principal points of the system coincide with the common vertices of the several thin lenses.

(*d*) Telescopic System.—An important case arises when two subordinate systems to be combined are so separated that the distance between their inner principal points is equal to the sum of their focal lengths. In such a case $f_1+f_2-\overline{P'_1P_2}$ (see equations (74), (75), and (76)) is zero, and the focal length becomes infinite with the principal points at the infinite points of the object and image space. Such a system is telescopic and equations (2) and (3) do not apply. In a telescopic system any point on the axis in the object space may be taken as an origin in the object space, provided that in the image space the conjugate point is taken as origin. The equations of imagery for a telescopic system are

$$\underline{u}'_{1,2}=\frac{1}{A^2_{1,2}}u_{1,2} \tag{78}$$

$$y'_{1,2}=\frac{1}{A_{1,2}}y_{1,2} \tag{79}$$

f_1 = focal length of first subordinate member of telescopic system.
f_2 = focal length of second subordinate member of telescopic system.
A = angular magnification produced by the telescopic system and is defined by the equation

$$\mathsf{A}=\frac{f_1}{f_2} \tag{80}$$

$u_{1,2}$ = projection on the axis of the distance from any selected axial origin to object point.
$u'_{1,2}$ = projection on the axis of the distance from origin in image space to image point. The origin selected in the image space must be the axial point conjugate to the origin selected in the object space and may be located by application of equations (2) and (3) in succession to the lenses of the system (see p. 117).
$u_{1,2}$ or $u'_{1,2}$ is positive if a generating point, when moving in the direction of the incident light, passes through the origin before it arrives at the object or image point.
$y_{1,2}$ = distance of object point from the axis.
$y'_{1,2}$ = distance of image point from the axis.
$y_{1,2}$ or $y'_{1,2}$ is positive if measured upward from the axis in the plane of the diagram.

3. CHARACTERISTICS OF PARAXIAL IMAGERY

For the nontelescopic system equations (2) and (3) are the fundamental equations of imagery whether the system is composed of one or several thin or thick lenses, together with reflecting prisms.

$$\frac{1}{s'}=\frac{1}{s}+\frac{1}{f} \tag{2}$$

$$y'=\frac{s'}{s}y \tag{3}$$

As s and s' are the distances from the principal points to object and image points, respectively, projected upon the axis, it is evident that all points in a plane perpendicular to the axis at the point $s=s_o$ go into a plane in the image space perpendicular to the axis at the point $s'=s'_o$. Planes perpendicular to the axis, therefore, go into a family of planes perpendicular to the axis. The plane perpendicular to the axis at the point $s=-f$ goes into the infinitely distant plane in the image space. The plane $s=-f$ is called the first focal plane of the system, and the point where it intersects the optic axis is termed the first focal point. The infinitely distant plane perpendicular to the axis in the object space goes into a plane perpendicular to the axis at the point $s'=f$. This point and plane are, respectively, the second focal point and focal plane. The distance measured along the axis from the vertex of the first surface of an optical system to the first focal point is the front focal length (F. F. L.) of the system. If $\overline{V_1P}_{1,m}$, is the distance from the first vertex to the first principal point of the system.

$$(\text{F.F.L.}) = \overline{V_1P}_{1,\text{m}} - f_{1,\text{m}} \tag{81}$$

Similarly, the distance from vertex of last surface to the second focal point is the back focal length (B.F.L.).

$$(\text{B.F.L.}) = \overline{V_\text{m}P'}_{1,\text{m}} + f_{1,\text{m}} \tag{82}$$

These lengths are illustrated in the upper diagram of Figure 21.

(*a*) Lateral Magnification.—A short element of length δy (see fig. 20) perpendicular to the axis at I will have its image $\delta y'$ perpendicular to the axis at I'.

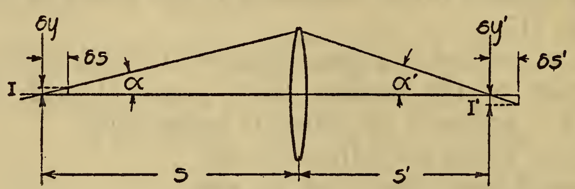

Fig. 20.—*Elements entering into the derivation of the values of the different magnifications*

The ratio of these two lengths is termed the lateral magnification and will be denoted by M. From equation (3)

$$\mathsf{M} = \lim_{\delta y \doteq o} \frac{\delta y'}{\delta y} = \frac{s'}{s} \tag{83}$$

Since M is a function of s and s', but not of y or y', it follows that object and image, when they lie in two conjugate planes perpendicular to the optic axis, are geometrically similar with the ratio of homologous lengths equal to M.

(*b*) Longitudinal Magnification.—Similarly, a short element of the optic axis in the neighborhood of I is imaged as a short element of the axis at I′. Let δs and $\delta s'$ represent these two lengths. The limit of $\frac{\delta s'}{\delta s}$ as δs approaches zero is termed the longitudinal magnification and will be denoted by L. Then from equation (2)

$$\mathsf{L} = \lim_{\delta s \doteq o} \frac{\delta s'}{\delta s} = \frac{s'^2}{s^2} \tag{84}$$

(*c*) Angular Magnification.—Assume that there is a line drawn through I (fig. 20) making an angle α with the optic axis. It is imaged as a line through I', making an angle α' with the optic axis. If $\frac{\delta y}{\delta s}$ is the tangent of α, $\frac{\delta y'}{\delta s'}$ is the tangent of α'. The limit of the ratio $\frac{\tan \alpha'}{\tan \alpha}$, as α approaches zero is termed the angular magnification and will be denoted by A. From equations (83) and (84)

$$\mathsf{A} = \lim_{\substack{\delta y \doteq o \\ \delta s \doteq o}} \frac{\delta y' \delta s}{\delta y \delta s'} = \frac{\mathsf{M}}{\mathsf{L}} = \frac{s}{s'} \tag{85}$$

In general, the longitudinal and lateral magnifications are not equal, and it follows that a spatial object and its image are not geometrically similar. If s approaches 0 as a limit, the ratio s'/s approaches 1 as a limit. For the two conjugate points $s=0$, $s'=0$—that is, for the two principal points—therefore, M and L are equal to 1, and this is the only pair of points in a nontelescopic system for which these two magnifications are equal. The planes perpendicular to the axis through these two points are planes of unit lateral and longitudinal magnification. A spatial object and its image in the neighborhood of the principal points are approximately similar and equal in size, the approximation, in general, becoming better as the object is decreased in size and brought nearer to the principal point. From equation (85) it is evident that the angular magnification is also unity at the principal points.

(*d*) Magnifications of an Optical System.—Equations (83), (84), and (85) give the values of the magnifications for a single lens. For a system of lenses the magnifications are equal to the products of the magnifications of the individual lenses. Thus, for a system of K lenses

$$\mathsf{M}_{1,\mathsf{K}} = \mathsf{M}_1 \mathsf{M}_2 \mathsf{M}_3 \; . \; . \; . \; \mathsf{M}_\mathsf{K} \tag{86}$$

with similar equations for the other magnifications.

(*e*) Definition of Focal Length in Terms of Tan α.—A pencil of parallel rays in the object space may be considered as a pencil of rays proceeding from a point of the infinitely distant plane. Since this plane is imaged as the plane $s'=f$, it follows that such a pencil of rays after refraction will come to a focus in the second focal plane. In Figure 21 assume that the pencil of parallel rays makes the angle α with the axis and consider the ray of the pencil which passes through the first principal point. The two principal points are conjugate; hence the refracted ray will pass through the second principal point. Furthermore, it will make an angle α with the axis, since the two principal points are points of unit angular magnification. But the distance from the second principal point to the second focal plane is

f. Therefore, the ray, after refraction by the system, will intersect the focal plane at the distance $f\tan\,\alpha$ from the axis. But the entire pencil of parallel rays making an angle α with the axis intersects the second focal plane at the same point. Hence, all rays in the object space, making an angle α with the axis, cut the second focal plane at a point distant $f\tan\,\alpha$ from the axis. This is sometimes used as a definition of the focal length, the equation being

$$f=\frac{y'}{\tan\,\alpha} \tag{87}$$

where

f = equivalent focal length,

α = angle between a ray in the object space and the optic axis,

y' = distance from optic axis to point where the ray after refraction intersects the second focal plane.

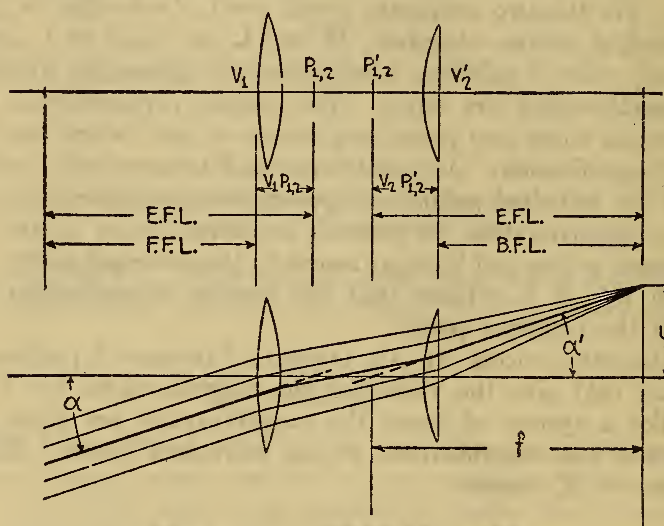

FIG. 21.—*Derivation of the definition of the focal length in terms y' and tan α*

Similarly

$$f=\frac{y}{\tan\,\alpha'} \tag{88}$$

where

α' = angle between a ray in the image space and the optic axis,

y = distance from optic axis to point where the ray before refraction intersects the first focal plane.

(*f*) MAGNIFICATION OF A TELESCOPIC SYSTEM.—For a telescopic system equations (78) and (79) are fundamental.

$$u'_{1,2}=\frac{1}{A^2_{1,2}}u_{1,2} \tag{78}$$

$$y'_{1,2}=\frac{1}{A_{1,2}}y_{1,2} \tag{79}$$

With further reference to Figure 20, assume as before that δy is a short element of length perpendicular to the axis at I and that $\delta y'$ is its image perpendicular to the axis at I'. Then, from equation (79)

$$\mathsf{M}=\lim_{\delta y \doteq o} \frac{\delta y'}{\delta y}=\frac{1}{\mathsf{A}} \tag{89}$$

Similarly from equation (78)

$$\mathsf{L}=\lim_{\delta u \doteq o} \frac{\delta u'}{\delta u}=\frac{1}{\mathsf{A}^2} \tag{90}$$

The angular magnification is A, and its value has already been given in equation (80). The telescopic system, therefore, differs from the nontelescopic system in that M, L, and A are constant for all pairs of conjugate points.

4. APPLICATION OF FIRST ORDER EQUATIONS TO DETERMINE THE POSITION OF STOPS AND THE FIELD OF VIEW

Before the third order equations can be applied to an optical system it is necessary to determine the entrance and exit pupils of the system from which one locates at once the entrance and exit pupils of each lens. In the optical system there may be one or more diaphragms spaced along the axis, and, in addition, the cell of each lens will be referred to as a diaphragm. Of this series there is, in general, some one which is the limiting diaphragm that determines the maximum aperture of an incident bundle which is entirely transmitted by the optical system. This particular diaphragm is the iris, and the diaphragms conjugate to it (see below) in the object and image spaces are the entrance and exit pupils, respectively.

(*a*) Method of Identifying the Entrance and Exit Pupil and Iris.—Frequently one can not determine by simple inspection which one of the several diaphragms is the iris. In such a case each diaphragm which is judged to be a potential iris is projected backward through all parts of the optical system lying between it and the object, and thus one locates in the object space areas each of which is conjugate to a material diaphragm of the system. Each one of these will be referred to as a conjugate object space diaphragm. Repeating the argument which was used in locating entrance and exit pupils of a single lens (see p.84), it is evident that each diaphragm of the system will transmit a bundle of rays which, in the object space, just clears its conjugate object space diaphragm. It is difficult to compare the diaphragming effect of the different apertures directly, as they lie in the different media, which are between the different components of the system, but the corresponding conjugate object space diaphragms can be compared directly, as they all lie in the same medium, which is the object space of the optical system.

Hence, it is seen that the conjugate object space diaphragm which restricts the entering bundle the most—that is, which subtends the smallest angle at the axial object point—is the entrance pupil of the system. The actual diaphragm conjugate to the entrance pupil is the iris, and it may be in the object or image space of the system or between any pair of components. The area in the image space conjugate to the entrance pupil (with respect to the entire system) and which will also be conjugate to the iris (with respect to the components of the system following the iris) is the exit pupil.

These relations are illustrated in Figure 22, which shows the optical parts of an elementary compound microscope with object at I, objective at L_1, diaphragm at EF, and eyelens at L_2. There are three potential irises in the system, the clear aperture of L_1 represented diagrammatically as limited by the diaphragm AB,

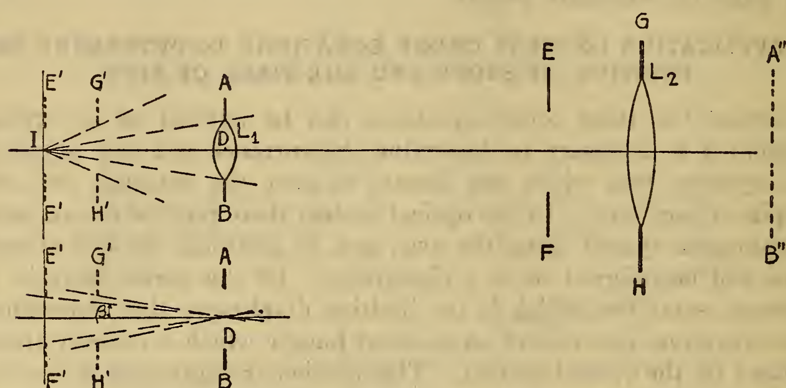

FIG. 22.—*Pupils and windows of an optical system*

the diaphragm EF, and the clear aperture of L_2 represented by diaphragm GH. As AB is in the plane of the objective, when projected into the object space, it is self-conjugate. The conjugate object space diaphragm corresponding to EF is $E'F'$ and that conjugate to GH is $G'H'$. It is apparent that AB subtends the smallest angle at I, and it is therefore the entrance pupil with entrance pupil point at D. In this case AB is also the iris. The conjugate image space diaphgram is $A''B''$ and is the exit pupil.

In the case illustrated, the tendency when using the microscope is to bring the entrance pupil of the eye (this is the virtual diaphragm in the object space conjugate to the iris of the eye and for many purposes can be assumed to be coincident with the iris) into coincidence with the exit pupil of the telescope because in this position the largest field of view is secured. This is the desirable condition, and when possible an instrument for visual use should be designed with a real exit pupil located sufficiently distant from the field lens to permit the entrance pupil of the eye to be brought into coincidence with it. But in some instruments, notably the Galilean telescope, the exit pupil is virtual; that is, it is in front of the eyelens, and the eye can not be brought into the desired position. In other cases, particularly with instruments in which a short focus eyepiece is employed, the exit pupil is real, but so close to the eyelens that mechanical interference by the eyelens cell prevents the desired positioning of the eye. In such a case the positon of the eye must be assumed and the entrance pupil of the eye must be treated as one of the potential irises of the system. In general, the end result will be that the entrance pupil of the eye is the exit pupil of the instrument, and it follows that the entrance pupil of the instrument is the conjugate diaphragm in the the object space which is conjugate to the entrance pupil of the eye.

(b) Entrance and Exit Windows and Field Diaphragm.—After the entrance pupil point has been located, one is in a position to determine the entrance window. Of the conjugate object space diaphragms, the one subtending the smallest angle at the entrance pupil point is the entrance window, as it is the one which limits the angular apertures of a pencil of chief rays, the vertex of which is at the entrance pupil point. The actual diaphragm to which it is conjugate will be referred to as the field diaphragm, and the corresponding conjugate image space diaphragm is the exit window (see p. 86). In Figure 22 (see lower diagram) $E'F'$ is the entrance window and EF is the field diaphragm. The exit window in this case is in the plane of the image and is not shown in the diagram. For a microscope it is ordinarily to the left of I.

As in the case of the single lens, for a combination the entrance pupil determines the maximum aperture of an incident bundle which is permitted to pass through the optical system. The exit pupil similarly limits the emergent bundle although as the two pupil points are conjugate a bundle which just fills the entrance window will not be further restricted by passage through the exit pupil. In a similar manner, the entrance and exit windows, respectively, limit the field of view in the object and image spaces. The entrance window is said to limit the true field. The angle at the entrance pupil point, subtended by the radius of the entrance window, is the maximum angle which any chief ray may have and pass through the entrance window. Double this maximum value of β_1 is defined as the true angular field. Similarly for the k^{th} lens the angle subtended at the exit pupil point by the radius of the exit window is the maximum value of β'_k, and is defined as half the apparent angular field. The value of β_1 is indicated in Figure 22. If the entrance window lies in the object plane, the exit window, which is conjugate, will lie in the image plane, and the extent of the image will be limited by a sharp, well-defined boundary which, in an instrument designed to be used visually, will be in focus when the eye is accommodated for the image. Such a condition is usually desirable. If the entrance window is not in the plane of the object, the image will be vignetted at the edge and the illumination of the field at the edge will gradually fall off as shown in Figure 3.

VI. ABERRATION EQUATIONS EXTENDED TO A SYSTEM OF THIN LENSES

Before applying the equations of the third order to a combination of lenses it is necessary to extend the definitions of the parameters which have been previously defined for a single thin lens. It will be assumed that the axial object point has been carried through the system step by step (see p. 117), and for each lens there is an s and an s'. Also, the entrance pupil of the system has been located (see p. 125), and its axial point has been carried through the system by the step-by-step method.

From the radii of curvature of each lens one determines the value of each shape factor (σ), from the values of the s's each position factor (π) is determined, and from the x's the eccentricity factor (ϵ) for each lens is found. The value of h for the first lens of the system has been defined by the equation

$$h_1 = \frac{s_1}{s_1 - x_1}, \tag{15}$$

and h_1 is seen to be the ratio of the height of incidence of ray on first lens to height of incidence of same ray on the plane of the entrance pupil of the first lens, which is also the entrance pupil for the combination of lenses. Any subsequent h, say h_k, is the ratio of the height of incidence of a ray on the k^{th} lens to the height of incidence of the same ray on the plane of the entrance pupil of the system of lenses. Consequently,

$$h_2 = h_1 \frac{s_2}{s'_1} \tag{91}$$

and, in general,

$$h_k = h_{k-1} \frac{s_k}{s'_{k-1}} = h_1 \frac{s_2 s_3 \cdot \cdot s_k}{s'_1 s'_2 \cdot \cdot s'_{k-1}} \tag{92}$$

These relations are illustrated in Figure 23.

In a similar manner the value of g for any lens of a system is the height of incidence of a chief ray divided by the tangent of the angle between the chief ray and axis at the entrance pupil point. Therefore,

$$g_1 = x_1 \tag{16}$$

$$g_2 = g_1 \frac{x_2}{x'_1} \tag{93}$$

$$g_k = g_{k-1} \frac{x_k}{x'_{k-1}} = g_1 \frac{x_2 x_3 \cdot \cdot x_k}{x'_1 x'_2 \cdot \cdot x'_{k-1}}. \tag{94}$$

These relations are illustrated in Figure 23.

Although h is a ratio and without dimensions, it may be interpreted physically as numerically equal to the height of incidence on the corresponding lens of a ray which originates at the axial object point and which passes through the plane of the entrance pupil at a point distant one unit from the axis. Of course, in general, the lenses of a system will not be large enough in diameter to transmit such a ray, but one can disregard this in applying the first order equations. In the case of g, one has not a ratio but an actual length which may be interpreted physically as the height of incidence of a chief ray which makes an angle of 45° with the axis at the entrance pupil point. These two hypothetical rays are sometimes referred to as the two auxiliary rays employed in applying the third order equations.

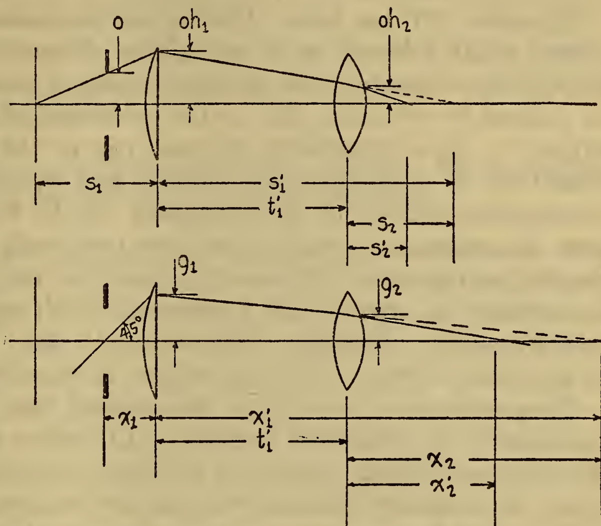

FIG. 23.—*Determination of g and h by the auxiliary rays traced through an optical system*

The equations governing the design of an optical system composed of thin lenses are[30]

$$S=\Sigma h\varphi \tag{95}$$

$$P=\Sigma\frac{\varphi}{n} \tag{96}$$

$$(\text{Ang. Lon. Chr.})=2o\Sigma\frac{h^2\varphi}{\nu} \tag{97}$$

$$(\text{Ang. Lat. Chr.})=-\tan\beta_1\Sigma gh\frac{\varphi}{\nu} \tag{98}$$

$$(\text{Ang. Sph.})=\frac{o^3}{4}\Sigma h^4\varphi^3A \tag{99}$$

$$(\text{Ang. Coma})=\frac{3}{8}o^2\tan\beta_1\Sigma(gh^3\varphi^3A-2h^2\varphi^2C) \tag{100}$$

$$(\text{Ang. Pri. Curv.})=o\tan^2\beta_1\left[P+3\Sigma\left(\varphi+g^2h^2\varphi^3\frac{A}{4}-gh\varphi^2C\right)\right] \tag{101}$$

[30] The above values of the angular aberrations are in radians. To reduce to minutes, multiply by 3,438. Log. 3,438=3.5363.

$$(\text{Ang. Sec. Curv.}) = o \tan^2 \beta_1 \left[P + \Sigma \left(\varphi + g^2 h^2 \varphi^3 \frac{A}{4} - g h \varphi^2 C \right) \right] \tag{102}$$

$$(\text{Ang. Dist.}) = \frac{1}{4} \tan^3 \beta_1 \Sigma \left(g^2 \varphi^2 T + g^3 h \varphi^3 \frac{B}{2} \right) \tag{103}$$

The condition for cementing two surfaces is

$$\varphi_1 = \frac{n_1 - 1}{n_{1+1} - 1} \frac{\varphi_{1+1}}{\varphi_1} \varphi_{1+1} + \frac{n_1 - 1}{n_{1+1} - 1} \frac{\varphi_{1+1}}{\varphi_1} + 1 \tag{104}$$

Equation (95) has been added to the aberration equations. If the object point selected is at an infinite distance or if the entrance pupil point coincides with the first principal point of the system, S, as defined by equation (95), is the reciprocal of the E. F. L. of the system[31]. In substantially all cases one or the other of these two conditions is approximately realized, and equation (95) serves for determining exactly or approximately the E. F. L. Its advantage over the expressions which have been previously given for the focal length (see equation (74)) lie in the fact that the right-hand member is expressed in terms of the parameters which are used in the aberration equations. Equation (96) defines P, the "Petzval curvature" of a system of lenses which enters later in equations (101) and (102).

Frequently in a lens system it is desired that two successive lens components be cemented together. In such a case it is necessary that the two adjacent faces have the same curvature. Equation (104) gives the necessary relation between the two shape factors if the *ith* and $(i+1)$*st* lenses are cemented. This equation is readily derived from equations (9).

The remaining equations of the group are extensions of those already given. The summation sign in each case indicates that one term is to be formed for each lens of the system. The angular value of the aberration, which is the value given by each equation, is the angle at the entrance pupil point, subtended by the appropriate linear dimension of the aberration disk when projected backward through the system into the object space.

For convenience the sign conventions of the aberrations will be given in condensed form.

(Ang.Lon.Chr.) is positive if the image of an axial point formed by the longer wave length (λ') is displaced with respect to a similar image formed by the shorter wave length (λ'') in the direction traveled by the incident light.

(Ang.Lat.Chr.) is positive for a positive value of $\tan \beta_1$ if the image of a marginal point formed by the longer wave length (λ') is farther from the axis than that formed by the shorter wave length (λ'').

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[31] Schwarzschild; Untersuchungen zur Geometrischen Optik, III Abh. d. K. Gesellschaft zu Göttingen, IV, No. 1, p. 9.

(Ang.Sph.) is positive if the image of an axial point formed by marginal rays is displaced with respect to that formed by paraxial rays in a direction opposed to the incident light.

(Ang.Coma) is positive for a positive value of tan β_1 if the diffuse flare of the comatic image extends from the axis.

A curvature which is positive indicates that the image of an oblique point is displaced from the image plane (a plane normal to the axis at the paraxial image of an axial point) in a direction opposed to that traveled by the incident light.

(Ang.Dist.) is positive for a positive value of tan β_1 if the actual image point is displaced from the axis with respect to the position of a distortion-free image point.

VII. APPLICATION OF THE THIRD ORDER THIN LENS EQUATIONS TO OPTICAL DESIGN

There are two applications of the aberration equations in lens design. In the first, and perhaps less important, the optical system is completely defined and one makes a straightforward application of the equations to determine the values of the several aberrations. In the second application the problem is the inverse of the first. The desired values of the aberrations are known, and certain parameters of the optical system, most frequently the shape factors, are left undetermined and introduced as unknowns in the equations. A solution then determines an optical system having the third order imagery of the desired characteristics.

1. DIRECT DETERMINATION OF ABERRATIONS OF A RAMSDEN EYEPIECE BY THE THIRD ORDER EQUATIONS

The aberrations of the eyepiece system shown in Figure 24 are to be determined. The upper drawing shows the course of a bundle of rays proceeding from an axial object point. The lower diagram shows the course of a bundle of chief rays which is brought to a focus at the center of the exit pupil. The constructional data of the system are

	∞	
1.51824		2.5
	-16.20	
1.00000		23.5
	$+16.20$	
1.51824		1.5
	∞	

An assembly of dimensions in this form will be commonly used to specify an optical system. All lengths are measured in millimeters.

The central column gives the radii of curvature in order beginning with the one which receives the incident light. The column to the left gives the index of the medium lying between each pair of surfaces, and the right-hand column gives the axial thickness. An air space is denoted by the appearance of the index 1.00000 in the left-hand column. The design is that of a typical Ramsden eyepiece with the lenses made of borosilicate crown.

(*a*) Application of First Order Equations.—In all the computations the thicknesses of the lenses will be neglected and lengths will be measured from the vertices of the curved surfaces. By equation (1)

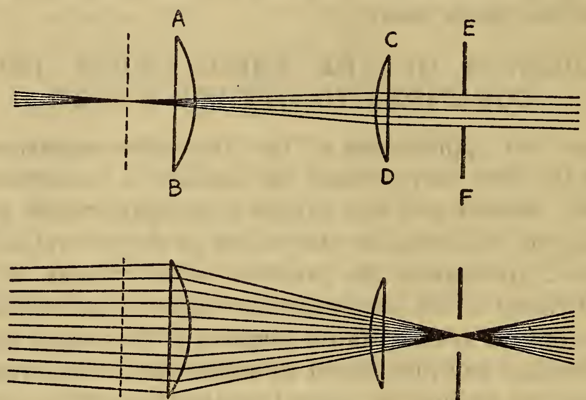

Fig. 24.—*Ramsden eyepiece*

The upper diagram shows the course of the rays from an axial object point. The lower diagram shows a bundle of chief rays.

$$f_1 = f_2 = +31.26$$

From equations (74), (75), (76), (81), and (82):

$$f_{1,2} = +25.04$$

$$\overline{V'_1P_{1,2}} = -\overline{V_2P'_{1,2}} = +18.83$$

$$\text{B.F.L.} = -\text{F.F.L.} = +6.21$$

It will be assumed that the eye is accomodated for parallel rays, in which case the object is placed at the first focal plane as indicated in the upper diagram of Figure 24 and

$$s_1 = -6.21$$

By equations (67) and (69)

$$s_1 = -6.21, \; s_2 = -31.25$$

$$s'_1 = -7.75, \; s'_2 = \infty$$

(*b*) Determination of Pupils and Windows.—In Figure 24 a diaphragm has been indicated at *EF*, which corresponds to the pupil of the eye. It is fairly obvious on inspection that this will be the iris of the system, but for purposes of illustration the iris and the entrance and exit pupils will be located by the method stated on pages 125 to 128. The three potential diaphragms to serve as the iris are the apertures of the two lenses *AB* and *CD* and the diaphragm *EF*. Each is to be projected into the object space by all components of the system lying in front of it. The radii of the apertures of the two lenses are 9 and 7 mm. It will be assumed that *EF* (in the position of the observer's eye) is 4 mm in diameter and 10 mm to the right of lens *CD*.

The cell *AB* projects into itself since it lies in the plane of the first lens. The angle subtended by its radius at the object point is $\tan^{-1} \frac{9}{6.21} = \tan^{-1} 1.45$.

The cell *CD* is to be projected backward into the object space of eyepiece by lens *AB*. The conjugate diaphragm lies 100.9 to the right of the object point. Its radius is 28.2 mm. The angle subtended by it at the object point is $\tan^{-1} \frac{28.2}{100.9} = \tan^{-1} 0.28$

The diaphragm conjugate to *EF* is found by projection backwards through both lenses of the system. It lies 165.8 mm to the left of the object point and its radius is 13.2 mm. At the object point the angle subtended is $\tan^{-1} \frac{13.2}{165.8} = \tan^{-1} 0.080$. It is evident that the diaphragm conjugate to *EF* subtends the smallest angle and it is therefore the entrance pupil. The diaphragm *EF* is both exit pupil and iris. The entrance window is the conjugate diaphragm in the object space which subtends the smallest angle at the entrance pupil point. The radius of *AB* subtends the angle $\tan^{-1} \frac{9}{172} = \tan^{-1} 0.05$ and for *CD* the angle subtended is $\tan^{-1} \frac{28}{266} = \tan^{-1} 0.11$. Therefore, *AB* is the entrance window and the true field of view is $2 \tan^{-1} \frac{9}{172} = 6°$. The exit window is the conjugate diaphragm in the image space of the eyepiece. It lies 104.7 mm to the left of the exit pupil point and the radius is 36.3 mm. The apparent field of view is $2 \tan^{-1} \frac{36.3}{104.7} = 38.2°$.

By equations (67) and (69)

$$x_1 = -172.0 \qquad x_2 = +14.70$$
$$x'_1 = +\ 38.2 \qquad x'_2 = +10.00$$

(c) EVALUATION OF THE PARAMETERS.—From the values of the s's and the x's the values of σ, π, ϵ g and h can be determined for each lens.

$\sigma_1 = -\ \ 1.00$	$\pi_1 = +9.06$	$\epsilon_1 = -0.64$
$\sigma_2 = +\ \ 1.00$	$\pi_2 = +1.00$	$\epsilon_2 = -5.26$
$g_1 = -172.0$	$h_1 = -0.0375$	$\varphi_1 = +0.0320$
$g_2 = -\ \ 66.2$	$h_2 = -0.1512$	$\varphi_2 = +0.0320$

The index, ν-value, reciprocal of ν and functions of n required in subsequent substitutions are given below:

$n_{\lambda=546}$	1.51824
$\nu = \dfrac{n_{546}-1}{n_F - n_C}$	64.30
$\dfrac{1}{\nu}$	0.01555
$\dfrac{n+2}{n(n-1)^2}$	8.63
$\dfrac{4(n+1)}{n(n-1)}$	12.80
$\dfrac{3n+2}{n}$	4.32
$\dfrac{n^2}{(n-1)^2}$	8.58
$\dfrac{n+1}{n(n-1)}$	3.20
$\dfrac{2n+1}{n}$	2.66
$\dfrac{1}{n}$	0.66

By direct substitution in equations (23), (64), (39), and (62):

$$\begin{aligned}
A_1 &= +255.84 \\
A_2 &= +\ \ 34.33 \\
B_1 &= +\ \ 27.17 \\
B_2 &= +\ \ 69.41 \\
C_1 &= +\ \ 20.90 \\
C_2 &= +\ \ \ \ 5.86 \\
T_1 &= -\ \ \ \ 3.62 \\
T_2 &= -\ \ \ \ 0.27
\end{aligned}$$

(d) APPLICATION OF THE THIRD ORDER EQUATIONS.—The value of o, the radius of the entrance pupil, has already been found to be 13.2 mm. The maximum value of β will be taken to be 1.52°, which corresponds to an apparent field of approximately 20°. This is approximately the limit of satisfactory definition in a system of this character. It has been found above that the actual angular field, transmitted is 38°, but in use, as the eyepiece of a telescopic system, a field stop in the front focal plane would be added which would cut down the field, the amount depending upon the character of definition considered satisfactory for the purpose at hand. There will now be given in some detail, the numerical substitutions in the various third order equations which lead to values of the aberrations for this Ramsden eyepiece.

From equation (97)
(Ang. Lon. Chr.)

$$=2(13.2)\begin{Bmatrix}(-0.0375)^2\ (0.0320)\ (0.01555)\\(-0.1512)^2\ (0.0320)\ (0.01555)\end{Bmatrix}\begin{matrix}=+0.000018\\=+0.000300\end{matrix}$$

$$\begin{matrix}\overline{+0.000318\text{ radians.}}\\=+1.09\text{ minutes.}\end{matrix}$$

From equation (98)
(Ang. Lat. Chr.)

$$=-0.0265\begin{Bmatrix}(-172.0)\ (-0.0375)\ (0.0320)\ (0.01555)\\(-\ 66.4)\ (-0.1512)\ (0.0320)\ (0.01555)\end{Bmatrix}\begin{matrix}=-0.000085\\=-0.000132\end{matrix}$$

$$\begin{matrix}\overline{-0.000217\text{ radians.}}\\=-0.75\text{ minutes.}\end{matrix}$$

From equation (99)
(Ang. Sph.)

$$=(0.25)\ (13.2)^3\begin{Bmatrix}(-0.0375)^4\ (0.0320)^3\ (255.84)\\(-0.1512)^4\ (0.0320)^3\ (\ 34.33)\end{Bmatrix}\begin{matrix}=+0.000010\\=+0.000338\end{matrix}$$

$$\begin{matrix}\overline{+0.000348\text{ radians.}}\\=+1.20\text{ minutes.}\end{matrix}$$

From equation (100)
(Ang. Coma)

$$=(0.375)\ (13.2)^2\ (0.0265)\begin{Bmatrix}(-172.0)\ (-0.0375)^3\ (0.0320)^3\ (255.84)\\-2(-0.0375)^2\ (0.0320)^2\ (20.90)\\(-66.2)\ (-0.1512)^3\ (0.0320)^3\ (34.33)\\-2(-0.1512)^2\ (0.0320)^2\ (5.86)\end{Bmatrix}\begin{matrix}=+0.000132\\=-0.000104\\=+0.000445\\=-0.000475\end{matrix}$$

$$\begin{matrix}\overline{-0.000002\text{ radians.}}\\=-0.01\text{ minutes.}\end{matrix}$$

From equation (101)
(Ang. Pri. Curv.)

$$=13.2(0.0265)^2\begin{Bmatrix}(0.0320)\ (0.66)\\3(0.0320)\\3(-172.0)^2\ (-0.0375)^2\ (0.0320)^3\ (255.84)\ (0.25)\\-3(-172.0)\ (-0.0375)\ (0.0320)^2\ (20.90)\\(0.0320)\ (0.66)\\3(0.0320)\\3(-66.2)^2\ (-0.1512)^2\ (0.0320)^3\ (34.33)\ (0.25)\\-3(-66.2)\ (-0.1512)\ (0.0320)^2\ (5.86)\end{Bmatrix}\begin{matrix}=+0.000196\\=+0.000890\\=+0.002425\\=-0.003839\\=+0.000196\\=+0.000890\\=+0.000783\\=-0.001670\end{matrix}$$

$$\begin{matrix}\overline{=-0.000129\text{ radians.}}\\=-0.44\text{ minutes.}\end{matrix}$$

If similar substitutions are made in equation 102, the corresponding terms are identical, except that the factor 3 is omitted in the second, third, fourth, sixth, seventh, and eighth terms. Summing the terms with this modification

$$\begin{matrix}(\text{Ang. Sec. Curv.})=+0.000218\text{ radians}\\=+0.75\text{ minutes}\end{matrix}$$

From equation (103)
(Ang. Dist) =

$$(0.0265)^3\ (0.25)\begin{Bmatrix}(-172.0)^2\ (0.0320)^2\ (-3.62)\\(-172.0)^3\ (-0.0375)\ (0.0320)^3\ (27.17)\ (0.5000)\\(-66.2)^2\ (0.0320)^2\ (-0.27)\\(-66.2)^3\ (-0.1512)\ (0.0320)^3\ (69.41)\ (0.5000)\end{Bmatrix}\begin{matrix}=-0.000510\\=+0.000395\\=-0.000006\\=+0.000232\end{matrix}$$

$$\begin{matrix}\overline{=+0.000110\text{ radians.}}\\=+0.38\text{ minutes.}\end{matrix}$$

The angular aberrations determined above are in the object space and represent the angle at the entrance pupil point subtended by the aberration area. It is desirable to determine the conjugate angles at the exit pupil point, as this gives the angular value of the aberration disk as presented to the eye. To determine these values, one multiplies those already obtained by the angular magnification between the two pupil points (see equation (86)).

$$A=\frac{(-172.0)\ (+14.70)}{(+38.20)\ (+10.00)}=-6.6$$

Following there is given in tabulated form the different aberrations, measured in minutes, in the object and image space.

TABLE 2.—*Third order abberrations of Ramsden eyepiece*

	Aberration in object space	Aberration in image space
(Ang. Lon. Chr.)	+1.09	+7.19
(Ang. Lat. Chr.)	−.75	−4.95
(Ang. Sph.)	+1.20	+7.92
(Ang. Coma)	−.01	−.07
(Ang. Pri. Curv.)	−.44	−2.90
(Ang. Sec. Curv.)	+.75	+4.95
(Ang. Dist.)	+.38	+2.51

In multiplying the aberrations in the object space by the angular magnification the sign of A is ignored. The negative sign indicates that the angle between chief ray and axis after passage through the eyepiece is the negative of the corresponding angle in the object space. Note, however, that this does not effect a change in the signs of the aberrations as defined on pages 130 and 131.

2. DESIGN OF A LENS SYSTEM WHICH HAS GIVEN ABERRATION CHARACTERISTICS

The second application of the third order equations in which the desired values of the aberrations are given and the constants of the lens system determined by a solution is the inverse of the first problem and from the theoretical standpoint is but little different. In practice, however, the inverse solution is the more useful, much the more difficult, and knowledge based on experience, together with many trials, are often necessary before a solution may be obtained. The difficulties arise from the fact that the roots of the equation must not only be real, but must fall within a limited numerical range if the solution is to serve for the design of a useful optical system.

(*a*) GIVEN CONDITIONS TO BE SATISFIED.—The designer begins with certain first order constants of the system which are given. If it is a photographic objective the focal length is specified, if a telescope the magnification is assigned, or if an eyepiece the focal length and the position of the exit pupil may be given. A second restriction arises from the limited varieties of optical glass available. Although the number of types of optical glass has been increased greatly since the founding of the glass plant at Jena, even yet the range of index and ν-values falls within a restricted region. And a further restriction is often imposed by the requirement that glass already at hand be utilized.

(*b*) Determination of the Focal Lengths and Spacing of the Components.—It will be noted that equations (95), (96), (97), and (98) contain as variables the power but not the shape factor of each lens. Consequently, the first step is to determine the power, spacing, and glass to be used for each lens such that the first order conditions alluded to earlier in the paragraph, and equations (95), (96), (97), and (98) are satisfied to the desired extent. If the system is sufficiently simple, this may be accomplished by a direct solution. In a more complicated system this requires a method of trial and successive approximations which is often facilitated by breaking the original system into a series of subordinate systems for each of which a direct solution may be obtained, after which the subordinate systems are combined. Equation (95) gives the curvature of the image surface if the completed system is anastigmatic. The condition $P=0$ is necessary, but not sufficient to determine a system giving a flat field free from third order astigmatism. If $P=0$ the field will be anastigmatic and flat, provided that the subsequent design of the system is such as to cause the quantity in parentheses in equations (101) and (102) to vanish. If $P\neq 0$, no subsequent modifications of the shape of the lens or diaphragm position will enable a flat anastigmatic field to be obtained so long as one is limited to third order imagery (see p. 110). Actually experience has shown that higher order aberrations materially modify the conclusions regarding curvature and astigmatism drawn from the third order equations and that in some cases large, sufficiently flat fields may be obtained even though the value of P departs considerable from zero. The equation

$$P=0 \tag{105}$$

is commonly referred to as the Petzval condition.

With this preliminary design completed one has given or can easily obtain the values of φ, s, s', x, x', g, h, π, and ϵ for each lens. The system is now completely specified except that σ for each lens remains undetermined.

(*c*) Determination of the Shapes of the Components.—Equations (99) to (103) give the values of the different monochromatic aberrations, and an equation is formed by the insertion of the values of the different known quantities for each aberration which is to be controlled. For each cemented pair of surfaces there will be an additional condition of the form of equation (104). The σ's will be the only unknowns in these equations, and it might seem a priori that with five or more components one has five equations and five unknowns from which a solution can be obtained in which all third order monochromatic aberrations are eliminated. For a useful solution, however, it is necessary that all the σ's be real, and in most cases the values obtained should lie within a relatively narrow range in the neighborhood of zero, the extent of range decreasing as ratio of diameter of component to focal length increases. This restriction greatly limits the possibilities open to the designer.

It follows that in laying out the powers and spacings of the different lenses of the system, not only must equations (95) to (98) be satisfied but at the same time the designer must have in view the determination of a design which will yield satisfactory roots when equations (99) to (104) are applied. It is here that a good physical picture of

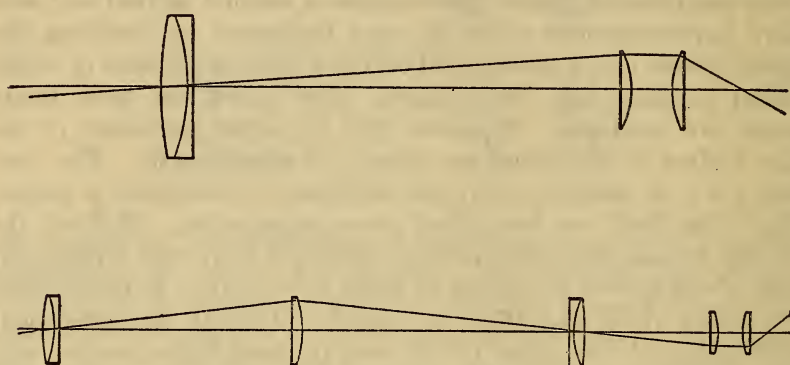

FIG. 25.—*Lenses for which* $g=0$

The upper diagram shows a simple telescopic system. For the components of the objective $g=0$ and hence by a change in their shape factors no control can be exercised upon lateral chromatic aberration, curvature, or distortion. The lower diagram illustrates the system of a periscope. For the objective and erecting lens $g=0$ and a similar limitation of control of aberrations exists.

the manner in which the aberrations arise, a knowledge of the characteristics of existing systems and experience in design become valuable.

(*d*) CONTROL EXERCISED BY A LENS FOR WHICH g IS SMALL.—Certain general conclusions may be drawn from the equations which are of help in arriving at a satisfactory design. Suppose one has a thin lens in the plane of the entrance or exit pupil or at one of the real images of the pupil formed within the system. Two such cases are illustrated in Figure 25. The first sketch shows the objective of a telescope in the plane of the entrance pupil, a case frequently realized. In the second case the erecting lens of a periscopic system is placed at the real image of the entrance pupil as formed by the collector lens. The auxiliary ray which serves for the determination of the g's is drawn, and it is evident that for each of the lenses referred to $g=0$. Reference to equations (101) and (102) shows that the contribution to the curvature of the entire system by either of these lenses is $o \tan^2\beta\left(\frac{\varphi}{n}+3\varphi\right)$ and $o \tan^2\beta\left(\frac{\varphi}{n}+\varphi\right)$ in the primary and secondary planes, respectively, and that it is entirely independent of the shape of the lens. Consequently, in any system having lenses for which $g=0$ the shape of the lens may be left undetermined and the shapes of the other lenses adjusted to give the required values to the two curvatures and distortion. In general, there will remain residual amounts of spherical and comatic aberrations. Now, if it is possible to determine the shape of the components for which $g=0$ in such a manner as to compensate for the spherical and comatic aberrations this can be done without disturbing the previously adjusted curvature. In this way the aberrations may be adjusted successively instead of simultaneously. This is an advantage, as the solution of the simultaneous equations

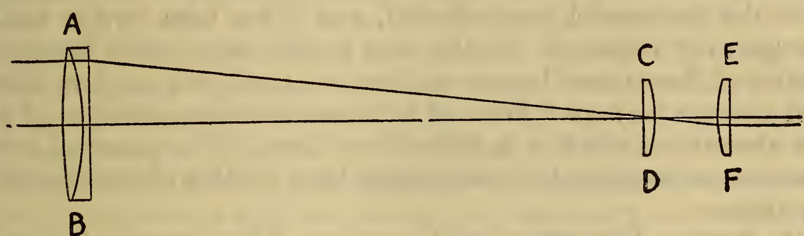

FIG. 26.—*Lenses for which* $h=0$

A simple telescopic system is shown in which the field lens of the eyepiece is placed in the plane of the image produced by the objective. The course of the auxiliary ray indicates that $h=0$ for the field lens. In such a case the lens contributes nothing to the magnification of the system. A change in the focal length of the field lens, therefore, will not change the power of the telescope (for this special case where $h=0$). A change in power, however, will change the positions of the exit pupil and with it the value of g for all following lenses. Indirectly, therefore, it offers a means of modifying the curvature of image of the system. Since $h=0$, a change in shape or power will have no effect upon the spherical aberration of the system.

is simpler, and one is more easily able to follow the physical significance of the modifications introduced in the successive trials which may be necessary.

(*e*) CONTROL EXERCISED BY A LENS FOR WHICH h IS SMALL.—A second typical case which it is useful to recognize is that in which a lens is placed in the plane of a real image of the object. The collector lens in the eyepiece of most telescopes very nearly satifies this condition. In Figure 26 it is represented as exactly satisfied for the lens CD and the tracing of the auxiliary ray shows that $h=0$. A lens in this position contributes nothing to the magnification of the system. This is clear from equation (95) and such freedom is useful, as the power of the lens may be changed at will without altering the size of image produced by the instrument. This lens does, however, control the position of the entrance and exit pupils as it acts with full effect upon the bundle of chief rays, and consequently a change in its power affects the value of g for all lenses and thus indirectly changes the contribution to coma, curvature, and distortion for each lens through the alteration in value of the term which contains g as a factor. A lens for which $h=0$, or is nearly zero, is commonly termed a collector or field lens, and it should be noted that such a lens makes no contribution, or only a small contribution, to either of the chromatic aberrations, to spherical aberration, or to coma. Furthermore, after its power has been fixed its contributions to the curvatures of the sytsem are completely determined and are $o \tan {}^2\beta \left(\frac{\varphi}{n}+3\varphi\right)$ and $o \tan {}^2\beta \left(\frac{\varphi}{n}+\varphi\right)$ just as in the case of the telescope objective. Consequently, modifying the shape of the collector lens offers opportunity to alter the distortion through the $g^2\varphi^2T$ term while leaving all other aberrations unchanged.

In the two special cases selected, g or h has been zero or small enough to be neglected. In the usual system many lenses belong to neither of these types, but are so placed that neither g nor h vanishes and a change in shape is followed by a complicated modification of all the aberrations which it is difficult to follow. The shapes of such lenses must, in general, be determined by a solution of simultaneous equations.

(*f*) Petzval Condition and Chromatic Aberration.—A system of thin lenses in contact is an important special case. For such a system both g and h have the same value for all components and may accordingly be placed before the summation sign. To satisfy the Petzval condition and eliminate the chromatic aberration simultaneously one must have

$$\Sigma \frac{\varphi}{n}=0 \tag{106}$$

$$\Sigma \frac{\varphi}{\nu}=0 \tag{107}$$

The first equation is identical with equation (105). The second is derived from equation (97). As n and ν are always positive, these two equations can only be satisfied if φ for some of the components is negative. If the complete system is to have a positive power, the combined power of the positive lenses must prevail over that of the negative components. If the Petzval condition is to be satisfied, equation (106) shows that the negative lenses must in the average have the smaller index of refraction. Similarly, if the chromatic condition is to be satisfied the negative lenses must in the average have the lower ν values.

Prior to the development of the new glasses by Abbe and Schott there were no pairs of optical glass available in which the glass of the lower index had the smaller ν-value, and consequently a thin lens contact combination which was achromatic could not simultaneously satisfy the Petzval condition. New glasses are now available in which the value of n and ν range in such a manner as to permit both conditions to be satisfied. It is customary to refer to such combinatons which are achromatic and satisfy the Petzval conditions as "new type achromats" in contradistinction to the older or so-called normal achromats.

But even the newer type glasses leave much to be desired. In a new type achromat the values of ν differ so slightly in a series of components that the sum of the powers of the negative lenses almost equals in absolute value the sum of the powers of the positive components. Consequently, if the net power of the combination is to be very great the individual components must be of very short focal length and this carries with it the necessity for steep curves, small aperture, and large higher order aberrations. In a thin lens combination if the Petzval condition is satisfied it is possible to have a flat field free from astigmatism provided that a diaphragm is so placed on the axis as to give a value of g for the components which will cause the quantity in parenthesis in equations (101) and (102) to vanish. If a thin lens contact combination has the diaphragm in the plane of the lens $g = 0$ for all components and the primary and secondary curvatures are $o \tan^2\beta_1\left(\Sigma\frac{\varphi}{n}+3\Sigma\varphi\right)$ and $o \tan^2\beta_1\left(\Sigma\frac{\varphi}{n}+\Sigma\varphi\right)$. Taylor has termed these the normal curvatures of a lens, and considers that these are always present but modified in the case of eccentric refraction ($g\neq 0$) by the superposition of the curvatures represented by the terms in g and g^2 in equations (101) and (102). It is impossible to design a lens system composed of thin lenses in contact which will give a flat field free from astigmatism (on the basis of third order theory) unless a diaphragm is properly placed without the plane of the lens.

The conflict between the Petzval condition and achromatization which exists in the case of optical systems consisting of thin lenses in contact is not so sharp when dealing with a system composed of thin lenses spaced along the axis. A positive and a negative lens having nearly the same absolute powers may be separated to produce a combination of considerable power. Assume that two glasses are used which satisfy the condition for achromatization. By hypothesis $\Sigma\varphi$ is relatively small in comparison with the power of the system, and as the values of n for optical glass fall within a relatively narrow range it follows that $\Sigma\frac{\varphi}{n}$ will also be small with reference to the power of the combination. In fact, with a separated system it is not difficult to satisfy the Petzval condition and achromatize with the older glasses. It is, of course, true that with a system of thin lenses in contact the older glasses may be used and P may be made small by making $\Sigma\varphi = 0$, but for the contact system this becomes a trivial case, as $\Sigma\varphi$ is the power of the complete system and the resulting system has insufficient power to be useful, whereas in the case of the separated lens system the power of the system may be many times greater than $\Sigma\varphi$. Taylor was the first to recognize the importance of this feature of the spaced system and used it in the design of the Cooke lens, which serves as a basis for many of the air-spaced photoghraphic objectives.

(*g*) Symmetrical Lens System.—Symmetry, either partial or complete, in an optical system permits useful generalizations to be

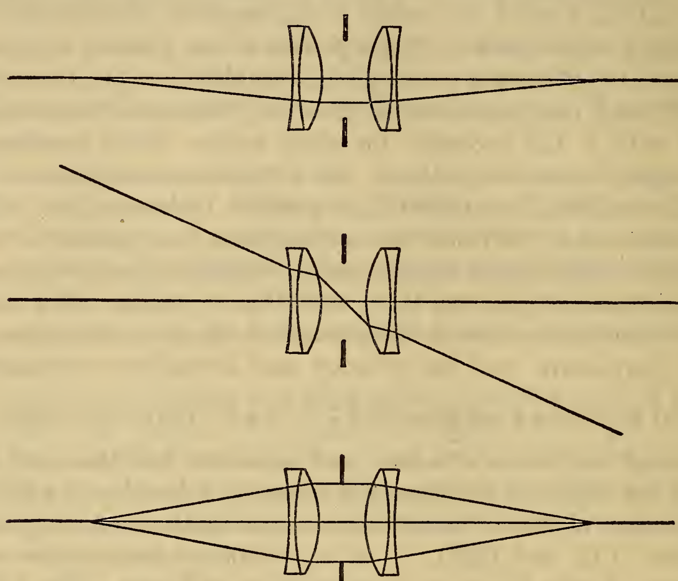

Fig. 27.—*A symmetrical photographic lens of the type commonly termed a rapid rectilinear*

By reason of the symmetry, the coma and distortion are small. They vanish completely for the pair of conjugate planes for which $M = -1$.

drawn. Figure 27 shows a symmetrical photographic lens, of the type usually termed a rapid rectilinear in this country, in which not only the components but also the object and image are symmetrical and symmetrically placed with respect to the central diaphragm, which is the iris of the system. The two lower drawings show the courses of the auxiliary rays for the determination of g and h. In the general case let it be assumed that there are k components, where k is necessarily an even number, symmetrically placed.

It is evident from inspection that

$$\varphi_1 = \varphi_k$$
$$\varphi_2 = \varphi_{k-1}$$

Similarly

$$g_1 = -g_k$$
$$h_1 = h_k$$

and reference to equations (8), (10), (12), (24), (39), (62), and (64) shows that

$$\begin{aligned} \sigma_1 &= -\sigma_k \\ \pi_1 &= -\pi_k \\ \epsilon_1 &= -\epsilon_k \\ A_1 &= A_k \\ B_1 &= B_k \\ C_1 &= -C_k \\ T_1 &= -T_k \end{aligned}$$

with similar expressions for each pair of similarly placed components. A consideration of the third-order equations shows that the symmetrical system will have no lateral chromatic aberration, coma, or distortion, as the terms after the summation sign will vanish in pairs. If the object and image are not symmetrically placed, the $g^2\varphi^2T$ term still vanishes in the equation for distortion, but the term in B does not vanish as $B_1 \neq B_k$. A neglect to appreciate the B term—that is, the distortion arising from the spherical aberration of the iris—has lead to the inference in many of the earlier books on optics that a symmetrical lens is entirely free from distortion without a definite statement that this is only true when object and image are symmetrically placed or when entrance pupil point is imaged at exit pupil point without spherical aberration. This has given the name "rapid rectilinear" to the symmetrical photographic objective, as rectilinearity of lines is a corollary to freedom from distortion.

(*h*) Hemisymmetrical Lens System.—The lens system shown in the upper diagram of Figure 27 is completely symmetrical about the iris. If now all lengths, object distances, image distances, spacings, radii of curvature, etc., on one side of the iris are multiplied by a constant, the parts on the two sides of the diaphragm remain geometrically similar, but not equal. Such a system is said to be hemisymmetrical. If m is the constant multiplier in such a system

$$\begin{aligned} mg_1 &= -g_k \\ h_1 &= h_k \\ \varphi_1 &= m\varphi_k \\ \sigma_1 &= -\sigma_k \\ \pi_1 &= -\pi_k \\ \epsilon_1 &= -\epsilon_k \\ A_1 &= A_k \\ B_1 &= B_k \\ C_1 &= -C_k \\ T_1 &= -T_k \end{aligned}$$

with similar expressions for each pair of similarly placed components. Such a system will have no lateral chromatic aberration or distortion. For the elimination of any of the other aberrations it is necessary that each half be corrected separately.

(*i*) Aberration Equations not Always Equated to Zero.—It is not always desirable that the third order aberrations be equated to zero. In fact, for the final system the third order aberrations should not be zero, but of such a value as to neutralize the higher order aberrations at some intermediate zone of the aperture. If one has at hand details of a system similar to the one to be designed its higher order aberrations may be determined by trigonometric ray tracing for the zone at which complete compensation of the aberrations is desired. As a first approximation one may then assume that the new system will have higher order aberrations of the same character and so may introduce compensating first order aberrations in the first solution. The assumption that the higher order aberrations will remain constant from system to system is, of course, doubtful and only to be justified empirically, and it becomes more hazardous as the difference between the two systems increases. But in the case where the only difference arises in the exhaustion of a melt of glass and the substitution of a new melt differing but slightly in index, a case continually arising in the production of a standardized product, such an assumption is especially helpful. When a similar system is not available, the third order design may at times proceed in two steps. First, the aberrations are equated to zero and a solution obtained trigonometrically. A second solution is then carried through in which the third order aberrations are set equal to the negative of the higher order aberrations at the zone where complete compensation is expected.

(*j*) Choice of Glass as an Independent Variable.—Usually the shape factors are reserved as the only unknowns but not always. Within some ranges of index it is possible to find several glasses having the same index for the D line (λ), but differing in ν-value. In such a case one may retain more degrees of freedom in the elimination of the monochromatic aberrations by retaining both ϕ and σ as unknowns for each component. Only in the simpler cases can a solution be obtained in a straightforward manner, as the s's and x's are functions of the ϕ's and must be so expressed, a feature which usually makes the equations unwieldy if there are more than two or three lenses. After the ϕ's and σ's have been determined the values of ν to eliminate the chromatic aberrations are determined. If the values fall within available limits, a solution has been made in which the choice of glass as well as shape factor has been made use of to correct the aberrations. Roughly this is the method employed by Harting[32] in his classic investigation of the two-lens cemented objective. With two shape factors as the independent variables one should expect to satisfy two conditions which might be

1. Freedom from spherical aberration,
2. That the inner surfaces have the same curvature

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[32] Harting, Zeits. Instrumentenk, **18**, p. 357; 1898.

in order that they may be cemented together. Harting showed that by the proper selection of available glass one can also cause the coma to vanish without giving up either of the two preceding conditions.

(*k*) Choice of Roots.—Some of the aberration equations are of the second degree in σ. Consequently, one may expect to usually obtain two sets of roots. By varying the types of glass additional solutions can often be obtained, all of which satisfy the third order equations equally well.

When two or more solutions are obtained, a criterion is necessary to determine which is to be selected for the actual design. The cost of manufacture, number of glass-air surfaces, weight, and similar characteristics must be given consideration varying in amount from instrument to instrument and can not be generalized. From the standpoint of the aberrations it is ordinarily considered that of two systems having equal residual third order aberrations the one having the less strongly curved surfaces, and more particularly the less curvature in the glass-air surfaces, is the better. Deeply curved surfaces generally imply large higher order aberrations, and this carries with it the implication that the final design must have large third order aberrations to compensate the higher order aberrations at the selected intermediate zone. Compensations can be accomplished at all zones only when the aberrations of each order vanish separately, and if the higher order aberrations are large the compensation at one zone will not be inconsistent with large aberrations at the other zones.

3. APPLICATION OF THE THIRD ORDER EQUATIONS TO THE DESIGN OF A KELLNER EYEPIECE

To illustrate the foregoing, a detailed numerical application of the equations to the design of an eyepiece will be made. The statement of the problem is as follows: An eyepiece is to be designed for which

		mm
E.F.L. of field lens	=	+31.30
E.F.L. of eye lens	=	+31.30
Distance between field lens and eye lens	=	23.50
Distance from eye lens to observer's eye	=	10.00

The eyepiece will be of the Kellner type in which the eye lens is composed of two components cemented together with the flint turned toward the observer's eye. It has already been shown that the field lens by a change of shape offers little opportunity for control of spherical aberration, coma, curvature of image, or astigmatism because of the small value of h. It will, therefore, be assumed that the field lens is a plano-convex lens with plane surface turned toward the object. This follows the usual practice and leads to economy in production. With the above data the focal length of the eyepiece will be 25.06 mm.

The characteristics of the glasses to be used in the eyepiece, together with such constants as are necessary in the solutions, are given below.

Optical glass to be used in eyepiece

	Borsilicate crown for field lens	Barium crown for eye lens	Light flint for eye lens
$n(\lambda=546)$	1.5180	1.5750	1.6240
ν	64.30	57.72	36.58
$\frac{1}{\nu}$	.01555	.01733	.02734
$\frac{n+2}{n(n-1)^2}$	8.637	6.865	5.731
$\frac{4(n+1)}{n(n-1)}$	12.809	11.373	10.357
$\frac{3n+2}{n}$	4.318	4.270	4.232
$\frac{n^2}{(n-1)^2}$	8.588	7.503	6.773
$\frac{n+1}{n(n-1)}$	3.202	2.8433	2.5894
$\frac{2n+1}{n}$	2.6588	2.6349	2.6158
$\frac{1}{n}$	.6588	.6349	.6158

With the above data decided upon there remain three degrees of freedom, the ratio of the powers of the two components of the eyelens, and the shape factors of the two components. The powers of the two components will be selected to give zero lateral chromatic aberration. The remaining degrees of freedom are required by the conditions that the two components are to be cemented and that the primary image surface be flat. In other words, equations (98) and (101) are to be satisfied when their left-hand members are zero and equation (104) is to be satisfied for $i=2$. This exhausts the degrees of freedom, but after a solution has been obtained an investigation of the other aberrations will be made to determine whether a compromise may not be obtained which will favor them.

(*a*) Determination of the *s*'s, *x*'s, *g*'s, and *h*'s.—The computation of the Ramsden eyepiece (p. 131) makes it evident that the pupil of the observer's eye is the exit pupil of the eyepiece. Consequently $x'_3=+10$; and, as it is assumed that the eye is accomodated for parallel light, $s'_3=\infty$. The two components of the eyelens are in contact and, for the initial computations, will be treated as a single lens denoted by the subscripts 2, 3. Then, by the first order equations one readily obtains the values

$$x_1 = -173.25 \qquad s_1 = -\ 6.244$$
$$x'_1 = +\ 38.20 \qquad s'_1 = -\ 7.800$$
$$x_{2,3} = +\ 14.695 \qquad s_{2,3} = -31.30$$
$$x'_{2,3} = +\ 10.00 \qquad s'_{2,3} = \infty$$

As the two components of the eyelens are in contact it further follows that $g_2 = g_3$ and $h_2 = h_3$. From equations (15), (16), (91), and (93)

$$g_1 = -173.25 \qquad h_1 = -0.03739$$
$$g_2 = -66.65 \qquad h_2 = -0.15004$$
$$g_3 = -66.65 \qquad h_3 = -0.15004$$

(*b*) Condition For Freedom From Lateral Color.—One now has the data necessary for substitution in equation (98). As the left-hand member is zero the factor tan β_1 may be omitted and the condition for achromatism becomes

$$0.003218 + 0.17330\varphi_2 + 0.27340\varphi_3 = 0 \qquad \text{(a)}$$

From the initial conditions of the problem

$$\varphi_2 + \varphi_3 = \frac{1}{31.30} = 0.03195 \qquad \text{(b)}$$

Solving, the eyepiece will be free from lateral chromatic color if

$$\varphi_2 = +0.11942$$
$$\varphi_3 = -0.08747$$

(*c*) Determination of the Constants of the Third Order Equations.—With the powers of the three components known, one is now in a position to determine all the constants of the third order equations.

$$x_1 = -173.25 \qquad s_1 = -6.244$$
$$x'_1 = +38.20 \qquad s'_1 = -7.800$$
$$x_2 = +14.695 \qquad s_2 = -31.30$$
$$x'_2 = +5.334 \qquad s'_2 = +11.432$$
$$x_3 = +5.334 \qquad s_3 = +11.432$$
$$x'_3 = +10.000 \qquad s'_3 = \infty$$

For convenience there is tabulated

$$g_1 = -173.25 \qquad h_1 = -0.03739 \qquad \varphi_1 = +0.03195$$
$$g_2 = -66.65 \qquad h_2 = -0.15004 \qquad \varphi_2 = +0.11942$$
$$g_3 = -66.65 \qquad h_3 = -0.15004 \qquad \varphi_3 = -0.08747$$

It is assumed that $\sigma_1 = -1$ and the other two shape factors will be the unknowns for which values are to be obtained. From equations (10) and (12)

$$\sigma_1 = -1 \qquad \epsilon_1 = -0.6387 \qquad \pi_1 = +9.026$$
$$\epsilon_2 = -2.1396 \qquad \pi_2 = -0.4650$$
$$\epsilon_3 = +3.286 \qquad \pi_3 = +1$$

From equations (23), (39), (62), and (64)

$$\begin{aligned}
A_1 &= +253.40\\
A_2 &= 6.865\sigma_2^2 - 5.288\sigma_2 + 8.426\\
A_3 &= 5.731\sigma_3^2 + 10.357\sigma_3 + 11.005\\
B_1 &= +27.167\\
B_2 &= 6.865\sigma_2^2 - 24.334\sigma_2 + 27.051\\
B_3 &= 5.731\sigma_3^2 + 34.03\sigma_3 + 52.47\\
C_1 &= +20.796\\
C_2 &= 2.8433\sigma_2 - 1.2252\\
C_3 &= 2.5894\sigma_3 + 2.6158\\
T_1 &= -3.623\\
T_2 &= 2.8433\sigma_2 - 13.584\\
T_3 &= 2.5894\sigma_3 + 2.0235
\end{aligned}$$

(*d*) Equations for the Determination of σ_2 and σ_3.—If the left-hand member of equation (101) is set equal to zero and the factor $o \tan^2 \beta_1$ omitted, the equation becomes

$$\begin{aligned}
0 = &+0.021049\\
&+0.09585\\
&+0.0010266(253.40)\\
&-0.019839(20.796)\\
&+0.07582\\
&+0.3583\\
&+0.12774(6.865\sigma_2^2 - 5.288\sigma_2 + 8.426)\\
&-0.4278(2.8433\sigma_2 - 1.2252)\\
&-0.05386\\
&-0.26241\\
&-0.05020(5.731\sigma_3^2 + 10.358\sigma_3 + 11.005)\\
&-0.22953(2.5894\sigma_3 + 2.6158)
\end{aligned} \qquad \text{(c)}$$

which further reduces to

$$0.8769\sigma_2^2 - 1.8919\sigma_2 - 0.28770\sigma_3^2 - 1.1142\sigma_3 + 0.5297 = 0 \qquad \text{(d)}$$

Equation (104) becomes

$$\sigma_2 = -0.6750\sigma_3 + 0.3250 \qquad \text{(e)}$$

Solving equations (d) and (e)

$$\sigma_3 = +0.0341 \text{ or } +1.9515$$

Rounding these values off to accord with the degree of significance of the solution the final solution becomes

$$\sigma_2 = +0.3020 \text{ or} -0.991$$
$$\sigma_3 = +0.034 \qquad +1.950$$

The specifications for these two eyepieces, presented in the usual manner, are

No. 1			No. 2		
$\sigma_2 = +0.3020$			$\sigma_2 = -0.991$		
$\sigma_3 = +0.034$			$\sigma_3 = +1.950$		
	∞			∞	
1.518		2.5	1.518		2.5
	−16.21			−16.21	
1.000		23.5	1.000		23.5
	+ 7.40			+1070	
1.575		2.3	1.575		2.3
	−13.80			− 4.84	
1.000		0.0	1.000		0.0
	−13.80			− 4.84	
1.624		1.0	1.624		1.0
	+14.77			−15.02	

The two systems are illustrated to scale in Figure 28. The thicknesses of the lenses have been determined to give sufficient mechanical strength for a field lens 16 mm and an eyelens 9 mm in diameter. It will be noted that for the first eyepiece the flint component is almost equiconcave, for the second the crown component is almost planoconvex.

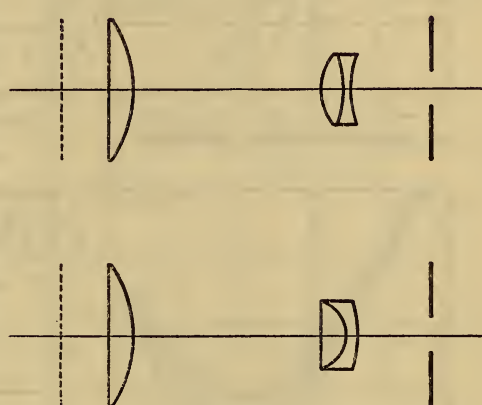


Fig. 28.—*The two Kellner eyepieces of the illustrative example for which $\sigma_3 = +0.034$ and $+1.950$, respectively*

(*e*) Third Order Characteristics of the Two Eyepieces.—In the subsequent calculations it will be assumed that $\beta_1 = 1.52°$ and $o = 13.2$ as was done for the Ramsden eyepiece already studied. If the missing factor $o \tan^2 \beta_1$ is multiplied into equation (*c*) and if the equation is multiplied by 3,438 to reduce the curvature to minutes, one obtains the equation

$$28.026\sigma_2^2 - 60.47\sigma_2 - 9.195\sigma_3^2 - 35.61\sigma_3 + 16.929 = (\text{Ang. Pri. Curv.}) \quad \text{(f)}$$

This is an equation in σ_2 and σ_3. Contour lines may, therefore, be appropriately plotted on a σ_2, σ_3 plane which will connect points for which the (Ang. Curv.) has equal values. In Figure 29 contour lines are shown for the values +10, 0, and −10 minutes. Similarly, the equations for the other monochromatic aberrations are

$$11.713\sigma_2^2 - 9.023\sigma_2 - 3.842\sigma_3^2 - 6.944\sigma_3 + 7.031 = (\text{Ang. Sph.}). \quad \text{(g)}$$

$$15.694\sigma_2^2 - 22.973\sigma_2 - 5.148\sigma_3^2 - 14.622\sigma_3 + 8.787 = (\text{Ang. Coma}). \quad \text{(h)}$$

$$4.172\sigma_2^2 - 11.894\sigma_2 - 1.3686\sigma_3^2 - 6.712\sigma_3 + 3.231 = (\text{Ang. Dist.}). \quad \text{(i)}$$

It is evident that each of these equations represents a family of hyperbolas and the absence of cross terms in σ_2, σ_3 indicates that they are not rotated with respect to the coordinate axes. The contour lines of the four equations are shown in Figure 29. Equation (e), the condition that the two components of the eyelens can be cemented, is plotted as the heavy-dashed straight line on each diagram. In each graph for a considerable distance the contour lines are approxi-

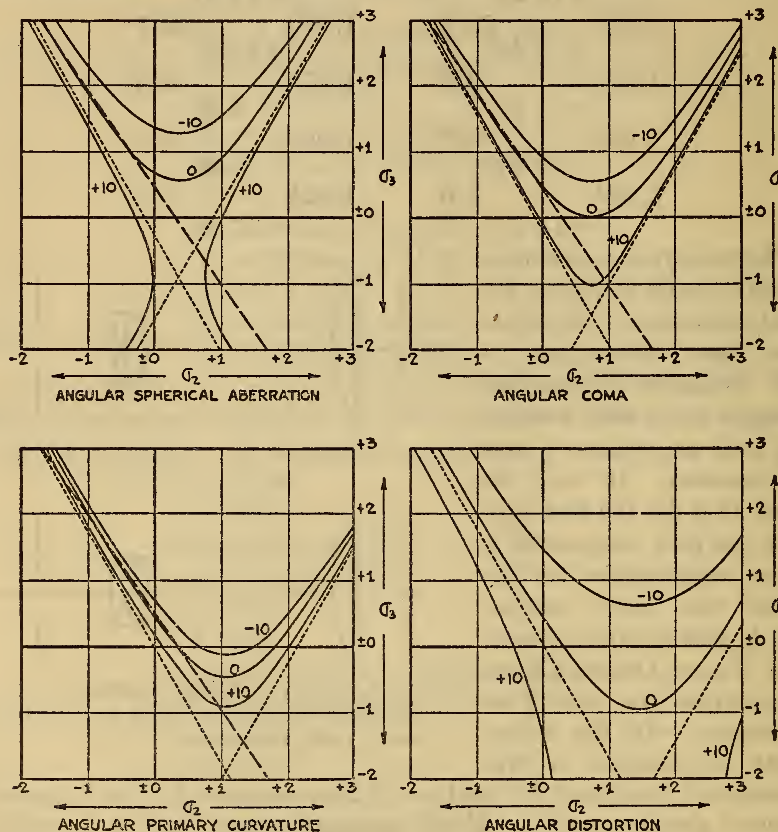


Fig. 29.—*Contour lines showing the manner in which the aberrations change as the shape factors of the components of eyelens of eyepiece of Figure 28 are changed*

The heavy dashed straight line is the locus of points for which the two components may be cemented together. For each diagram the hyperbolas connect points for which the respective aberration has the values +10, 0, and −10 minutes.

mately parallel to the line of cemented components. This indicates that the eyepiece is relatively insensitive to a change in shape factor so long as one remains on the line of cemented components.

This last characteristic is, perhaps, better shown by Figure 30. If, in equations (f), (g), (h), and (i) one eliminates σ_2 by equation (e), the equations are obtained which give the values of the aberrations as functions of σ_3 for eyepieces in which the two components of the eyelens are cemented. In Figure 30 these values are plotted against σ_3 and the values corresponding to eyepieces Nos. 1 and 2 ($\sigma_3 = +0.034$ and $+1.950$, respectively) may be readily read from the diagram. For convenience the points corresponding to eyepiece No. 1 are indicated by the small solid circles, those corresponding to No. 2 by the small open circles. An examination of Figure 30 shows at once that spherical aberration is very much better corrected in eyepiece No. 1 than in No. 2, and that the coma correction is also somewhat better in No. 2. Curvature of image is, of course, equally well corrected, and there is not much difference in the correction for distortion of the two systems, although No. 1 is slightly the better. The curvatures of the glass-air surfaces of No. 2 are much less than of No. 1, and this is a favorable condition, as it carries with it the implication that the higher order aberrations are less. On the other hand, the cemented surface of No. 2 has a very steep curve which not only suggests that it should be examined for the presence of higher order aberrations, but which also increases the difficulty of manufacture and sets a definite limit for the maximum possible diameter of eyelens.

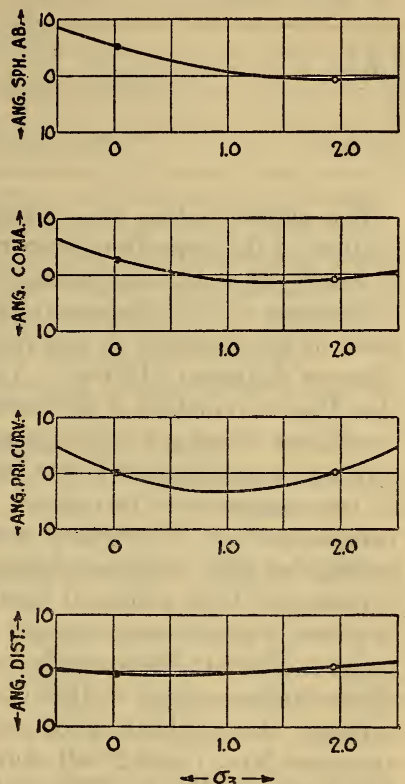


FIG. 30.—*Variations of aberrations with change in shape of eyelens*

If it is assumed that the shape factors of the two components of eyelens of eyepiece shown in Figure 28 are permitted to vary only in such a manner as is compatible with cementing them together, the aberrations may be plotted as functions of σ_3. These diagrams may be considered as sections of the aberration surfaces of which the contours are shown in Figure 29, made by a plane perpendicular to the σ_3, σ_3 plane and containing the heavy dashed straight line.

(*f*) A COMPARISON OF THE EYEPIECES BASED UPON THICK LENS AND TRIGONOMETRIC COMPUTATION.—To discuss the method of trigonometric computation as applied to these eyepieces is beyond the scope of this paper. However, the results of such work will be given for comparison with the results of the third order computations. Preliminary to this it is interesting to see the manner in which the first order constants of the eyepieces have been modified by the introduction of the necessary thicknesses.

TABLE 3.—*First order constants of eyepieces Nos. 1 and 2*

	Eyepiece No. 1		Eyepiece No. 2	
	Based on thick lens computation	Based on thin lens computation	Based on thick lens computation	Based on thin lens computation
E. F. L.	22.40	25.06	26.54	25.06
F. F. L.	−2.29	−6.24	−2.92	−6.24
x_1	−173.25	−173.25	−173.25	−173.25
$x'_{3'}$	+6.13	+10.00	+9.01	+10.00
s_1	−2.29	−6.24	−2.92	−6.24
$s'_{3'}$	∞	∞	∞	∞
Ang. Mag	7.63	6.66	6.42	6.66

The values[33] of x_1, $x'_{3'}$, s_1 and $s'_{3'}$ are measured from the nearer vertices of the respective components. In the thin lens computation, of which the results are tabulated in the third and fifth columns, the thicknesses of the components were ignored. In the original conditions of the problem it was required that E. F. L. = 25.06 and $x'_{3'}$ (the eye distance) = 10 mm. A comparison of the data above shows that the introduction of thicknesses has caused departures from these conditions which are much greater for No. 1 than for No. 2. In the thin lens computations it has been assumed that the eyelens consists of two components in contact. In the thick lenses, although the components are in physical contact, nevertheless from the optical standpoint they are spaced along the axis as the optical separation is measured from principal point to principal point. If one surface of a lens is much more strongly curved than the other, the principal point is closer to the strongly curved surface, and in the special case where the one surface is plane the one principal point lies in the curved surface. An application of this consideration to the eyelenses of eyepieces Nos. 1 and 2 will show that the separation is much less in No. 2 than in No. 1. This probably accounts for the better agreement, which has been already noted, between the two computations for No. 2.

If the departure of the thick lens system from the original specification, which has been tabulated above, is too great an adjustment may be secured by one of the two following methods:

(*a*) All dimensions of the eyepiece may be multiplied by the same ratio. This will leave the aberrations unchanged in angular value and will enable any desired first order constant to be given the preassigned value. For example, if it is necessary that the eye distance be exactly 10 mm, all radii and thicknesses of No. 2 may be multiplied by $\frac{10.00}{9.01}$. This, however, will at the same time change all other lengths in the same ratio and in the present case will make the E.F.L. 29.46 mm, which is still farther away from the value initially

[33] The primed subscript indicates the second surface of the corresponding component.

desired. In general, it will be impossible to correct more than one of the first order values by the ratio method.

(b) As an illustrative example of the second method, let it be assumed necessary to hold the focal length at 25.06 mm and at the same time have the eye distance 10. The present design may be considered as a first step. A second algebraic computation might be made in which the preassigned values are

$$\text{E.F.L.} = 25.06 - (26.54 - 25.06)$$

$$x'_{3'} = 10.00 - (9.01 - 10)$$

If the changes caused by the introduction of thicknesses are the same in the second design as before, the thick lens computation corresponding to eyepiece No.2 will have the focal length 25.06 and eye distance +10.00. Even if the change is different a better approximation will be obtained. Usually there is sufficient flexibility in the initial conditions to make the ratio method given above quite sufficient.

The values obtained by the trigonometric computations are given below:

TABLE 4.—*Aberrations of eyepieces Nos. 1 and 2*

	Eyepiece No. 1		Eyepiece No. 2	
	Trigonometric	Third order	Trigonometric	Third order
Ang. Lon. Chr.	−0.19	−0.59	−0.57	−0.59
Ang. Lat. Chr.	−.24	0	−.04	0
Ang. Sph.	+8.00	+5.09	−.70	−.67
Ang. Coma	+2.18	+2.77	−1.94	−1.13
Ang. Pri. Curv.	−1.22	−.08	+.99	−.03
Ang. Dist.	−.42	+.81	+.02	−.21

The trigonometric values are based on computations in which 4/5 significant figures [34] are employed. It is important to note that the trigonometric as well as the third order computations show that the design of eyepiece No. 2 is the better. The algebraic results given above show the values of the third order aberrations for eyepieces composed of components of zero thickness, the trigonometric results give the third order aberrations of a thick lens system plus the higher order aberrations which are sufficiently large to appear in computations of the given precision. Consequently, these differences between the two parallel columns arise not only from the introduction of higher order aberrations, but also from the introduction of thickness. The agreement for eyepiece No. 2 is again the better and for the same reason; that is, the thicknesses introduced have not done so great violence to the first order constants upon which the third order equations were based.

[34] See p. 158.

(*g*) Compensation of the Aberrations of the Objective System.—The reader familiar with eyepiece design will, perhaps, be somewhat surprised to note that for each eyepiece the surface of the flint component which is nearer the eye is strongly curved, whereas in most of the examples one sees this surface is nearly flat. It must, however, be recalled that one of the original conditions imposed was that the primary image surface be flat. Such a condition might be imposed on an optical system designed to read a scale and a vernier. The usual eyepiece, however, is used with an objective system as in the microscope and telescope. The image presented to the eyepiece by the objective system, in general, possesses positive curvature. It is therefore necessary to have an eyepiece with negative curvature if the complete optical system is to present an image free from curvature. An examination of Figure 30 shows that the negative curvature of maximum absolute value corresponds to $\sigma_3 = +1.0$ and this, in turn, indicates a flint component with the outer surface plane. It follows that eyepieces of this type with negative curvature will have the external surface of the flint flat or nearly so.

In a telescopic system the aperture of the objective is commonly the entrance pupil. Therefore, in designing an eyepiece for a telescope the exit pupil of the objective system is the entrance pupil of the eyepiece. Furthermore, the position of the exit pupil of the eyepiece must be so determined as to provide ample clearance for the observer's eye. Even though the entrance pupil of the eyepiece is fixed by the location of the objective, the position of the exit pupil can be controlled by varying the power or position of the field lens.

(*h*) Modifications Which May be Introduced to Secure Additional Degrees of Freedom.—It must not be assumed that Figure 30 shows the entire possibilities of our choice of glass in the design of Kellner eyepieces. Assume that none of the combinations of aberrations as there indicated appear to offer a satisfactory compromise. One may start out anew with a different choice of ratio of power of eyelens to field lens and with the spacing altered to hold the focal length constant. If the distance to the exit pupil can be changed, this changes the values of the g's and ϵ's but leaves the h's and π's as before. Consequently, this will change all aberrations except spherical, which is not a function of the g's or ϵ's. To investigate the entire range of possibilities of so simple a system as a Kellner eyepiece of specified glass becomes, therefore, rather a long task, and the amount of work required to completely study a system increases rapidly as the number of components becomes greater.

(*i*) Sensitiveness of the Design to Small Departures From Specifications.—Other things being equal the design is superior which shows the smallest variation in its essential characteristics as a result of the unavoidable changes which arise in actual production. For a study of this feature the algebraic equations are particularly useful.

From equation (f) one may write to a sufficiently good approximation

$$\frac{\partial\,(\text{Ang. Curv.})}{\partial\sigma_2}=56\ \sigma_2-60 \tag{j}$$

and

$$\frac{\partial\,(\text{Ang. Curv.})}{\partial\sigma_3}=-18\ \sigma_3-36 \tag{k}$$

Substituting the values of σ_2 and σ_3 for the two eyepieces one obtains the following values:

	No. 1	No. 2
$\frac{\partial\,(\text{Ang. Curv.})}{\partial\sigma_2}$	-43	-116
$\frac{\partial\,(\text{Ang. Curv.})}{\partial\,\sigma_3}$	-37	-71

Roughly speaking, a change of 0.01 in σ_2 or σ_3 makes a change of from 0.5 to 1 minute in the curvature. Design No. 1 shows the greater stability.

From equation (8)

$$\frac{\partial\,\sigma}{\partial\,r}=\frac{2r'}{(r'-r)^2},\qquad \frac{\partial\,\sigma}{\partial r'}=\frac{-2r}{(r'-r)^2} \tag{l}$$

Substituting values from eyepieces Nos. 1 and 2 one has

	No. 1	No. 2
$\frac{\partial\sigma_2}{\partial r_2}$	-0.061	0
$\frac{\partial\sigma_2}{\partial r'_2}$	-0.033	-0.002
$\frac{\partial\sigma_3}{\partial r_3}$	$+0.036$	-0.287
$\frac{\partial\sigma_2}{\partial r'_3}$	$+0.034$	$+0.092$

Combining this with the results already given

	No. 1	No.2
$\frac{\partial\,(\text{Ang. Curv.})}{\partial r_2}$	$+2.6$	0
$\frac{\partial\,(\text{Ang. Curv.})}{\partial r'_2}$	$+1.4$	$+0.2$
$\frac{\partial\,(\text{Ang. Curv.})}{\partial r_3}$	-1.3	$+21.9$
$\frac{\partial\,(\text{Ang. Curv.})}{\partial r'_3}$	-1.3	-6.5

and it is evident that for a given change in the radii of curvature design No. 1 shows much the greater constancy. Such values as these are often very useful, as they show the possibility of using radii of curvature for which tools and test plates are already at hand instead of the exact values as prescribed by the computation. This results in a great saving in the cost of production.

From equations (101) and (c) it is apparent that

$$\frac{\partial(\text{Ang. Pri. Curv.})}{\partial n_2}=3{,}438\ o\ \tan^2\beta_1\left[\frac{\partial}{\partial n_2}\left(\frac{\varphi_2}{n_2}\right)+3\frac{\partial\varphi_2}{\partial n_2}+.128\frac{\partial A_2}{\partial n_2}-.428\frac{\partial C_2}{\partial n_2}\right] \tag{m}$$

and

$$\frac{\partial(\text{Ang. Pri. Curv.})}{\partial n_3}=3{,}438\ o\ \tan^2\beta_1\left[\frac{\partial}{\partial n_3}\left(\frac{\varphi_3}{n_3}\right)+3\frac{\partial\varphi_3}{\partial n_3}-.502\frac{\partial A_3}{\partial n_3}-.230\frac{\partial C_3}{\partial n_3}\right] \tag{n}$$

The following values are obtained with sufficient accuracy by the use of a slide rule. The derivatives of the functions of n are readily obtained from the tabulated differences of Appendix 4.

$$\frac{\partial A_2}{\partial n_2}=-26\ \sigma_2{}^2+10.5\ \sigma_2-17.2$$

$$\frac{\partial C_2}{\partial n_2}=-5.6\ \sigma_2+0.18$$

$$\frac{\partial A_3}{\partial n_3}=-20\ \sigma_3{}^2-19\ \sigma_3-14$$

$$\frac{\partial C_3}{\partial n_3}=-4.7\sigma_3-0.3$$

Substituting in a straightforward manner it can be shown that a change of 0.0001 in n_2 changes the curvature for eyepieces Nos. 1 and 3 by 0.002 and 0.03 minute, respectively. The variations resulting from a similar change in n_3 are 0.02 and 0.2 minute. In this comparison design No. 2 appears the more unfavorable for production. In a similar manner the stability of the design with respect to variations in the other aberrations can be studied.

(*j*) Precision Necessary in the Computations Involving the Third Order Equations.—At best the computations necessary in the design of a moderately complicated optical system are long and tedious, and it is desirable to shorten the labor as much as possible. If many computations similar in nature are to be made, it is accordingly well worth while to carefully determine the number of significant figures required in the final result and to limit the precision of the numerical work in accordance with this finding. It is at once apparent that there are at least three considerations, any one of which may determine the number of significant figures necessary in the computations. They are: (1) The limiting precision with which control is to be exercised during manufacture which will determine the fidelity with which the finished system corresponds to the computed system, (2) the demands to be made upon the instrument in use, and (3) the approximation in the equations involved in the neglect of the higher order terms.

Of these three the last will usually be the limiting one which determines the precision of the computations. Optical tests of surfaces are precise and simple to make, and, except in the very cheapest instruments, it is feasible to carry the control to a precision beyond that of the third order equations. The second consideration will rarely apply as it is, in general, as easy and as economical to construct the system according to the best design (from a given number of components) as it is to begin with an imperfect design. But so long as one is restricted to third order equations the approximations involved in their derivation set a definite limit of precision beyond which the numerical computations should not be carried. In the majority of better grade instruments the third order computations will be followed by empirical adjustment in accordance with the results of trigonometric ray tracing, and here again the computations should be carried out to the limit of precision of the equations in order to make the trigonometric adjustment as simple as possible.

An exact appraisal of the precision of the third order equations is difficult to make and can not be made sufficiently general to justify its inclusion here. However, a rough estimate can be made on the following basis: In the development of the third order equations the approximation arises from the fact that series developments of the sines and tangents of the half aperture and half field angles are broken off with the second term. A value of 0.1 radian is a fair approximation of the maximum value of these angles for any lens in an optical system. The next term is of the fifth order, and hence as a first assumption (but perhaps the best that can be made without a special consideration of each case) the error in any one of the products of equations (99) to (103) may be considered as of the order of 1 per cent (the ratio of the fifth power of the angle to the third). The significance of this will be evident from an application to the illustrative example already given. The entire expression for the (Ang. Pri. Curv.) will consist of 12 terms, 4 for each lens. For systems Nos. 1 and 2, expressed in minutes, they are as follows:

No. 1	No. 2	
+0.67	+0.67	Field lens.
+3.06	+3.06	
+8.31	+8.31	
−13.19	−13.19	
+2.42	+2.42	Crown component of eyelens.
+11.45	+11.45	
+30.43	+83.30	
+5.01	+55.28	

No. 1	No. 2	
−1.72	−1.72	Flint component of eyelens.
−8.39	−8.39	
−18.23	−85.00	
−19.83	−56.23	
−00.01 minutes.	−00.04 minutes.	

In No. 1 the largest single contribution is of the order of 30 minutes, and it may accordingly be estimated that the error from this particular approximation will be of the order of 0.3 minute. Similarly for No. 2 the minimum value will be of the order of 0.9 minute. It must be understood that from this consideration one can form no conclusion regarding the maximum error to be expected, as there are other important sources of error which will be considered later, but the minimum error to be expected is desired at present as it sets the limit to the precision which may be usefully employed in the computations. Referring to the values of $\frac{\partial(\text{Ang. Curv.})}{\partial\sigma_2}$ and $\frac{\partial(\text{Ang. Curv.})}{\partial\sigma_3}$ given in the preceding section, it is evident that σ_2 and σ_3 should be determined to the nearest hundredth, and the computation should be planned with this purpose. The computations are rather long, and there are usually places where subtractions are encountered in which there is considerable loss of significance. One can determine the number of significant figures in the final result by a careful scrutiny of each step, but for the example already worked the degree of significance retained in the final result was determined by carrying three computations along in parallel. In the first, in all results, both intermediate and final, only three significant figures were retained except when the first digit was 1 or 2, in which case four were retained. This will be referred to subsequently as a computation in 3/4 significant figures. Similarly in the second computation, of which the results are given in the preceding text, four and five figures were retained and in the third 5/6 significant figures were preserved. The values obtained for σ_2 and σ_3 in the computations are given below.

	3/4	4/5	5/6
σ_2	+0.300	+0.3021	+0.3021
	or −0.971	or −0.9916	or −0.9924
σ_3	+0.036	+0.0339	+0.0340
	or +1.916	or +1.9505	or +1.9520

The above results show that the 3/4 computation is not sufficiently precise and that the 5/6 computation involves more labor than is useful. The above computations were carried out on a computing machine, but the 3/4 work may be considered to be approximately the same, as regards precision, as work with a four-place logarithm table. The 4/5 computation corresponds to the use of a five-place table. A more critical examination of the computation for the illustrative example would, of course, show that the number of significant figures should be varied at different stages of the computation, and the designer having a very large amount of work to do may find an investigation along this line profitable.

When an isolated computation is made, it is probably simpler to err in the direction of too many significant figures. But if a number of computations on systems of the same general character are to be made, the importance of determining carefully the number of significant figures to be retained can not be overemphasized. For eyepiece work of the character of the illustrative example it appears that 4/5 significant figures are sufficient for final third order computations. For preliminary work or for a general reconnaissance, in which the glasses or other factors are to be varied, good qualitative results may be obtained by the 20-inch slide rule or four-place logarithm table, and the saving of time effected by their use will be very great. This last point can hardly be overemphasized. Unfortunately, it seems to be the general impression that a very large number of significant figures are necessary in all optical computations, and as a result very interesting problems and researches are made to appear most unattractive.

In the foregoing consideration of the precision of the third order equations the only source of error which has been treated was that arising from the breaking off of the series developments of the trigonometric functions. This sets a minimum value for the error to be expected, and it at once suggests the desirability of obtaining an estimate of the maximum value. But this is more difficult and can only be dealt with in a very general manner. Before the third order equations are applied the thick lens system is replaced by an idealized thin lens system. If one is dealing with a telescope objective, consisting of two or three thin components cemented together, the approximation yielded by the thin lens system is good. But if one has thick components with large separations the first order differences between the thick lens and thin lens combinations may be large. The eyepiece is particularly unfavorable, and Table 3, page 152, well illustrates this. As the first order constants are the basis of the third order equations, this in turn introduces large variations in the values of the aberrations as evidenced by Table 4, page 153. These errors will be detected in the first stage of the trigonometric adjustment or can be estimated with considerable accuracy if the third order equations are applied to the equivalent system of elements as defined by Taylor.[35]

[35] Taylor, H. D., A System of Applied Optics, Section II. Macmillan & Co. (Ltd.), London; 1906.

VIII. DESIGN OF OPTICAL SYSTEMS WHICH CONTAIN THIN LENSES AND PLANE PARALLEL PLATES OR REFLECTING PRISMS

In the equations which have been given thus far the thicknesses of the lens components have been considered as negligible. If the optical system contains plates of glass having plane parallel surfaces placed normal to the axis of the system and of thicknesses comparable in magnitude to those of the lens components, their aberrations may, in general, be neglected as of the same order of magnitude as the thickness aberrations of the lenses. But if the plates are of great thickness or if the system contains reflecting prisms, aberrations are introduced by the excessive length of glass path which can not be ignored.

1. ABERRATIONS OF PLANE PARALLEL PLATES

The determination of the aberrations of a plane parallel plate proceeds in two steps.

1. The plane parallel plates are replaced by equivalent air thicknesses and the first order equations applied. This gives the values of s's and x's which are subsequently used to determine the values of g, h, σ, π, and ϵ for each component.

2. The third order equation for the plane parallel plate is applied. For this the s, x, g, and h as determined in (1) are used, but the actual thickness of the plate and not the equivalent air thickness is applied in the equation.

(*a*) Replacement of Plane Parallel Plates by Equivalent Air Thicknesses.—For first order imagery each plane parallel plate may be removed from the system and replaced by an equivalent path length in air, after which the first order equations are applied in the usual manner. Figure 31 will make the manner of substitution clear. In the upper diagram a system is shown which comprises two plates and two lenses. The first plate is between the lens L_1 and the object at I. The second between L_1 and L_2. In first order imagery a plate of thickness d is equivalent to an air thickness d/n where

$$d = \text{thickness of plate,}$$

$$n = \text{index of refraction of plate.}$$

The simplified system obtained when the plates are replaced by the equivalent air thicknesses is shown in the lower diagram, and the manner of replacement is evident from the equations written on the dimension lines.

If one has given a system with plane parallel plates and wishes to apply the first order equations, it is only necessary to pass from the given system to one without plates in the manner indicated, after which the first order equations are applied in the usual manner. Conversely, if a system has been designed without plane parallel plates and it is desired to determine the distances between the lenses and their conjugate planes after the introduction of the plates, the alteration is made which is indicated in passing from the lower to the upper drawing of Figure 31. Any optical path measured along the axis in which the ray is transmitted by the glass is increased by the length $d\left(\frac{n-1}{n}\right)$. But equations (1) and (2) can only be applied to distances in which the air equivalents are inserted as in the lower diagram of Figure 31. Distances measured on the upper diagram in which the actual thicknesses of the glass plate appear are not to be used in the first order equations, but are only applied in the determination of the lengths to be used in designing the mechanical parts. A plane parallel plate may be considered as a lens of zero power and does not affect the magnification of a system. Hence, images in homologous conjugate planes of the two systems of Figure 31 are the same size and the values of the y's in the two systems are identical.

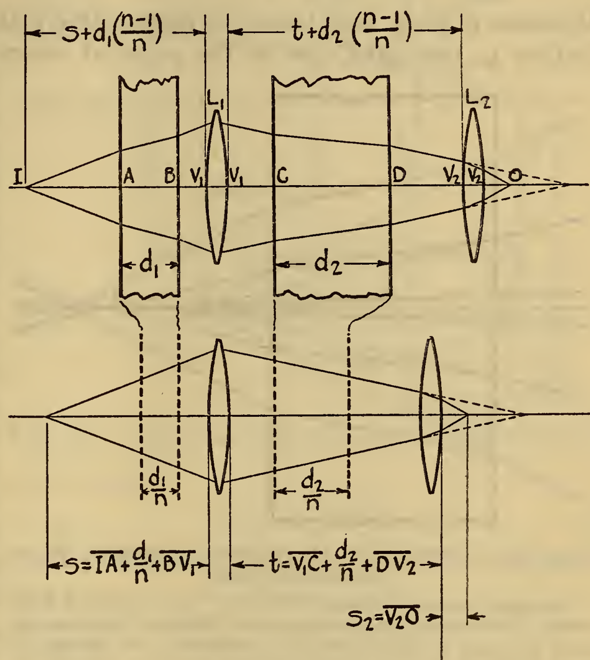


FIG. 31.—*Manner in which plane parallel plates of glass are replaced by equivalent air thicknesses for first order computations*

In first order imagery a plate of thickness d is equivalent to an air thickness d/n where

d=thickness of plate.

n=index of refraction of plate.

In the preceding paragraphs it has been assumed that the axial ray is normal to the plane parallel plate. If this is not the case, the thickness is to be measured along the path of the axial ray and not normal to the surface of the plate.

(*b*) THIRD ORDER ABERRATIONS OF PLANE PARALLEL PLATES.—The imagery as defined by the first order equations is identical for the two systems of Figure 31. But each plate of the upper system introduces third order aberrations and consequently to the terms of equations (96) to (103) there must be added one term for each plate present. The manner in which a plane parallel plate introduces spherical aberration is indicated by Figure 32. Assume that a bundle of rays proceeds from an optical system, not shown, lying to the left of the plate and that an axial image free from aberration is formed at I'. When the glass plate is introduced, on the basis of first order imagery, the paraxial rays form a new image at I'' where the length $I'\,I''$ is $d\left(\frac{n-1}{n}\right)$. A marginal ray passes through a greater thickness of glass, and, hence, its intersection with the axis is displaced farther to the right than is the paraxial image. The net result is that the image at I'' has negative spherical aberration. If in a similar manner a point off the axis is considered, it will be found that all aberrations, both chromatic and monochromatic, are introduced by the plate. The equations giving the magnitudes of these aberrations may be readily obtained from the Seidel equations [36] by a method similar to that used to determine the spherical aberration of a thin lens. The terms for two successive surfaces are written, after which one writes $r=r'=\infty$ and applies reductions similar to those used in the derivation of the lens equations. If one does this for the plane parallel plate lying between lenses k and $k+1$ and having the surfaces a and b as illustrated in Figure 33, the equations will contain the ratio $\frac{h_a}{s_a}$ or $\frac{h_b}{s'_b}$ according to the manner in which the elimination is carried out. But any ray which enters a plane parallel plate after emergence is traveling in a direction parallel to that which it originally had. It follows that

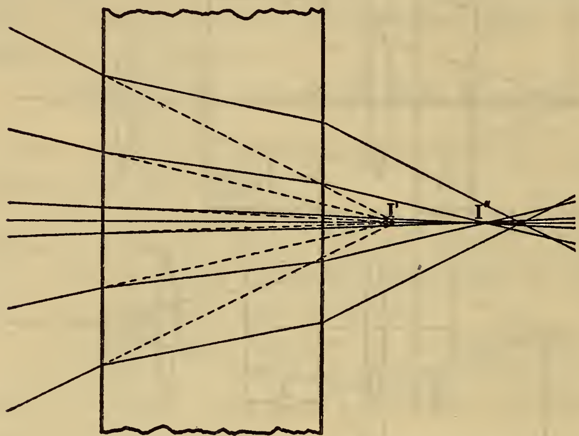


FIG. 32.—*Introduction of spherical aberration by a plane parallel plate*

The paraxial image is transferred from I' to I'' when the plate is introduced. The marginal rays travel a greater distance in the glass, and hence the image produced by them is transferred to the right of I''. At I'', therefore, there is negative spherical aberration although the original image at I' was aberration free.

$$\frac{h_a}{s_a}=\frac{h_b}{s'_b}=\frac{h_k}{s'_k}=\frac{h_{k+1}}{s_{k+1}} \tag{108}$$

[36] See Appendix 2.

In the first order equations, which will have already been applied to the system, all plates will have been replaced by equivalent air thicknesses. The values of h_a, h_b, s_a, s'_b can only be determined by an additional computation, while h_k, h_{k+1}, s'_k, s_{k+1} will be already available. Consequently, in the formulas the last two ratios of equation (108) will be used. In the terms for the other aberrations a similar elimination will be made in the ratios of g and x which occur. As a corollary it follows that the aberrations introduced by a plane parallel plate are independent of the axial position of the plate so long as it remains between any two given lenses.

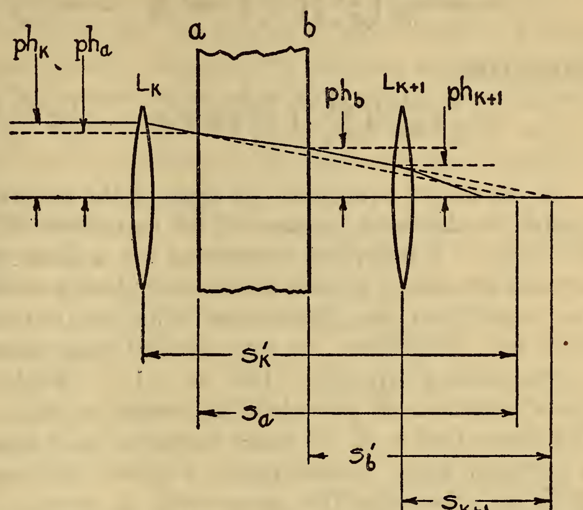


FIG. 33.—*A plane parallel plate between lenses K and K+1*

It is evident that

$$\frac{h_a}{s_a}=\frac{h_b}{s'_b}=\frac{h_k}{s'_k}=\frac{h_{k+1}}{s_{k+1}}$$

(*c*) THIRD ORDER ABERRATION EQUATIONS OF PLANE PARALLEL PLATES.—Following are the formulas for the different aberrations produced by a plane parallel plate placed normal to the optic axis of thickness d and index n:

	Aberrations referred to preceding lens	Aberrations referred to following lens	
(Ang. Lon. Chr.)	$= -2od\dfrac{\Delta n}{n^2}\left(\dfrac{h_k}{s'_k}\right)^2$	$= -2od\dfrac{\Delta n}{n^2}\left(\dfrac{h_{k+1}}{s_{k+1}}\right)^2$	(109)
(Ang. Lat. Chr.)	$= -d\dfrac{\Delta n}{n^2}\left(\dfrac{g_k}{x'_k}\right)\left(\dfrac{h_k}{s'_k}\right)\tan\beta_1$	$= -d\dfrac{\Delta n}{n^2}\left(\dfrac{g_{k+1}}{x_{k+1}}\right)\left(\dfrac{h_{k+1}}{s_{k+1}}\right)\tan\beta_1$	(110)
(Ang. Sph.)	$= -o^3d\dfrac{n^2-1}{n^3}\left(\dfrac{h_k}{s'_k}\right)^4$	$= -o^3d\dfrac{n^2-1}{n^3}\left(\dfrac{h_{k+1}}{s_{k+1}}\right)^4$	(111)
(Ang. Coma)	$= -\dfrac{3}{2}o^2d\dfrac{n^2-1}{n^3}\left(\dfrac{g_k}{x'_k}\right)\left(\dfrac{h_k}{s'_k}\right)^3\tan\beta_1$	$= -\dfrac{3}{2}o^2d\dfrac{n^2-1}{n^3}\left(\dfrac{g_{k+1}}{x_{k+1}}\right)\left(\dfrac{h_{k+1}}{s_{k+1}}\right)^3\tan\beta_1$	(112)

Aberrations referred to preceding lens. Aberrations referred to following lens.

(Ang. Pri. Curv.)

$$=-3od\frac{n^2-1}{n^3}\left(\frac{g_k}{x'_k}\right)^2\left(\frac{h_k}{s'_k}\right)^2\tan^2\beta_1 \quad =-3od\frac{n^2-1}{n_3}\left(\frac{g_{k+1}}{x_{k+1}}\right)^2\left(\frac{h_{k+1}}{s_{k+1}}\right)^2\tan^2\beta_1 \tag{113}$$

(Ang. Sec. Curv.)

$$=-od\frac{n^2-1}{n^3}\left(\frac{g_k}{x'_k}\right)^2\left(\frac{h_k}{s'_k}\right)^2\tan^2\beta_1 \quad =-od\frac{n^2-1}{n^3}\left(\frac{g_{k+1}}{x_{k+1}}\right)^2\left(\frac{h_{k+1}}{s_{k+1}}\right)^2\tan^2\beta_1 \tag{114}$$

(Ang. Dist.)

$$=-\frac{1}{2}d\frac{n^2-1}{n^3}\left(\frac{g_k}{x'_k}\right)^3\left(\frac{h_k}{s'_k}\right)\tan^3\beta_1 \quad =-\frac{1}{2}d\frac{n^2-1}{n^3}\left(\frac{g_{k+1}}{x_{k+1}}\right)^3\left(\frac{h_{k+1}}{s_{k+1}}\right)\tan^3\beta_1 \tag{115}$$

In the above equations the signs of the aberrations are to be interpreted in the same manner as for equations (95) to (103). (See pp. 130–131.) A complete expression for a third order aberration of a system composed of thin lenses and plane parallel plates will contain one term from the appropriate equation (95) to (103) for each thin lens, and, in addition, for each parallel plate there will be a term from corresponding equation (109) to (115). Each of the plane parallel plate equations is preceded by a negative sign. This indicates that the aberration is of the same character as is generally introduced by a negative lens. Accordingly, a system of lenses and plates which gives an aberration-free image will, in general, be undercorrected in the absence of the plates. The aberrations increase as the ratio $\frac{h}{s}$ or $\frac{g}{x}$ increases, and it follows that a plane parallel plate introduces the more aberration accordingly as the apertures of the transmitted bundle of rays or the divergence of the chief rays is large. In all cases the aberration is directly proportional to the thickness. If $s=\infty$; that is, if a plane parallel plate is introduced in a portion of the system when the bundles of rays are parallel, there is no contribution to the third order aberrations.

2. THIRD ORDER ABERRATIONS OF PRISMS

Plane parallel plates are commonly used as windows to protect the surfaces of prisms or other large components from scratching or mechanical injury. For this purpose they are often used in military fire-control instruments and other instruments subjected to particularly rough usage. In the compound microscope the cover glass placed between objective and object forms an important part of the optical system. In the majority of such cases, however, the plates are of so little thickness that the aberrations when determined are found to be negligibly small in comparison with the aberrations of the lenses. In fact, it will be remembered that the thicknesses of the lenses are neglected in the thin lens equations, and the errors which arise from the neglect of the aberrations of plane parallel plates of approximately equal thickness will be of the same order of magnitude. But it will be shown below that a reflecting prism is equivalent, as far as aberrations are concerned, to a plane parallel plate, the thickness of which equals the length of path of axial ray. Consequently, prisms are equivalent to plane parallel plates of great thickness, and the application of equations (109) to (115) is necessary to secure a satisfactory approximation.

(*a*) Replacement of a Prism by an Equivalent Thick Plate.—In Figure 34 there is illustrated the course of two bundles of rays through a 90° reflecting prism. In the lower diagram the prism and the emergent bundles have been reflected in the reflecting surface of the prism. In the two illustrations it will be seen that the angles of incidence and refraction are identical at the corresponding surfaces. As reflection at a plane surface introduces no aberrations, but only a change in direction, it follows that the third order aberrations of the prism in the upper design and the thick parallel plate in the lower diagram are identical. When a system contains prisms, the process is therefore as follows:

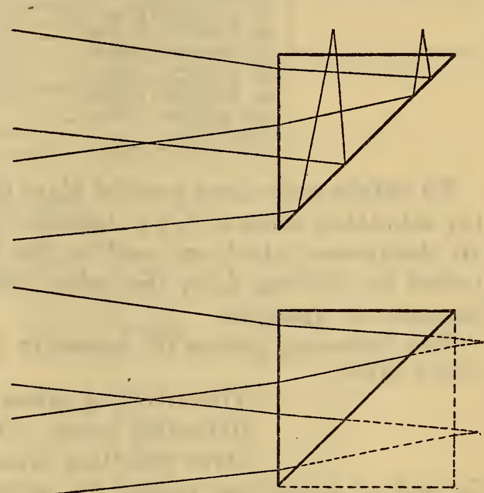


Fig. 34.—*Equivalence in third order imagery of a reflecting prism and a plane parallel plate*

1. Each prism is considered as replaced by a thick plate which has a thickness identical with the path length of the axial ray within the prism.

2. The thick plate thus obtained is replaced by the equivalent air thickness and the first order equations applied. The s's and x's obtained in this stage are used to determine the π's, ϵ's, g's, and h's of the lenses.

3. In the formation of the third order equations one term is introduced for each prism by applying equations (109) to (115) to the equivalent thick plate of (1).

(*b*) Principal Types of Prisms Used in Optical Instruments With the Thicknesses of the Equivalent Plane Parallel Plates.—In Appendix 3 there is a series of plates prepared by O. K. Kaspereit and which formed a part of the revised 1924 edition of "Elementary Optics and Applications to Fire Control Instruments," United States Army Ordnance Document No. 1065. Below, for convenience of reference, there is tabulated a list of the different prisms, together with the values of the lengths of light paths for unit aperture, which will be found represented on each plate by the symbol d_p. These lengths are the thicknesses of the plane parallel plates equivalent to the prisms.

TABLE 4.—*Equivalent parallel plate thicknesses of reflecting prisms*

Type of prism	d_p=equivalent parallel plate thickness for prism of unit aperture
Inverting prisms:	
No. 1 by Abbe	5.20
No. 2 by Abbe	4.30
Hensolt	5.33
Porro	4.50
Leman	5.20
Amici	1.71
Pentagon	3.41
Camera Lucida	4.83
40° deviation "D-40"	3.14
45° deviation "D-45-a"	3.41
45° deviation "D-45-b"	7.44
50° deviation "D-50"	3.75
60° deviation "D-60-a"	3.46
60° deviation "D-60-b"	5.46
120° deviation "D-120"	3.46
Rhomboid	3.00
Right angle reflecting	1.00

To obtain equivalent parallel plate thickness for any prism, multiply tabulated value of d_p by diameter of aperture. The equivalent air thicknesses which are used in the first order equations are obtained by dividing d_p by the index of refraction and multiplying by diameter of aperture.

The following prisms of Appendix 3 have not been listed in the above table:

The inverting prism by Wirth,
Reflecting prism "D-90,"
Dove reflecting prism.

In each of these three prisms the angles of incidence and emergence at the two glass air surfaces are not zero. The equivalent plane parallel plate which replaces the prism is not a plate placed normal to the axis of the system, but one inclined to give the same angle of incidence as that of prism. Such a plate of a thickness equivalent to the prism, if placed in a position where the pencils are convergent, introduces aberrations so large that the system is useless for most purposes. Consequently, these prisms can only be used where the rays within a bundle proceeding from any point of the object are parallel; that is, $s=\infty$. One finds such prisms used either in front of the objective or back of the eyepiece of a telescopic system. In such a position no aberration is introduced, and the equivalent glass thickness is of no import as it is a finite length to be subtracted from the relatively very large or infinite s.

The method of the replacement of prisms by parallel plates can not be applied to prisms which are used to introduce dispersion as in the spectrograph or spectrometer. Obviously, these prisms can not be replaced by equivalent plane parallel plates, since the latter introduce no dispersion. Dispersion components must be dealt with by a method of ray tracing.

3. APPLICATION OF THIRD ORDER THEORY OF IMAGERY TO A TELESCOPIC SYSTEM WHICH CONTAINS A PARALLEL PLATE AND A PRISM

A telescopic system is to be designed for which the following data hold:

Diameter of objective	25 mm.
E. F. L. of objective	125 mm.
E. F. L. of eyepiece	25 mm.
Angular magnification	5 times.

Such a thin lens system is shown in the upper diagram of Figure 35. A window of borosilicate glass 4 mm thick is to be placed in front of the objective to prevent mechanical injury. An Amici inverting prism, commonly referred to as a roof prism, is to be placed back of the objective. The insertion of these two components is shown in the lower diagram.

The window is indicated with its inner surface 10 mm in front of the first vertex of the objective. As this is a telescopic system, $s_1 = \infty$, and the path length within the window can, therefore, be ignored. The inverting prism is to be placed with its first surface 10 mm back of second principal point of the objective. The aperture of the prism is 25 mm. Reference to the table, page 166, shows that the equivalent parallel plate thickness of this prism will be $25 \times 1.71 = 42.75$ mm. It will be assumed that the prism is made of borosilicate glass, the constants of which are given on page 146. The equivalent air thickness for light of wave length 546 mμ is $42.75 \div 1.518 = 28.16$ mm. In the upper diagram the distance from objective to image $(s = \infty) = 125$ mm. In the lower diagram

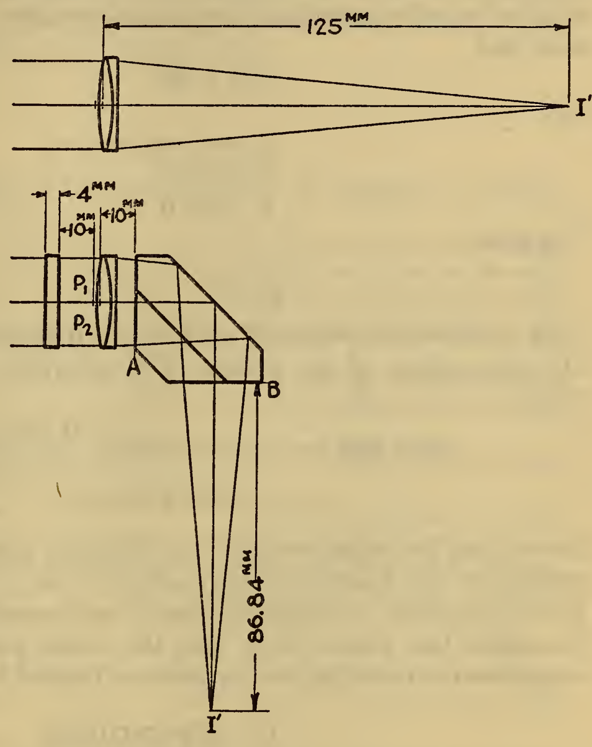


FIG. 35.—*Telescopic system with plane parallel plates of the illustrative example*

$$\overline{P'_{1,2}A} + \text{equivalent air thickness of prism} + \overline{BI'} = 125 \text{ mm}$$

and on substituting $BI' = 86.84$ mm. The details of the eyepiece will not be entered into, but this gives the necessary data for design of mechanical parts so far as the objective system is concerned.

It is evident that the window will introduce no aberrations, as it is placed in front of the objective where $s_1 = \infty$. It will be assumed that the entrance pupil is in the plane of the objective. Then

$$s_1 = \infty \qquad x_1 = 0$$

The values of s'_1 and s_2 will depend upon the manner in which the power is divided between the two components of the objective, and this, in turn, is dictated by the choice of glass and the condition for elimination of longitudinal chromatic aberration. But it can easily be seen that

$$s'_2 = +125$$

also

$$x_1 = x'_1 = x_2 = x'_2 = 0$$
$$h_1 = h_2 = 1,$$
$$g_1 = g_2 = 0$$

The ratio

$$\frac{g_2}{x'_2} = 1$$

This provides the necessary data for the formation of the term giving the aberrations of the prism. For spherical aberration (equation (111)

$$(\text{Ang. Sph.}) = -(12.5)^3\,(42.75)\,\frac{(1.518)^2 - 1}{(1.518)^4}\left(\frac{1}{125}\right)^4$$
$$= -0.000084 \text{ radians}$$

In writing the equations for the spherical aberration of the entire system to the 5 terms corresponding to the 5 lens components (2 in the objective, 3 in the eyepiece) from equation (98) there would be added the above term and the whole equated to zero. The equations for the other aberrations are formed in a similar manner.

IX. APPENDIXES

APPENDIX 1.—NOTATION AND SIGN CONVENTIONS

Below there is given a list of the characters which have been used, together with their significations. In the second column there will be found the equivalent characters as used by Taylor.[37]

The lenses of an optical system are assumed to be numbered in order, beginning with the one which first receives the light from the object. Subscripts attached to a symbol relate it to the lens bearing the same number. Unprimed characters refer to magnitudes in the object space, primed characters to the homologous magnitudes in the image space of a lens. An exception has been made in the case of

[37] Taylor, A System of Applied Optics. Published by Macmillan & Co. (Ltd.), London; 1906.

the primes affixed to λ (see below), but this will lead to no confusion. Additional subscripts may be added to n to indicate the wave length for which the index of refraction is indicated. The subscript may be a letter, in which case it refers to the Fraunhofer designation of the line; or it may be a number, in which case it refers to the wave length measured in mμ. A double subscript applied to characters other than n indicates that the magnitude is not referred to a single component, but to the optical system composed of the lenses from the first to the second subscripts, inclusive. Thus $s_{1,5}$ is the axial object distance for the system composed of the first five components and would be measured from the first principal point of this system to the projection of the object point.

Present notation	Taylor notation	
A	A'	=the coefficient of spherical aberration. It is defined by equations (23) and (24).
A		=the angular magnification. It is the limit of the ratio of the angle between a ray and the optic axis in the image space to the angle between the conjugate ray in the object space and the optic axis when the last-mentioned angle approaches zero as a limit. It is defined by equation (85).
B	B'	=the coefficient of distortion which is a measure of the distortion produced by the aberration of the pupil points, and is defined by equations (64) and (65).
b		=the deformation coefficient for an aspherical surface. It is defined by equation (122) in Appendix 2.
C	C'	=the coefficient of coma. It is a measure of the amount of normal coma present and is defined by equations (39) and (40).
d		=the thickness of a plane parallel plate or path length in a prism. It differs from t (q. v.) in that it is a length measured in glass.
f		=the focal length. For a thin lens it is defined by equation (1).
g		=the height of incidence of a chief ray divided by the tangent of the angle between the axis and the conjugate chief ray in the *object space of system.* It is defined by equations (16), (93), and (94).
h		=the height of incidence of a ray from an axial object point divided by the height of incidence of conjugate ray in *entrance pupil plane of optical system.* It is a dimensionless quantity defined by equations (15), (91), and (92).
L		=the longitudinal magnification. It is the limit of the ratio of a short length of the optic axis in the image space to the conjugate segment in the object space when the last approaches zero as a limit. It is defined by equation (84).
M		=the lateral magnification. It is the limit of the ratio of a short length normal to the optic axis in the image space to the conjugate segment in the object space when the last approaches zero as a limit. It is defined by equation (83).
n	μ	=index of refraction of any optical medium relative to air which is assumed to be the common medium surrounding the optical components.

Present notation	Taylor notation	
Δn	$\Delta \mu$	$=n\lambda'' - n\lambda'$. If not otherwise qualified, it usually refers to $n_F - n_C$.
o	A	= the radius of the entrance pupil of the optical system.
P		= the Petzval curvature. It is defined by equation (96).
p		= the distance measured in the entrance pupil plane from axis to point of incidence of a ray proceeding from an axial object point. The maximum value of p is o.
$Q_{s,i}$		= the invariant of the object point for the ith surface. It is defined by equation (123) in Appendix 2.
$Q_{x,i}$		= the invariant of the pupil points for the ith surface. It is defined by equation (124) in Appendix 2.
r	r	= the radius of curvature of first surface of lens; that is, the surface which receives the incident light.
r'	s	= the radius of curvature of second surface of lens; that is, the surface from which the light emerges. r and r' are positive when the surfaces are convex toward the incident light, negative if concave toward it.
S_{I} S_{II} S_{III} S_{IV} S_{V}		These are the Seidel sums and are defined by equations (116) to (120), inclusive, in Appendix 2.
s	u	= the "axial distance" of object point, which is the distance from the common vertex of the two surfaces of a thin lens (for a thick lens or lens system s is measured from the first principal point) to the projection on the axis of the object point.
s'	v	= the "axial distance" of image point, which is the distance from the common vertex of the two surfaces of a thin lens (for a thick lens or lens system s', is measured from the second principal point) to the projection on the axis of the image point. s or s' is positive if a generating point, when moving in the direction of the incident light, passes through the common vertex of the two surfaces (or principal point in the case of thick lenses or lens systems) before it arrives at the object or image point, respectively.
T	T'	= a second coefficient of distortion, which is a measure of the distortion that arises from lack of coplanarity of intersections of chief rays, and is defined by equations (62) and (63).
t	s	= the distance between two optical components. It differs from d (q. v.) in that it is a length measured in air.
u		= the axial distance of object point for a telescopic system. It may be measured from any arbitrary origin in the object space on the axis to the projection on the axis of the object point.
u'		= the axial distance of image point for a telescopic system. It must be measured from an origin conjugate to the origin arbitrarily selected in the object space to the projection on the axis of the image point. u or u' is positive if a generating point, when moving in the direction of the incident light, passes through the origin before it arrives at the object point or image point, respectively.

Present notation	Taylor notation	
x	U	=the axial distance of the entrance pupil point. It is the distance from the common vertex of the two surfaces of a thin lens (for a thick lens or lens system x is measured from the first principal point) to the entrance pupil point.
x'	V	=the axial distance of the exit pupil point. It is the distance from the common vertex of the two surfaces of a thin lens (for a thick lens or lens system x' is measured from the second principal point) to the exit pupil point. x or x' is positive if a generating point, when moving in the direction of the incident light, passes through the common vertex of the two surfaces (or principal point in the case of thick lenses or lens systems) before it arrives at the pupil point.
y		=the "lateral distance" of the object point and is the distance from optic axis to the object point.
y'		=the "lateral distance" of the image point and is the distance from optic axis to the image point. When indicated on a diagram, y or y' is positive if measured upward.

Greek characters

Present notation	Taylor notation	
α		=the angle in the object space between a ray and the optic axis.
β	φ	=the angle between the chief ray and the axis in the object space. With the subscript 1 it becomes the angle between the chief ray and the axis in the object space of the optical system. The vertex of this angle will be at the entrance pupil point.
ϵ	β	=the eccentricity factor, a dimensionless parameter applied in the Taylor system of third order equations, which is a measure of the eccentricity of point of incidence of chief ray. It is defined by equation (12).
λ'		=the longer wave length of the two for which chromatic aberration is eliminated in equations (97) and (98). In instruments for visual use it is commonly the wave length of the C spectrum line.
λ		=the intermediate wave length for which the monochromatic aberrations are corrected. In instruments for visual use it is commonly the wave length of the mean of the D spectrum lines. The value 5,460.7, corresponding to the mercury line, may also be adopted for λ.
λ''		=the shorter wave length of the two for which chromatic aberration is eliminated in equations (97) and (98). In instruments for visual use it is commonly the wave length of the F spectrum line.
ν		$=\dfrac{n_\lambda-1}{n_{\lambda''}-n_{\lambda'}}$ If not otherwise qualified it is usually equal to $\dfrac{n_D-1}{n_F-n_C}$
π	α	=the position factor, a dimensionless parameter applied in the Taylor system of third order equations, which is a function of the axial distances of object and image. It is defined by equation (10).
σ	x	=the shape factor, a dimensionless parameter, applied in the Taylor system of third order equations, which indicates the manner in which the power of the lens is divided between the two surfaces. It is defined by equation (8).
φ		=the power of a lens and is the reciprocal of the focal length. For a thin lens it is defined by equation (1).

APPENDIX 2.—THIRD ORDER EQUATIONS AS GIVEN BY SCHWARZSCHILD

The five Seidel sums in the form employed by Schwarzschild[38] are given below without change other than that necessary to make notation and sign convention uniform with body of article.

$$S_{\mathrm{I}}=\frac{1}{2}\Sigma h_1^4\left\{\frac{b_1}{r_1^3}(n'_1-n_1)+Q^2_{s,1}\left(\frac{1}{n'_1s'_1}-\frac{1}{n_1s_1}\right)\right\} \tag{116}$$

$$S_{\mathrm{II}}=\frac{1}{2}\Sigma g^2_1h^2_1\left\{\frac{b_1}{r_1^3}(n'_1-n_1)+Q^2_{x,1}\left(\frac{1}{n'_1s'_1}-\frac{1}{n_1s_1}\right)\right\} \tag{117}$$

$$S_{\mathrm{III}}=\frac{1}{2}\Sigma g^2_1h^2_1\left\{\frac{b_1}{r_1^3}(n'_1-n_1)+Q_{s,1}Q_{x,1}\left(\frac{1}{n'_1s'_1}-\frac{1}{n_1s_1}\right)\right.$$
$$\left.-Q_{s,1}(Q_{s,1}-Q_{x,1})\left(\frac{1}{n'_1x'_1}-\frac{1}{n_1x_1}\right)\right\} \tag{118}$$

$$S_{\mathrm{IV}}=\frac{1}{2}\Sigma g^3_1h_1\left\{\frac{b_1}{r_1^3}(n'_1-n_1)+Q^2_{x,1}\left(\frac{1}{n'_1s'_1}-\frac{1}{n_1s_1}\right)\right.$$
$$\left.-Q_{x,1}(Q_{s,1}-Q_{x1})\left(\frac{1}{n'_1x'_1}-\frac{1}{n_1x_1}\right)\right\} \tag{119}$$

$$S_{\mathrm{V}}=\frac{1}{2}\Sigma g_1h^3_1\left\{\frac{b_1}{r_1^3}(n'_1-n_1)+Q_{s,1}Q_{x,1}\left(\frac{1}{n'_1s'_1}-\frac{1}{n_1s_1}\right)\right\} \tag{120}$$

The summation indicates that one term is to be taken for each surface in the system. If, as in the following, the subscript i is assumed to change from *lens component* to *lens component*, with the second surface of each component denoted by a prime, then i must take on the values

$$i=1,\ 1',\ 2,\ 2',\ 3,\ 3',\ \text{etc.}$$

With this notation a cemented surface—say, between the second and third components—will be considered as two surfaces, 2′ and 3, separated by a layer of air of zero thickness.

The new symbols to be defined are given below:

b = deformation coefficient. The equation of a spherical surface with origin at the vertex, if the development is broken off with the fourth order term is

$$x=\frac{y^2+z^2}{2r}+\frac{(y^2+z^2)^2}{8r^3} \tag{121}$$

where x, y, and z are rectangular coordinates.

$$x=\frac{y^2+z^2}{2r}+\frac{(y^2+z^2)^2}{8r^3}(1+b) \tag{122}$$

[38] Schwarzschild, K., Untersuchungen zur geometrischen Optik. I. Einleitung in die Fehlertheorie optischer Instrumente auf Grund des Eikonalbegriffs. Abh. der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Math-Phys. Kl. Neue Folge. 4, No. 1; 1905.

is the equation of a "deformed" surface of revolution for which b, the "deformation coefficient," is taken as a measure of the departure from sphericity. Only spherical surfaces have been dealt with in this article, for which $b=0$. If $b=-1$ the surface becomes a paraboloid of revolution. If $-1<b<0$, the section of the surface revolution is an ellipse lying between the circle and the parabola; if $b<-1$, the surface is a hyperboloid of revolution. For any value of b the radius of curvature of the surface at the vertex is r.
n_i is the index of refraction of the medium preceding the ith surface. The index following this surface is n_i' or n_{i+1}.
$Q_{s,i}$ is an optical invariant of the surface defined by the equation

$$Q_{s,i}=n_i\left(\frac{1}{r_i}-\frac{1}{s_i}\right)=n_i'\left(\frac{1}{r_i}-\frac{1}{s'_i}\right) \tag{123}$$

$Q_{x,i}$ is similarly defined by the equation

$$Q_{x,i}=n_i\left(\frac{1}{r_i}-\frac{1}{x_i}\right)=n_i'\left(\frac{1}{r_i}-\frac{1}{x_i'}\right) \tag{124}$$

The angular values of the aberrations in the object space are defined by the equations

$$(\text{Ang. Sph.}) \quad =2\, o^3\, S_{\text{I}} \tag{125}$$

$$(\text{Ang. Coma}) \quad =3\, o^2\, S_{\text{v}} \tan\beta_1 \tag{126}$$

$$(\text{Ang. Pri. Curv.})=2\, o(2\, S_{\text{II}}+S_{\text{III}}) \tan^2\beta_1 \tag{127}$$

$$(\text{Ang. Sec. Curv.})=2\, o\, S_{\text{III}} \tan^2\beta_1 \tag{128}$$

$$(\text{Ang. Dist.}) \quad =S_{\text{IV}} \tan^3\beta_1 \tag{129}$$

The third order equations of imagery for a single lens may be obtained by a straightforward but rather tedious elimination between the appropriate aberration equation and the following equations descriptive of a single lens of zero thickness, index of refraction n, in a medium of index 1.

The aberration equation is to be applied to the two surfaces 1 and 1′.

$$\left.\begin{aligned} n_1=n'_1&=1 \\ n_1'=n_{1'}&=n \end{aligned}\right\} \tag{130}$$

$$\left.\begin{aligned} &\frac{1}{r_1}=\frac{1}{r} \\ &\frac{1}{r_1'}=\frac{1}{r'}=\frac{1}{r}-\frac{1}{n-1}\frac{1}{f} \end{aligned}\right\}. \tag{131}$$

$$\left.\begin{aligned} \frac{1}{s_1}&=\frac{1}{s} \\ \frac{1}{s_1{}'}=\frac{1}{s_{1'}}&=\frac{1}{ns}+\frac{n-1}{nr} \\ \frac{1}{s_{1'}}&=\frac{1}{s}+\frac{1}{f} \end{aligned}\right\} \tag{132}$$

$$h_1=h_{1'}=h \tag{133}$$

As an illustrative example the substitution will be made in the equation giving the spherical aberration.

$$Q_{s,1}=\frac{1}{r}-\frac{1}{s} \tag{134}$$

$$Q_{s,1'}=\frac{1}{r}-\frac{1}{s}-\frac{n}{n-1}\ \frac{1}{f} \tag{135}$$

$$\frac{1}{n'_1s'_1}-\frac{1}{n_1s_1}=\frac{1-n^2}{n^2}\ \frac{1}{s}+\frac{n-1}{n^2}\ \frac{1}{r} \tag{136}$$

$$\frac{1}{n'_{1'}s'_{1'}}-\frac{1}{n_{1'}s_{1'}}=-\left(\frac{1-n^2}{n^2}\ \frac{1}{s}+\frac{n-1}{n^2}\ \frac{1}{r}-\frac{1}{f}\right) \tag{137}$$

It follows that

$$\begin{aligned}(\text{Ang. Sph.}) &=o^3h^4\left[\left(\frac{1}{r}-\frac{1}{s}\right)^2\left(\frac{1-n^2}{n^2}\ \frac{1}{s}+\frac{n-1}{n^2}\ \frac{1}{r}\right)-\right. \\ &\left(\frac{1}{r}-\frac{1}{s}-\frac{n}{n-1}\ \frac{1}{f}\right)\left(\frac{1-n^2}{n^2}\ \frac{1}{s}+\frac{n-1}{n^2}\ \frac{1}{r}-\frac{1}{f}\right)\end{aligned} \tag{138}$$

This can be reduced to the expression

$$\begin{aligned}(\text{Ang. Sph.}) &=o^3h^4\left[\left(\frac{n}{n-1}\right)^2\varphi^3+\frac{3n+1}{n-1}\ \frac{\varphi^2}{s}+\frac{3n+2}{n}\ \frac{\varphi}{s^2}-\right. \\ &\left.\frac{2n+1}{n-1}\ \frac{\varphi^2}{r}+\frac{n+2}{n}\ \frac{\varphi}{r^2}-\frac{4(n+1)}{n}\ \frac{\varphi}{rs}\right] \\ &=\frac{o^3}{4}h^4\varphi^3A. \quad \text{(See equations (21) and (24).)}\end{aligned} \tag{139}$$

If, at any stage of the substitution $\frac{1}{r}$ and $\frac{1}{s}$ had been eliminated by introducing their equivalent expressed in terms of π and σ (equations (9) and (11)), the value of A would have been similarly obtained for the Taylor system of notation. The generalization by which equation (79), applicable to a system of thin lenses, is obtained is evident. Also, in a manner similar to that employed above, one can obtain the equation for any aberration of a plane parallel plate of thickness d and with surfaces of zero curvature.

APPENDIX 3.—SPECIFICATIONS FOR REFLECTING PRISMS

The following 20 plates were prepared by Otto Kaspereit, optical designer, Frankford Arsenal, and are given in Elementary Optics and Applications to Fire Control Instruments, Document No. 1065, revised edition, January, 1924, Ordnance Department, United States Army, and are here republished through permission kindly granted by the War Department.

Drawings are given of substantially all types of reflecting prisms in general use. For each prism A_p is the diameter of a cylindrical beam of light traveling along the "optic axis" which is wholly transmitted by the prism. Literal expressions for each dimension in terms of A_p and numerical values of each dimension for $A_p=1$ are given. The length d_p is the thickness of the thick plane parallel plate equivalent to the prism of unit aperture. If d_p is multiplied by diameter of prism aperture—that is, A_p—and divided by the index of refraction, one obtains the equivalent path length in air.

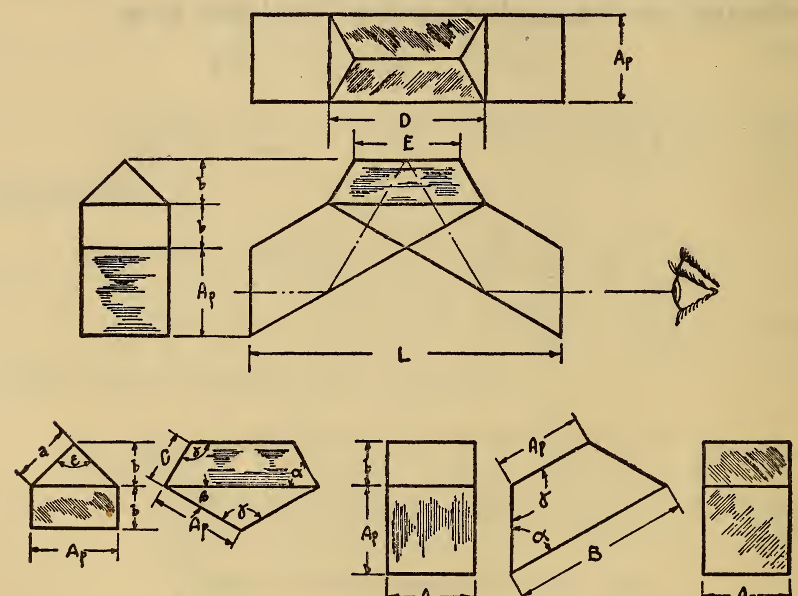
DESIGNING OF PRISMS AND PRISM-SYSTEMS

A_p = CLEAR APERTURE OF PRISM = 1.00 IN ALL EXAMPLES.
d_p = LENGTH OF PATH OF RAY THROUGH PRISM (GLASS).
d_f = LENGTH OF PATH OF RAY IN PRISM TO IMAGE PLANE. (RANGE-FINDER HALVING PRISMS)
n_p = REFRACTIVE INDEX OF PRISM GLASS = 1.5165 FOR ALL EXAMPLES.
u_p = PARALLEL DISPLACEMENT OF INTERSECTION DISTANCE CAUSED BY PRISM.
u_f = PARALLEL DISPLACEMENT OF INTERSECTION DISTANCE (IN IMAGE-PLANE) CAUSED BY PRISM.

INVERTING PRISM-SYSTEM NO. 1 BY ABBE

THIS PRISM IS MADE OUT OF THREE PIECES WHICH ARE CEMENTED TOGETHER WITH CANADA BALSAM. THE PRISM MADE BY ZEISS CONSISTED OF ONLY TWO PIECES, TWO OF THE THREE PARTS SHOWN BELOW WERE COMBINED INTO ONE PIECE AND WERE NOT CEMENTED.

THIS PRISM INVERTS THE IMAGE COMPLETELY BUT NEITHER DEVIATES NOR DISPLACES THE AXIS.



SIZE OF PRISMS

$\alpha = 60°$ $\beta = 30°$ $\gamma = 120°$ $\varepsilon = 90°$

$B = 2A_p = 2.00$ $C = \frac{A_p}{2\cos\beta} = 0.5774$ $D = A_p \text{ COTAN}\beta = 1.732$

$E = \frac{A_p}{\cos\beta} = 1.1547$ $L = 2A_p \text{ COTAN}\beta = 3.464$ $a = \frac{A_p}{2 \sin 45°} = .7071$

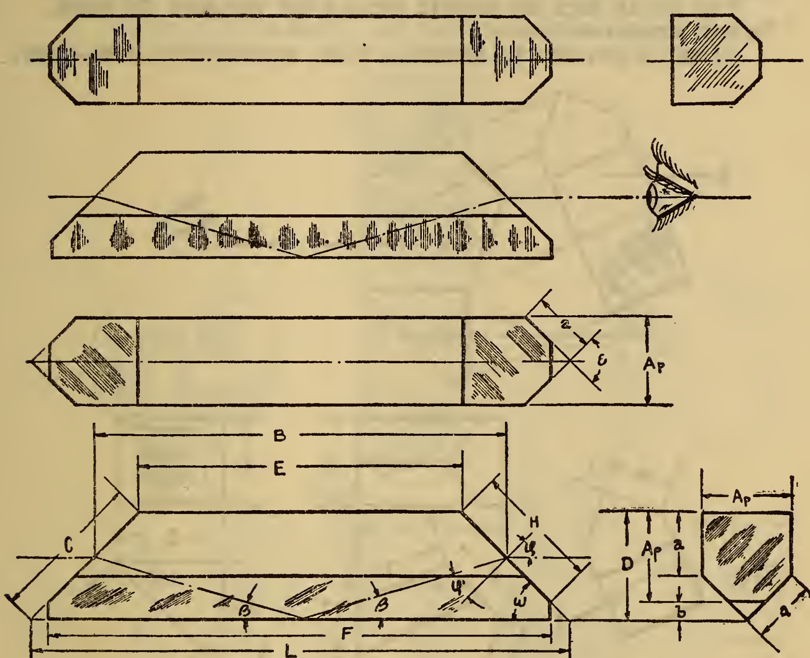
$b = \frac{A_p}{2} = 0.50$ $d_p = 3A_p \text{ COTAN}\beta = 5.196$ $u_p = \frac{(n_p - 1) d_p}{n_p} = 1.7697$

FIG. 36.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

INVERTING PRISM BY WIRTH

THIS PRISM IS MADE OUT OF ONE PIECE OF GLASS. IT INVERTS THE IMAGE COMPLETELY BUT NEITHER DEVIATES NOR DISPLACES THE AXIS. IT IS RARELY USED ON ACCOUNT OF ITS GREAT LENGTH.



SIZE OF PRISM

$\varphi = 45°$ $\quad \varepsilon = 90°$ $\quad \omega = 45°$

$\varphi' = \sin^{-1} \dfrac{\sin\varphi}{n_p} = 27°47'34''.09$ $\qquad \beta = \varphi - \varphi' = 17°12'25''.91$

$a = A_p \sin\omega = 0.7071$ $\qquad b = a - \dfrac{A_p}{2} = 0.2071$

$B = \dfrac{2A_p \sin\omega}{\tan\beta} = 4.5665$ $\qquad C = A_p + a = 1.7071$

$D = a + \dfrac{A_p}{2} = 1.2071$ $\qquad E = B - A_p = 3.5665$

$F = A_p + B = 5.5665$ $\qquad H = \dfrac{A_p}{\sin\omega} = 1.4141$

$d_p = \dfrac{2A_p \sin\omega}{\sin\beta} = 4.7805$ $\qquad u_p = \dfrac{(n_p - 1)a_p}{n_p} = 1.6282$

$L = A_p + B + 2b = 5.9807$

FIG. 37.

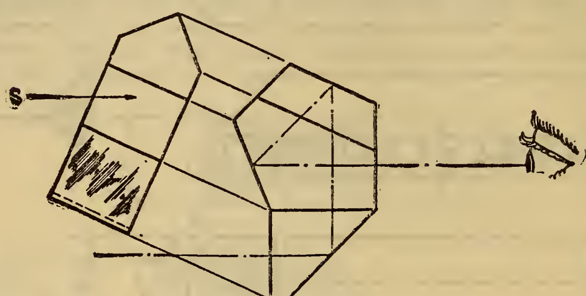
DESIGNING OF PRISMS AND PRISM-SYSTEMS

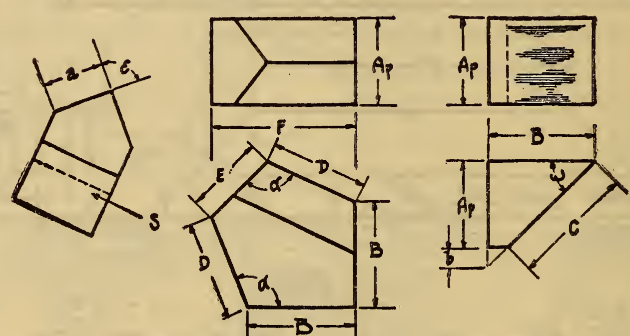
INVERTING SYSTEM BY HENSOLT

THIS SYSTEM CONSISTS OF TWO PRISMS ONE OF WHICH IS A RIGHT ANGLE PRISM. THIS RIGHT ANGLE PRISM IS CEMENTED TO A "PENTA-PRISM." ONE REFLECTING SURFACE OF THIS PRISM IS GROUND TO THE SHAPE OF A ROOF OR AT 90°. THIS SYSTEM COMPLETELY INVERTS THE IMAGE.

THIS SYSTEM DOES NOT DEVIATE THE AXIS BUT DISPLACES THE SAME TO THE AMOUNT OF "B".

HENSOLT & SONS USE THIS SYSTEM IN THEIR BINOCULAR FIELD-GLASSES.





SIZE OF PRISMS

$\alpha = 112°30'$ $\quad \varepsilon = 90°$ $\quad \omega = 45°$

$a = A_p \sin\omega = 0.7071$ $\quad b = a - \frac{A_p}{2} = 0.2071$ $\quad B = a + \frac{A_p}{2} = 1.2071$

$C = \frac{A_p}{\sin\omega} = 1.4142$ $\quad \beta = \alpha - \varepsilon = 22°30'$ $\quad D = \frac{A_p}{\cos\beta} = 1.0824$

$E = \frac{A_p \tan\beta}{\sin\omega} + (A_p - a) = 0.8787$ $\quad F = B + A_p \tan\beta = 1.6213$

$d_p = 4A_p \sin\omega + 2.5A_p = 5.3284$ $\quad u_p = \frac{(n_p - 1)d_p}{n_p} = 1.8148$

SILVER THE REFLECTING SURFACE MARKED "S".

FIG. 38.

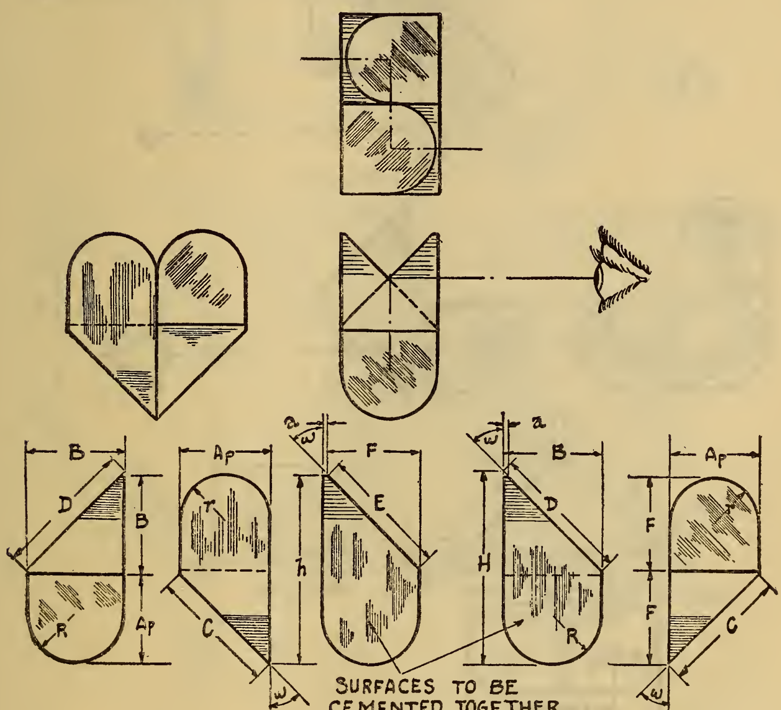
DESIGNING OF PRISMS AND PRISM-SYSTEMS

INVERTING PRISM-SYSTEM NO. 2 BY ABBE

THIS SYSTEM CONSISTS OF TWO PRISMS OF LIKE CONSTRUCTION CEMENTED TOGETHER. THE SHARP CORNERS ARE REMOVED BY MAKING THE ENDS OF PRISMS ROUND WHICH WILL REDUCE THE WEIGHT AND ALSO LESSEN THE CHANCE OF BREAKING.

THIS SYSTEM INVERTS THE IMAGE COMPLETELY; IT DOES NOT DEVIATE THE AXIS BUT DISPLACES THE SAME TO THE AMOUNT OF A_p.

SCHÜTZ & CO. IN KASSEL USE THIS SYSTEM IN THEIR BINOCULAR FIELD-GLASSES.



SIZE OF PRISMS

$a = 0.05$ $\omega = 45°$

$B = A_p + 2a = 1.10$ $C = \frac{A_p}{\sin\omega} = 1.4142$ $D = \frac{B}{\sin\omega} = 1.5556$

$E = \frac{A_p + a}{\sin\omega} = 1.4849$ $F = A_p + a = 1.05$ $h = 2F = 2.10$

$H = a + h = 2.15$ $r = \frac{A_p}{2} = 0.50$ $R = \frac{B}{2} = 0.55$

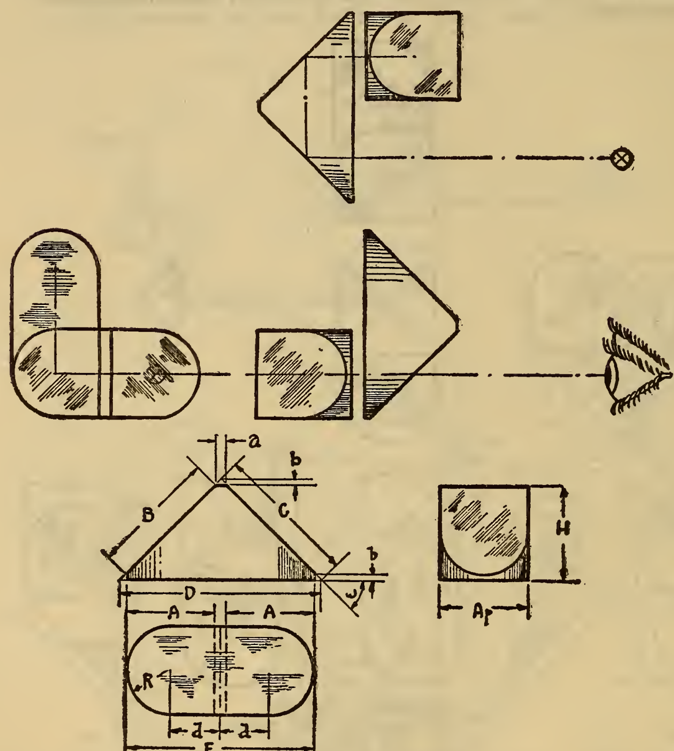
$d_p = 2H = 4.30$ $U_p = \frac{(n_p - 1)\, d_p}{n_p} = 1.4645$

FIG. 39.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

INVERTING SYSTEM BY PORRO

THE PORRO PRISM SYSTEM CONSISTS OF TWO PRISMS OF LIKE CONSTRUCTION. IT IS ONE OF THE OLDEST ERECTING SYSTEMS EVER USED. THERE ARE NUMEROUS MODIFICATIONS OF THIS TYPE OF WHICH ABBE'S DESIGN (SHOWN ON DRWG. 8-10178) IS PROBABLY THE BEST. THE PORRO SYSTEM INVERTS THE IMAGE COMPLETELY. IT DOES NOT DEVIATE THE AXIS BUT DISPLACES THE SAME UP (OR DOWN) AND AT THE SAME TIME TO THE RIGHT (OR LEFT) TO THE AMOUNT OF $(A_p + a)$.



SIZE OF PRISMS

$a = 0.125$ $b = \frac{a}{2} = 0.0625$ $C = \frac{A_p + a}{\sin\omega} = 1.591$ $\omega = 45°$

$B = \frac{A_p}{\sin\omega} = 1.414$ $D = 2(A_p + a) = 2.25$ $E = 2A_p + a = 2.125$

$F = A_p + a = 1.125$ $H = A_p + b = 1.0625$ $R = \frac{A_p}{2} = 0.50$

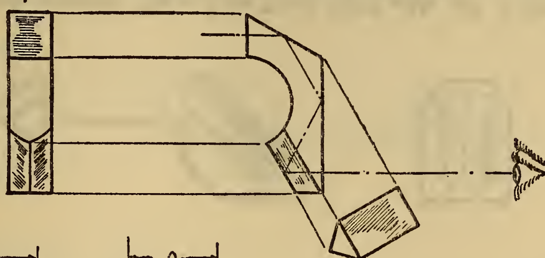
$d = \frac{A - a}{2} = 0.5625$ $d_p = 4(A_p + a) = 4.50$ $U_p = \frac{(n_p - 1)d_p}{n_p} = 1.5325$

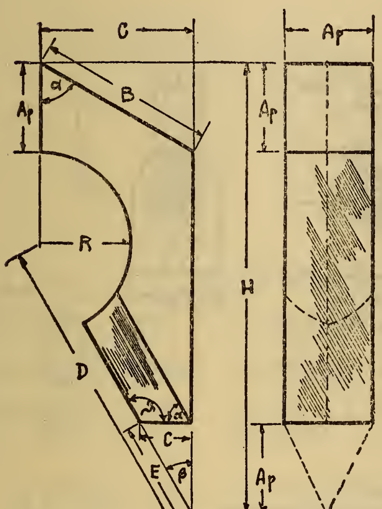
FIG. 40.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

INVERTING PRISM BY LEMAN

THIS PRISM CONSISTS OF ONE PIECE OF GLASS AND TRANSMITS MORE LIGHT THAN ANY OF THE OTHER PRISMS ALREADY DESCRIBED. IT INVERTS THE IMAGE COMPLETELY AND DOES NOT DEVIATE THE AXIS BUT DISPLACES THE SAME TO THE AMOUNT OF $3A_p$.





SIZE OF PRISM

$\alpha = 60^\circ$ $\beta = 30^\circ$ $\vartheta = 120^\circ$

$\varepsilon = 90^\circ$ $\omega = 45^\circ$

$C = A_p \text{ COTAN } \beta = 1.732$

$D = \dfrac{3A_p}{\text{COS } \beta} = 3.464$

$E = \dfrac{A_p}{\text{COS } \beta} = 1.155$

$H = 5A_p = 5.00$

$R = A_p = 1.00$

$a = \dfrac{A_p}{2 \text{ SIN } \omega} = 0.707$

$b = \dfrac{A_p}{\text{SIN } \beta} - \dfrac{A_p}{2} = 1.50$

$c = A_p \text{ TAN } \beta = 0.577$

$d_p - 3A_p \text{ COTAN } \beta = 5.196$

$u_p = \dfrac{(D^{n_p} - 1) d_p}{D^{n_p}} = 1.770$

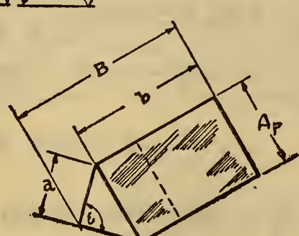
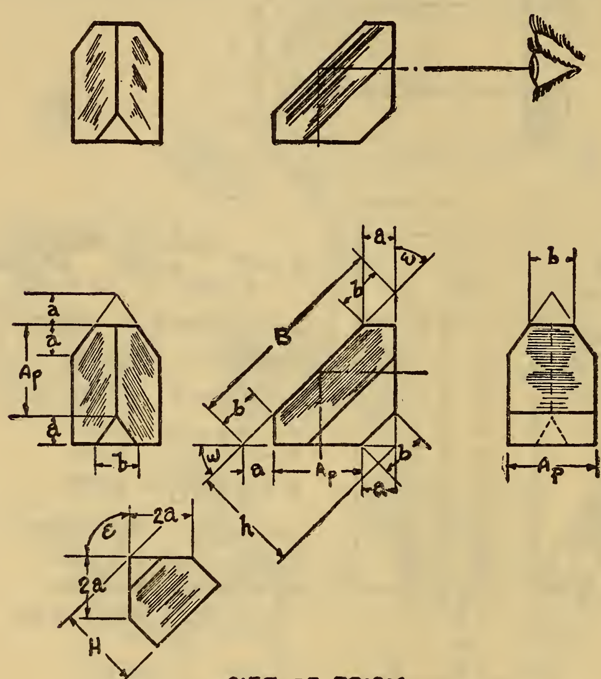


FIG. 41.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

INVERTING PRISM BY AMICI

THIS PRISM CONSISTS OF ONE PIECE OF GLASS AND TRANSMITS EVEN MORE LIGHT THAN THE INVERTING PRISM BY LEMAN. IT IS BETTER KNOWN BY THE NAME OF "ROOF-ANGLE" PRISM ON ACCOUNT OF ITS SHAPE. IT DEVIATES THE AXIS THROUGH AN ANGLE OF 90° AND INVERTS THE IMAGE COMPLETELY.



SIZE OF PRISM

$\varepsilon = 90°$ $\omega = 45°$

$a = \frac{A_p}{4 \sin \omega} = 0.354$ $b = \frac{A_p}{2} = 0.50$ $B = \frac{A_p}{\sin \omega} + A_p = 2.414$

$h = \frac{B}{2} = 1.207$ $H = h - \frac{b}{2} = 0.957$ $d_p = A_p \sin \omega + A_p = 1.707$

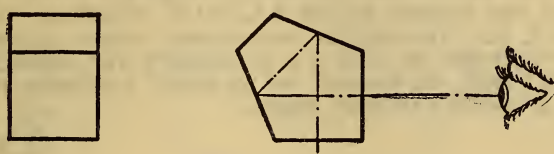
$$u_p = \frac{(n_p - 1) d_p}{n_p} = 0.581$$

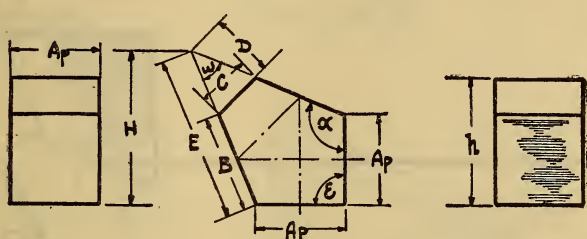
FIG. 42.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE "PENTAGON" PRISM BY PRANDL

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DEVIATES THE AXIS THROUGH AN ANGLE OF 90° BUT DOES NOT CHANGE THE IMAGE. THE IMAGE REMAINS STATIONARY WHEN TURNING THE PRISM HENCE THE PRISM IS CALLED THE "OPTICAL-SQUARE". THE TWO REFLECTING SURFACES OF THIS PRISM HAVE TO BE SILVERED.





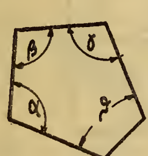
SIZE OF PRISM

$\alpha = 112°30'$ $\beta = 22°30'$ $\varepsilon = 90°$ $\omega = 45°$

$B = \dfrac{A_P}{\cos\beta} = 1.082$ $C = \dfrac{A_P \tan\beta}{\sin\omega} = 0.586$ $D = \dfrac{A_P}{2\sin\omega} = 0.707$

$E = \dfrac{A_P \sin\omega}{\sin\beta} = 1.848$ $H = \dfrac{A_P \sin\omega}{\tan\beta} = 1.707$ $L = \dfrac{A_P}{\tan\beta} = 2.414$

$h = \dfrac{A_P}{\sin\omega} = 1.414$ $d_P = \dfrac{A_P}{\sin\omega} + 2A_P = 3.414$ $u_P = \dfrac{(n_P - 1)d_P}{n_P} = 1.163$



IF THIS PRISM IS TO BE USED IN CONNECTION WITH "WEDGE PRISMS" IN RANGE-FINDERS THEN THE ANGLES WILL HAVE TO BE CHANGED TO A SMALL AMOUNT. FOR INSTANCE, A RANGE-FINDER PRISM, MADE BY BAUSCH & LOMB-ZEISS, HAD THE FOLLOWING ANGLES:

$$\left.\begin{aligned} \alpha &= 112°\,23' \pm 4' \\ \beta &= 90°\,9'\,30'' \pm 2' \\ \gamma &= 112°\,22'\,45'' \pm 4' \\ \delta &= 45°\,4'\,45'' \pm 30' \end{aligned}\right\}$$

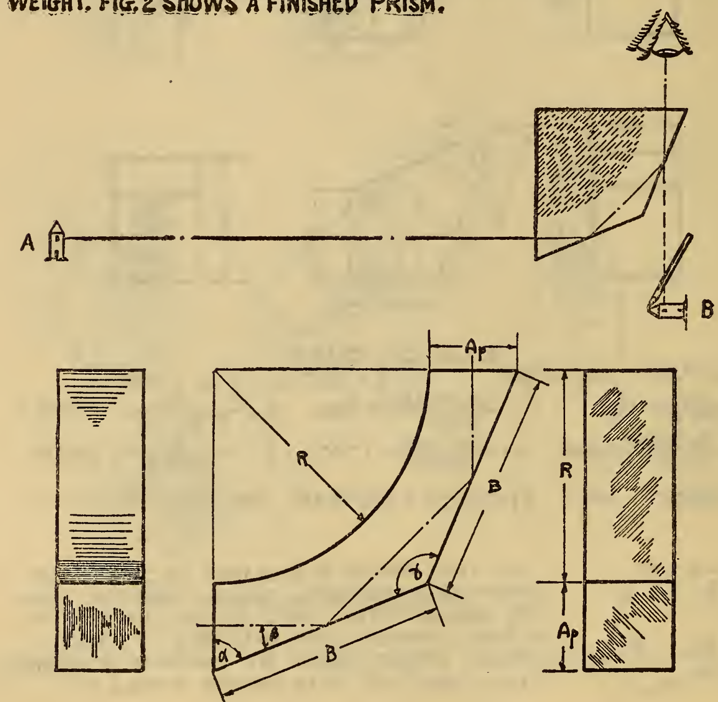
FIG. 43.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

CAMERA LUCIDA OR CAMERA CLARA

THIS CAMERA CONSISTS OF ONLY ONE PRISM AS SHOWN IN SKETCH BELOW. WHEN LOOKING THROUGH THIS PRISM AT AN OBJECT "A" THE IMAGE "B" WILL BE ERECT AND APPEAR IN FRONT OF THE OBSERVER. IF IT IS THE DESIRE OF THE OBSERVER TO MAKE A PENCIL SKETCH OF THIS OBJECT HE WILL PLACE A PIECE OF WHITE PAPER WHERE IMAGE "B" APPEARS. IN ORDER THAT HE MAY SEE THE IMAGE AND PENCIL SIMULTANEOUSLY HE WILL HAVE TO MOVE THE EYE A LITTLE OVER THE EDGE OF THE PRISM; AFTER A LITTLE PRACTICE THE OBSERVER WILL SEE THE OBJECT AND PENCIL VERY SHARP.

IF THIS PRISM IS TO BE USED IN A TELESCOPE THE SHADED PORTION IN FIG. I WILL BE GROUND OFF IN ORDER TO REDUCE THE WEIGHT. FIG. 2 SHOWS A FINISHED PRISM.



$\alpha = 67°30'$ $\quad$ $\beta = \frac{\alpha}{3} = 22°30'$ $\quad$ $\gamma = 2\alpha = 135°$

$B = \frac{A_p}{\sin\beta} = 2.613$ $\quad$ $R = A_p \operatorname{cotan}\beta = 2.414$ $\quad$ $d_p = 2A_p \operatorname{cotan}\beta = 4.828$

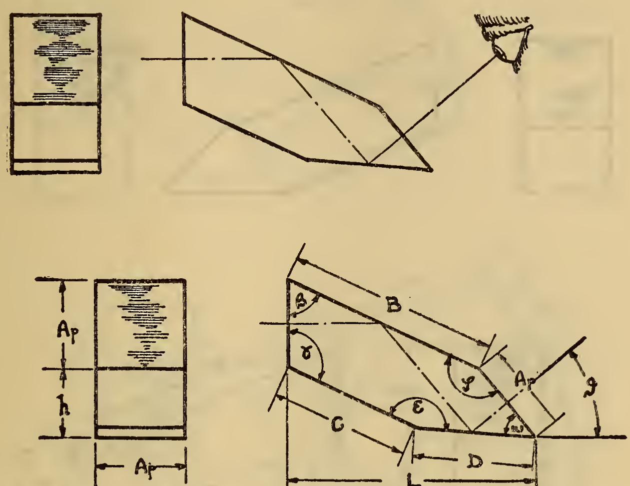
$$u_p = \frac{(n_p - 1)d_p}{n_p} = 1.645$$

FIG. 44.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-40."

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 40°.



SIZE OF PRISM

$\alpha = 25°$ $\gamma = 115°$ $\varepsilon = 160°$ $\eta = 5°$

$\beta = 65°$ $\vartheta = 40°$ $\varphi = 155°$ $\omega = 45°$

$B = \dfrac{A_p}{\sin\alpha} = 2.366$ $C = \dfrac{A_p \sin\vartheta}{\sin\alpha} = 1.521$ $D = \dfrac{A_p}{\sin\omega} = 1.414$

$h = A_p \sin(\eta + \omega) = 0.766$ $L = \dfrac{A_p}{\tan\alpha} + A_p \sin\vartheta = 2.787$

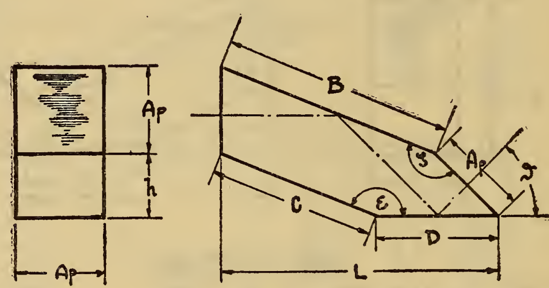
$d_p = \dfrac{A_p}{\tan\alpha} + A_p = 3.145$ $u_p = \dfrac{(n_p - 1)d_p}{n_p} = 1.071$

FIG. 45.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-45-a."

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 45°.



SIZE OF PRISM

$\alpha = 67°\,30'$ $\quad \gamma = 112°\,30'$ $\quad \varepsilon = 157°\,30'$

$\beta = 22°\,30'$ $\quad \vartheta = 45°$ $\quad \delta = 157°\,30'$ $\quad \omega = 45°$

$B = \frac{A_p}{\sin\beta} = 2.613$ $\quad C = \frac{A_p \sin\omega}{\sin\beta} = 1.848$ $\quad D = \frac{A_p}{\sin\omega} = 1.414$

$h = A_p \sin\omega = 0.707$ $\quad L = D - 2A_p \cos^2\beta = 3.121$

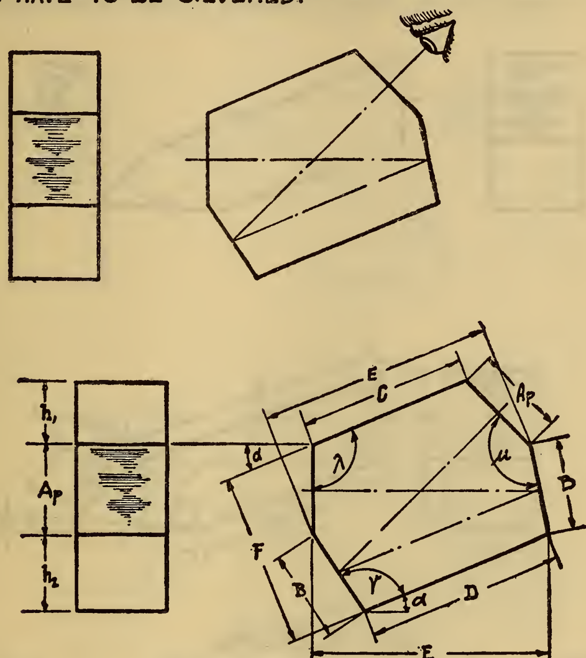
$d_p = A_p + A_p \operatorname{cotan}\beta = 3.414$ $\quad a_p = \frac{(n_p - 1)\, d_p}{n_p} = 1.163$

FIG. 46.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-45-b".

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 45°. THE TWO REFLECTING SURFACES WILL HAVE TO BE SILVERED.



SIZE OF PRISM

$\alpha = 22°30'$ $\beta = 33°45'$ $\gamma = \alpha + \beta = 56°15'$ $\vartheta = 2\alpha = 45°$

$\varepsilon = \vartheta - \beta = 11°15'$ $\lambda = 112°30'$ $\mu = 146°15'$ $\nu = 101°15'$

$B = \dfrac{A_p}{\cos\varepsilon} = 1.0196$ $C = \dfrac{A_p \sin\vartheta}{\sin\alpha} = 1.848$ $D = \dfrac{A_p \sin\gamma}{\sin\alpha\cos\varepsilon} = 2.215$

$E = \dfrac{A_p}{\sin\alpha} = 2.613$ $F = A_p + A_p\cos\alpha = 1.924$ $h_1 = A_p \sin\vartheta = 0.707$

$h_2 = \dfrac{A_p \sin\gamma}{\cos\varepsilon} = 0.848$ $d_p = \dfrac{A_p}{\sin\alpha} + 2A_p \operatorname{cotan}\alpha = 7.442$

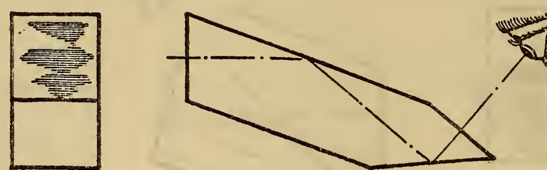
$$u_p = \frac{(n_p - 1)d_p}{n_p} = 2.534$$

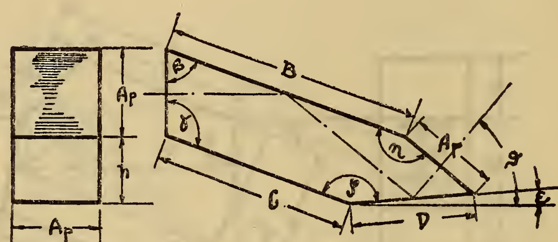
FIG. 47.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-50"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 50.°





SIZE OF PRISM

$\alpha = 20^\circ$ $\beta = 70^\circ$ $\gamma = 110^\circ$ $\vartheta = 50^\circ$

$\delta = 155^\circ$ $\eta = 160^\circ$ $\omega = 45^\circ$ $\varepsilon = 5^\circ$

$B = \frac{A_P}{\sin\alpha} = 2.924$ $C = B - 2A_P \sin\alpha = 2.240$ $D = \frac{A_P}{\sin\omega} = 1.414$

$h = C \sin\alpha = 0.766$ $d_P = A_P + \frac{A_P}{\tan\alpha} = 3.748$

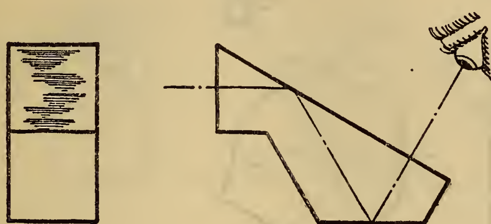
$$u_P = \frac{(n_P - 1)d_P}{n_P} = 1.276$$

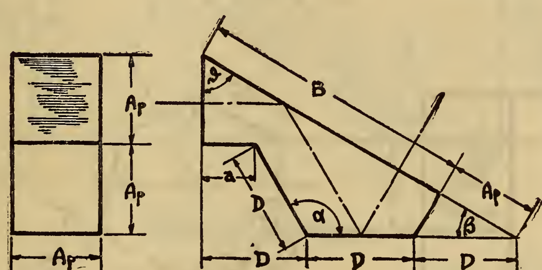
FIG. 48.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-60-a"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 60°. THE LOWER REFLECTING SURFACE WILL HAVE TO BE SILVERED.





SIZE OF PRISM

$\alpha = 120°$ $\beta = 30°$ $\vartheta = 60°$

$a = A_p \tan\beta = 0.577$ $B = 3A_p = 3.00$ $a_p = C = 3.464$

$$C = \frac{2A_p}{\tan\beta} = 3.464 \qquad D = \frac{A_p}{\cos\beta} = 1.155$$

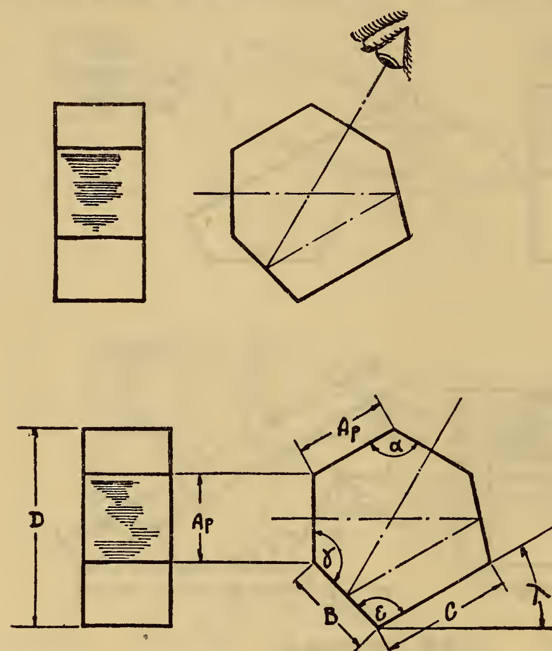
$$u_p = \frac{(n_p - 1)a_p}{n_p} = 1.180$$

FIG. 49.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-60-b"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 60°. THE TWO REFLECTING SURFACES WILL HAVE TO BE SILVERED.



SIZE OF PRISM

$\alpha = 120°$ $\beta = 15°$ $\vartheta = 60°$

$\gamma = 135°$ $\varepsilon = 105°$ $\omega = 45°$ $\lambda = 30°$

$B = \frac{A_p}{\cos\beta} = 1.035$ $C = \frac{2A_p \sin\omega}{\cos\beta} = 1.464$ $D = \frac{3A_p}{2} + \frac{A_p \sin\omega}{\cos\beta} = 2.232$

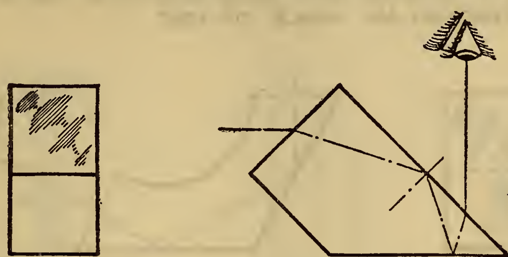
$d_p = \frac{2A_p \sin\omega}{\sin\beta} = 5.464$ $u_p = \frac{(n_p - 1)d_p}{n_p} = 1.861$

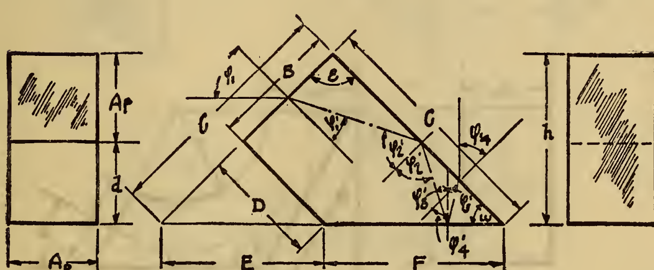
FIG. 50.

DESIGNING OF PRISMS AND PRISM-SYSTEMS.

THE REFLECTING PRISM "D-90."

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 90.° ITS LOWER REFLECTING SURFACE MUST BE SILVERED.





SIZE OF PRISM

$\varepsilon = 90°$ $\qquad \omega = 45°$ $\qquad \varphi_1 = 45°$

$\varphi_1' = \text{SIN}^{-1} \dfrac{\text{SIN}\,\varphi_1}{n_p} = +27°47'34''.09 \qquad \varphi_2 = -\varphi_2' = \varphi_1' - \varepsilon = 62°12'25''.91$

$\varphi_3 = -\varphi_3' = \varphi_2' - \omega = +17°12'25''.91 \qquad \varphi_4 = (180° + \varphi_3') - (\varepsilon + \omega) = +27°47'34.09$

$\varphi_4' = \text{SIN}^{-1}\, n_p\, \text{SIN}\,\varphi_4 = +45° \qquad \vartheta$ = TOTAL DEVIATION

$\vartheta = (\varphi_1 - \varphi_1') + (\varphi_2 - \varphi_2') + (\varphi_3 - \varphi_3') + (\varphi_4 - \varphi_4') = (\varphi_1 - \varphi_4') - 2(\varepsilon + \omega) + 180° = -90°$

$B = \dfrac{A_p}{\text{SIN}\,\omega} = 1.414 \qquad C = \dfrac{A_p}{\text{TAN}\,\varphi_1'\,\text{SIN}\,\omega} = 2.683 \qquad D = C - B = 1.269$

$E = \dfrac{D}{\text{SIN}\,\omega} = 1.795 \qquad F = \dfrac{A_p}{\text{SIN}^2\,\omega} = 2.00 \qquad h = \dfrac{E}{2} = 1.897$

$d = D\,\text{SIN}\,\omega = 0.897 \qquad d_p = \dfrac{d}{\text{SIN}(\omega - \varphi_1')} = 3.033$

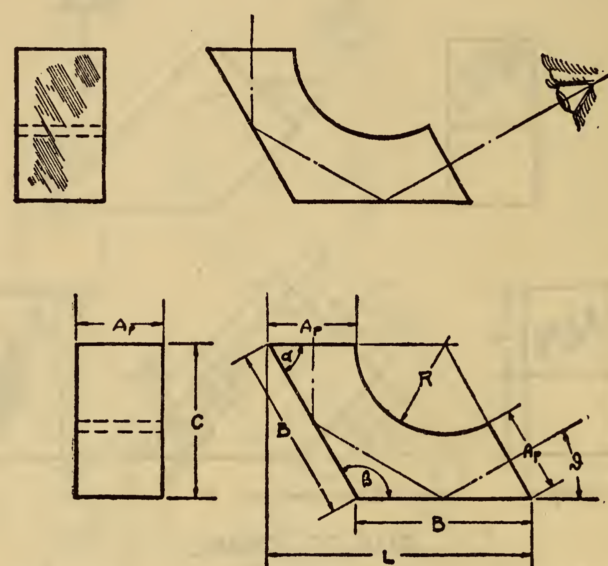
$u_p = \dfrac{(n_p - 1)\,d_p}{n_p} = 1.033$

FIG. 51.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE REFLECTING PRISM "D-120"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE BUT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 120°.



SIZE OF PRISM

$\alpha = 60°$ $\beta = 120°$ $\vartheta = 30°$

$B = \frac{A_p}{\cos\alpha} = 2A_p = 2.00$ $C = A_p \tan\alpha = 1.732$

$L = 3A_p = 3.00$ $R = A_p = 1.00$

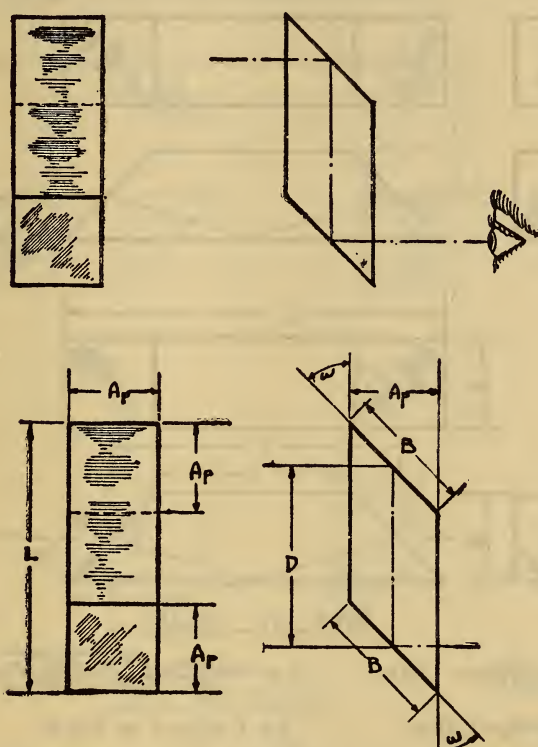
$d_p = 2A_p \tan\alpha = 3.464$ $v_p = \frac{(n_p - 1)\, d_p}{n_p} = 1.180$

Fig. 52.

DESIGNING OF PRISMS AND PRISM-SYSTEMS.

THE "RHOMBOID PRISM"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT DOES NOT CHANGE THE IMAGE NOR DEVIATE THE AXIS AND IS ONLY USED IN INSTRUMENTS WHERE IT IS FOUND NECESSARY TO DISPLACE THE AXIS.



SIZE OF PRISM.

$\omega = 45°$ D = 2.00 (ASSUMED). $B = \frac{A_P}{\sin\omega} = 1.414$

$L = A_p + D = 3.00$ $d_p = L = 3.00$

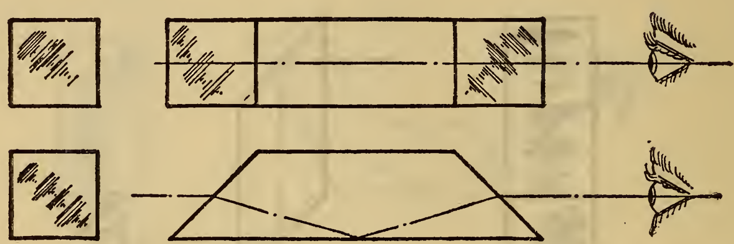
$$u_p = \frac{(n_p - 1)\,d_p}{n_p} = 1.022$$

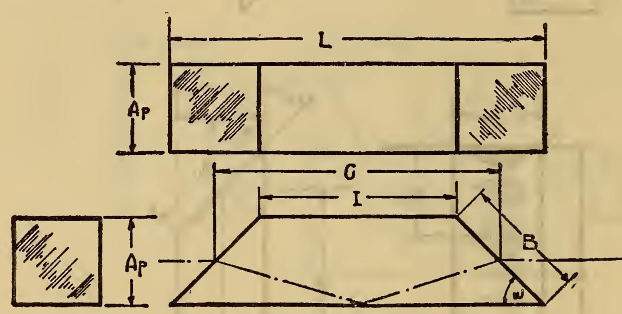
FIG. 53.

DESIGNING OF PRISMS AND PRISM-SYSTEMS

THE "DOVE REFLECTING PRISM"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT WILL NEITHER DEVIATE NOR DISPLACE THE AXIS. THE IMAGE WILL APPEAR ERECT BUT THE SIDES REVERSED WHEN PRISM IS HELD AS SHOWN IN FIG. 1 AND THE IMAGE WILL APPEAR INVERTED BUT THE SIDES IN THEIR CORRECT POSITIONS WHEN PRISM IS HELD AS SHOWN IN FIG. 2





SIZE OF PRISM

$$B = \frac{A_p}{\sin\omega} = 1.414 \qquad C = \frac{A_p[\sin\omega + \sqrt{n_p^2 - \sin^2\omega}]}{\sqrt{n_p^2 - \sin^2\omega} - \sin\omega} = 3.229$$

$$I = C - A_p = 2.229 \qquad L = C + A_p = 4.229$$

$$d_p = \frac{n_p A_p}{\sin\omega[\sqrt{n_p^2 - \sin^2\omega} - \sin\omega]} = 3.380 \qquad u_p = \frac{(n_p - 1)d_p}{n_p} = 1.151$$

THE FOLLOWING TABLE GIVES THE DIMENSIONS OF PRISMS MADE OUT OF DIFFERENT GLASSES. IN EACH CASE THE APERTURE OF THE PRISM $A_p = 1.00$ AND, THEREFORE $B = 1.414$

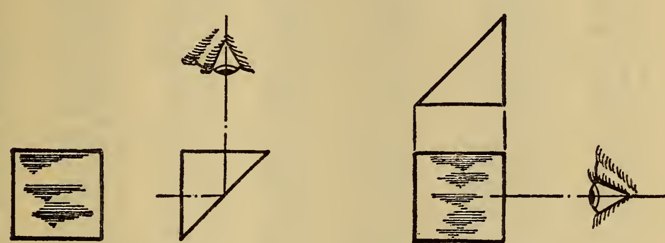
$n_p =$	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
C =	4.685	3.822	3.297	2.942	2.686	2.492	2.339	2.215
I =	3.685	2.822	2.297	1.942	1.686	1.492	1.339	1.215
L =	5.685	4.822	4.297	3.942	3.686	3.492	3.339	3.215

FIG. 54.

DESIGNING OF PRISMS AND PRISM-SYSTEMS.

THE "RIGHT ANGLE REFLECTING PRISM"

THIS PRISM CONSISTS OF ONE PIECE OF GLASS. IT WILL DEVIATE THE AXIS THROUGH AN ANGLE OF 90° AND THE IMAGE WILL BE INVERTED, BUT SIDES WILL REMAIN IN THEIR CORRECT POSITIONS, WHEN PRISM IS HELD AS SHOWN IN FIG. 1; THE IMAGE WILL APPEAR ERECT BUT THE SIDES REVERSED, WHEN PRISM IS HELD AS SHOWN IN FIG. 2.



SIZE OF PRISM.

$\omega = 45°$

$$B = \frac{A_p}{\sin \omega} = 1.414 \qquad u_p = \frac{(n_p - 1)\, d_p}{n_p} = 0.341$$

$$d_p = A_p = 1.00$$

FIG. 55.

APPENDIX 4.—FUNCTIONS OF INDEX OF REFRACTION REQUIRED IN FORMING ABERRATION COEFFICIENTS

The following table, computed by H. U. Graham, gives the values of the several functions of n which enter into the aberration coefficients for values of n differing by 0.001 and extending from $n=1.4$ to $n=1.75$. On each page, at the foot of each column, there will be found tables of proportional parts to facilitate interpolation. Values for which the first significant figure is 1 or 2 are given to 5 significant figures. Other values are, in general, given to 4 significant figures. This accuracy will be found sufficient for substantially all computations with third order equations.

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.400	15.179	82	17.143	48	4.429	1	12.250	44	4.2857	119	0.7143	5
.401	.097	82	.095	48	.428	1	.206	43	.2738	119	38	5
.402	.015	80	.047	47	.427	1	.163	43	.2619	119	33	5
.403	14.935	80	.000	47	.426	1	.120	43	.2500	118	28	5
.404	.855	80	16.953	47	.425	2	.077	42	.2382	117	23	6
1.405	14.775	79	16.906	47	4.423	1	12.035	42	4.2265	116	.7117	5
.406	.696	78	.859	46	.422	1	11.993	42	.2149	116	12	5
.407	.618	78	.813	46	.421	1	.951	42	.2033	116	07	5
.408	.540	77	.767	46	.420	1	.909	41	.1917	114	02	5
.409	.463	76	.721	46	.419	1	.868	41	.1803	115	.7097	5
1.410	14.387	76	16.675	45	4.418	1	11.827	41	4.1688	113	.7092	5
.411	.311	75	.630	45	.417	1	.786	40	.1575	113	87	5
.412	.236	75	.585	45	.416	1	.746	41	.1462	113	82	5
.413	.161	74	.540	45	.415	1	.705	40	.1349	112	77	5
.414	.087	74	.495	45	.414	1	.665	39	.1237	111	72	5
1.415	14.013	73	16.450	44	4.413	1	11.626	40	4.1126	111	.7067	5
.416	13.940	72	.406	44	.412	1	.586	39	.1015	111	62	5
.417	.868	72	.362	44	.411	1	.547	39	.0904	109	57	5
.418	.796	72	.318	44	.410	1	.508	39	.0795	110	52	5
.419	.724	71	.274	43	.409	1	.469	38	.0685	108	47	5
1.420	13.653	70	16.231	44	4.408	1	11.431	38	4.0577	108	.7042	5
.426	.583	70	.187	43	.407	1	.393	38	.0469	108	37	5
.422	.513	69	.144	42	.406	1	.355	38	.0361	107	32	5
.423	.444	69	.102	43	.405	1	.317	38	.0254	107	27	5
.424	.375	68	.059	42	.404	0	.279	37	.0147	106	22	4
1.425	13.307	68	16.017	43	4.404	1	11.242	37	4.0041	105	.7018	5
.426	.239	68	15.974	42	.403	1	.205	37	3.9936	105	.13	5
.427	.171	66	.932	42	.402	1	.168	36	.9831	105	08	5
.428	.105	67	.890	41	.401	1	.132	36	.9726	104	03	5
.429	.038	66	.849	42	.400.	1	.096	37	.9622	103	.6998	5
1.430	12.972	65	15.807	41	4.399	1	11.059	35	3.9519	103	.6993	5
.431	.907	65	.766	41	.398	1	.024	36	.9416	103	88	5
.432	.842	64	.725	41	.397	1	10.988	35	.9313	102	83	5
.433	.778	64	.684	40	.396	1	.953	36	.9211	102	78	4
.434	.714	64	.644	41	.395	1	.917	35	.9109	101	74	5
1.435	12.650	63	15.603	40	4.394	1	10.882	34	3.9008	100	.6969	5
.436	.587	62	.563	40	.393	1	.848	35	.8908	100	64	5
.437	.525	63	.523	40	.392	1	.813	34	.8808	100	59	5
.438	.462	61	.483	39	.391	1	.779	34	.8708	99	54	5
.439	.401	62	.444	40	.390	1	.745	34	.8609	99	49	5
1.440	12.339	61	15.404	39	4.389	1	10.711	34	3.8510	98	.6944	4
.441	.278	60	.365	39	.388	1	.677	33	.8412	98	40	5
.442	.218	60	.326	39	.387	1	.644	34	.8314	97	35	5
.443	.158	60	.287	39	.386	1	.610	33	.8217	97	30	5
.444	.098	59	.248	39	.385	1	.577	33	.8120	97	25	5
1.445	12.039	58	15.209	38	4.384	1	10.544	32	3.8023	96	.6920	4
.446	11.981	59	.171	38	.383	1	.512	33	.7927	95	16	5
.447	.922	58	.133	38	.382	1	.479	32	.7832	95	11	5
.448	.864	57	.095	38	.381	1	.447	32	.7737	95	06	5
.449	.807	57	.057	38	.380.	1	.415	32	.7642	94	01	4
.450	.750		.019		.379		.383		.7548		97	

P.P.	80	75	70	48	46	44		42	40	38	118	113	108	6	5	4
1	8	8	7	5	5	4		4	4	4	12	11	11	1	0	0
2	16	15	14	10	9	9		8	8	8	24	23	22	1	1	1
3	24	23	21	14	14	13		13	12	11	35	34	32	2	2	1
4	32	30	28	19	18	18		17	16	15	47	45	43	2	2	2
5	40	38	35	24	23	22		21	20	19	59	57	54	3	2	2
6	48	45	42	29	28	26		25	24	23	71	68	65	4	3	2
7	56	52	49	34	32	31		29	28	27	83	79	76	4	4	3
8	64	60	56	38	37	35		34	32	30	94	90	86	5	4	3
9	72	68	63	43	41	40		38	36	34	106	102	97	5	4	4

P.P.	66	63	59	42	40	38		36	35	33	104	100	96	
1	7	6	6	4	4	4		4	4	3	10	10	10	
2	13	13	12	8	8	8		7	7	7	21	20	19	
3	20	19	18	13	12	11		11	10	10	31	30	29	
4	26	25	24	17	16	15		14	14	13	42	40	38	
5	33	32	30	21	20	19		18	18	17	52	50	48	
6	40	38	35	25	24	23		22	21	20	62	60	58	
7	46	44	41	29	28	27		25	24	23	73	70	67	
8	53	50	47	34	32	30		29	28	26	83	80	77	
9	59	57	53	38	36	34		32	32	30	94	90	86	

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.450	11.750	57	15.019	37	4.379	1	10.383	32	3.7548	94	0.6897	5
.451	.693	56	14.982	38	.378	1	.351	32	.7454	93	92	5
.452	.637	56	.944	37	.377	1	.319	31	.7361	93	87	5
.453	.581	56	.907	37	.376	0	.288	31	.7268	93	82	4
.454	.525	55	.870	37	.376	1	.257	31	.7175	92	78	5
1.455	11.470	55	14.833	36	4.375	1	10.226	31	3.7083	91	.6873	5
.456	.415	54	.797	37	.374	1	.195	30	.6992	92	68	5
.457	.361	54	.760	36	.373	1	.165	31	.6900	91	63	4
.458	.307	54	.724	36	.372	1	.134	30	.6809	90	59	5
.459	.253	53	.688	36	.371	1	.104	30	.6719	90	54	5
1.460	11.200	53	14.652	36	4.370	1	10.074	30	3.6629	90	.6849	4
.461	.147	53	.616	36	.369	1	.044	30	.6539	89	45	5
.462	.094	52	.580	35	.368	1	.014	29	.6450	89	40	5
.463	042	52	.545	36	.367	1	9.985	30	.6361	88	35	4
.464	10.990	51	.509	35	.366	1	.955	29	.6273	88	31	5
1.465	10.939	52	14.474	35	4.365	1	9.926	29	3.6185	88	.6826	5
.466	.887	50	.439	35	.364	1	.897	29	.6097	87	21	4
.467	.837	51	.404	35	.363	1	.868	29	.6010	87	17	5
.468	.786	50	.369	34	.362	1	.839	28	.5923	86	12	5
.469	.736	50	.335	35	.361	0	.811	29	.5837	87	07	4
1.470	10.686	49	14.300	34	4.361	1	9.782	28	3.5750	85	.6803	5
.471	.637	50	.266	34	.360	1	.754	28	.5665	86	.6798	5
.472	.587	48	.232	34	.359	1	.726	28	.5579	85	93	4
.473	.539	49	.198	34	.358	1	.698	28	.5494	84	89	5
.474	.490	48	.164	34	.357	1	.670	27	.5410	84	84	4
1.475	10.442	48	14.130	33	4.356	1	9.643	28	3.5326	84	.6780	5
.476	.394	48	.097	34	.355	1	.615	27	.5242	84	75	5
.477	.346	47	.063	33	.354	1	.588	27	.5158	83	70	4
.478	.299	47	.030	33	.353	1	.561	27	.5075	83	66	5
.479	.252	46	13.997	33	.352	1	.534	27	.4992	82	61	4
1.480	10.206	47	13.964	33	4.351	1	9.507	27	3.4910	82	.6757	5
.481	.159	46	.931	33	.350	0	.480	26	.4828	82	52	4
.482	.113	46	.898	32	.350	1	.454	27	.4746	81	48	5
.483	.067	45	.866	32	.349	1	.427	26	.4665	81	43	4
.484	.022	45	.834	33	.348	1	.401	26	.4584	81	39	5
1.485	9.977	45	13.801	32	4.347	1	9.375	26	3.4503	80	.6734	5
.486	.932	45	.769	32	.346	1	.349	26	.4423	80	29	4
.487	.887	44	.737	32	.345	1	.323	25	.4343	80	25	5
.488	.843	44	.705	31	.344	1	.298	26	.4263	79	20	4
.489	.799	44	.674	32	.343	1	.272	25	.4184	79	16	5
1.490	9.755	43	13.642	31	4.342	1	9.247	26	3.4105	79	.6711	4
.491	.712	43	.611	32	.341	1	.221	25	.4026	78	07	5
.492	.669	43	.579	31	.340	0	.196	25	.3948	78	02	4
.493	.626	43	.548	31	.340	1	.171	25	.3870	78	.6698	5
.494	.583	42	.517	31	.339	1	.146	24	.3792	77	93	4
1.495	9.541	42	13.486	31	4.338	1	9.122	25	3.3715	77	.6689	5
.496	.499	42	.455	30	.337	1	.097	24	.3638	77	84	4
.497	.457	41	.425	31	.336	1	.073	25	.3561	76	80	4
.498	.416	42	.394	30	.335	1	.048	24	.3485	76	76	5
.499	.374	41	.364	31	.334	1	.024	24	.3409	76	71	4
.500	.333		.333		.333		.000		.3333		.67	

P. P.	55	52	50	37	36	34		31	30	28	92	89	86	5	4
1	6	5	5	4	4	3	----------	3	3	3	9	9	9	0	0
2	11	10	10	7	7	7	----------	6	6	6	18	18	17	1	1
3	16	16	15	11	11	10	----------	9	9	8	28	27	26	2	1
4	22	21	20	15	14	14	----------	12	12	11	37	36	34	2	2
5	28	26	25	19	18	17	----------	16	15	14	46	45	43	2	2
6	33	31	30	22	22	20	----------	19	18	17	55	53	52	3	2
7	39	36	35	26	25	24	----------	22	21	20	64	62	60	4	3
8	44	42	40	30	29	27	----------	25	24	22	74	71	69	4	3
9	50	47	45	33	32	31	----------	28	27	25	83	80	77	4	4

P. P.	47	45	43	33	32	31		27	26	25	83	80	77	
1	5	4	4	3	3	3	----------	3	3	2	8	8	8	----------
2	9	9	9	7	6	6	----------	5	5	5	17	16	15	----------
3	14	14	13	10	10	9	----------	8	8	8	25	24	23	----------
4	19	18	17	13	13	12	----------	11	10	10	33	32	31	----------
5	24	23	22	16	16	16	----------	14	13	12	42	40	38	----------
6	28	27	26	20	19	19	----------	16	16	15	50	48	46	----------
7	33	32	30	23	22	22	----------	19	18	18	58	56	54	----------
8	38	36	34	26	26	25	----------	22	21	20	66	64	62	----------
9	42	40	39	30	29	28	----------	24	23	22	75	72	69	----------

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.500	9.333	40	13.333	30	4.333	1	9.000	24	3.3333	75	0.6667	5
.501	.293	41	.303	30	.332	0	8.976	24	.3258	75	62	4
.502	.252	40	.273	30	.332	1	.952	23	.3183	75	58	5
.503	.212	40	.243	30	.331	1	.929	24	.3108	74	53	4
.504	.172	40	.213	29	.330	1	.905	23	.3034	75	49	4
1.505	9.132	39	13.184	30	4.329	1	8.882	24	3.2959	73	.6645	5
.506	.093	40	.154	29	.328	1	.858	23	.2886	74	40	4
.507	.053	39	.125	29	.327	1	.835	23	.2812	73	36	5
.508	.014	39	.096	30	.326	1	.812	23	.2739	73	31	4
.509	8.975	38	.066	29	.325	0	.789	23	.2666	73	27	4
1.510	8.937	38	13.037	29	4.325	1	8.766	22	3.2593	72	.6623	5
.511	.899	38	.008	29	.324	1	.744	23	.2521	72	18	4
.512	.861	38	12.979	28	.323	1	.721	23	.2449	72	14	5
.513	.823	38	.951	29	.322	1	.698	22	.2377	72	09	4
.514	.785	37	.922	28	.321	1	.676	22	.2305	71	05	4
1.515	8.748	37	12.894	29	4.320	1	8.654	22	3.2234	71	.6601	5
.516	.711	37	.865	28	.319	1	.632	22	.2163	70	.6596	4
.517	.674	37	.837	28	.318	0	.610	22	.2093	71	92	4
.518	.637	36	.809	28	.318	1	.588	22	.2022	70	88	5
.519	.601	37	.781	28	.317	1	.566	22	.1952	69	83	4
1.520	8.564	36	12.753	28	4.316	1	8.544	21	3.1883	70	.6579	4
.521	.528	36	.725	27	.315	1	.523	22	.1813	69	75	5
.522	.492	35	.698	28	.314	1	.501	21	.1744	69	70	4
.523	.457	36	.670	27	.313	1	.480	21	.1675	69	66	4
.524	.421	35	.643	28	.312	1	.459	21	.1606	68	62	5
1.525	8.386	35	12.615	27	4.311	0	8.438	21	3.1538	68	.6557	4
.526	.351	34	.588	27	.311	1	.417	21	.1470	68	53	4
.527	.317	35	.561	27	.310	1	.396	21	.1402	68	49	4
.528	.282	34	.534	27	.309	1	.375	21	.1334	67	45	5
.529	.248	34	.507	27	.308	1	.354	20	.1267	67	40	4
1.530	8.214	34	12.480	27	4.307	1	8.334	21	3.1200	67	.6536	4
.531	.180	34	.453	26	.306	1	.313	20	.1133	66	32	5
.532	.146	34	.427	27	.305	0	.293	21	.1067	67	27	4
.533	.112	33	.400	26	.305	1	.272	20	.1000	66	23	4
.534	.079	33	.374	27	.304	1	.252	20	.0934	65	19	4
1.535	8.046	33	12.347	26	4.303	1	8.232	20	3.0869	66	.6515	5
.536	.013	33	.321	26	.302	1	.212	20	.0803	65	10	4
.537	7.980	32	.295	26	.301	1	.192	20	.0738	65	06	4
.538	.948	33	.269	26	.300	0	.172	19	.0673	65	02	4
.539	.915	32	.243	26	.300	1	.153	20	.0608	64	.6498	4
1.540	7.883	32	12.217	25	4.299	1	8.133	19	3.0544	65	.6494	5
.541	.851	32	.192	26	.298	1	.114	20	.0479	64	89	4
.542	.819	31	.166	25	.297	1	.094	19	.0415	63	85	4
.543	.788	32	.141	26	.296	1	.075	19	.0352	64	81	4
.544	.756	31	.115	25	.295	1	.056	20	.0288	63	77	5
1.545	7.725	31	12.090	25	4.294	0	8.036	19	3.0225	63	.6472	4
.546	.694	31	.065	25	.294	1	.017	19	.0162	63	68	4
.547	.663	31	.040	25	.293	1	7.998	18	.0099	63	64	4
.548	.632	30	.015	25	.292	1	.980	19	.0036	62	60	4
.549	.602	31	11.990	25	.291	1	.961	19	2.9974	62	56	4
.550	.571		.965		.290		.942		.9912		52	

P. P.	40	38	36	30	29	28		24	23	22	74	72	70	4	5
1	4	4	4	3	3	3		2	2	2	7	7	7	0	0
2	8	8	7	6	6	6		5	5	4	15	14	14	1	1
3	12	11	11	9	9	8		7	7	7	22	22	21	1	2
4	16	15	14	12	12	11		10	9	9	30	29	28	2	2
5	20	19	18	15	14	14		12	12	11	37	36	35	2	2
6	24	23	22	18	17	17		14	14	13	44	43	42	2	3
7	28	27	25	21	20	20		17	16	15	52	50	49	3	4
8	32	30	29	24	23	22		19	18	18	59	58	56	3	4
9	36	34	32	27	26	25		22	21	20	67	65	63	4	4

P. P.	34	32	30	27	26	25		21	20	19	68	66	64	
1	3	3	3	3	3	2		2	2	2	7	7	6	
2	7	6	6	5	5	5		4	4	4	14	13	13	
3	10	10	9	8	8	8		6	6	6	20	20	19	
4	14	13	12	11	10	10		8	8	8	27	26	26	
5	17	16	15	14	13	12		10	10	10	34	33	32	
6	20	19	18	16	16	15		13	12	11	41	40	38	
7	24	22	21	19	18	18		15	14	13	48	46	45	
8	27	26	24	22	21	20		17	16	15	54	53	51	
9	31	29	27	24	23	22		19	18	17	61	59	58	

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.550	7.571	30	11.965	25	4.290	1	7.942	18	2.9912	62	0.6452	5
.551	.541	30	.940	25	.289	0	.924	19	.9850	61	47	4
.552	.511	30	.915	24	.289	1	.905	18	.9789	62	43	4
.553	.481	29	.891	25	.288	1	.887	19	.9727	61	39	4
.554	.452	30	.866	24	.287	1	.868	18	.9666	61	35	4
1.555	7.422	29	11.842	24	4.286	1	7.850	18	2.9605	61	.6431	4
.556	.393	29	.818	24	.285	0	.832	18	.9544	60	27	4
.557	.364	30	.794	24	.285	1	.814	18	.9484	60	23	5
.558	.334	28	.770	24	.284	1	.796	18	.9424	60	18	4
.559	.306	29	.746	24	.283	1	.778	18	.9364	60	14	4
1.560	7.277	29	11.722	24	4.282	1	7.760	18	2.9304	60	.6410	4
.561	.248	28	.698	24	.281	1	.742	17	.9244	59	06	4
.562	.220	28	.674	24	.280	0	.725	18	.9185	59	02	4
.563	.192	28	.650	23	.280	1	.707	17	.9126	59	.6398	4
.564	.164	28	.627	24	.279	1	.690	18	.9067	59	94	4
1.565	7.136	28	11.603	23	4.278	1	7.672	17	2.9008	58	.6390	4
.566	.108	27	.580	23	.277	1	.655	17	.8950	58	86	4
.567	.081	28	.557	24	.276	0	.638	17	.8892	58	82	4
.568	.053	27	.533	23	.276	1	.621	17	.8834	58	78	5
.569	.026	27	.510	23	.275	1	.604	17	.8776	58	73	4
1.570	6.999	27	11.487	23	4.274	1	7.587	17	2.8718	57	.6369	4
.571	.972	27	.464	23	.273	1	.570	17	.8661	57	65	4
.572	.945	27	.441	22	.272	1	.553	17	.8604	57	61	4
.573	.918	26	.419	23	.271	0	.536	17	.8547	57	57	4
.574	.892	27	.396	23	.271	1	.519	16	.8490	57	53	4
1.575	6.865	26	11.373	22	4.270	1	7.503	17	2.8433	56	.6349	4
.576	.839	26	.351	23	.269	1	.486	16	.8377	56	45	4
.577	.813	26	.328	22	.268	1	.470	17	.8321	56	41	4
.578	.787	26	.306	22	.267	0	.453	16	.8265	56	37	4
.579	.761	25	.284	23	.267	1	.437	16	.8209	55	33	4
1.580	6.736	26	11.261	22	4.266	1	7.421	16	2.8154	56	.6329	4
.581	.710	25	.239	22	.265	1	.405	16	.8098	55	25	4
.582	.685	26	.217	22	.264	1	.389	16	.8043	55	21	4
.583	.659	25	.195	22	.263	0	.373	16	.7988	55	17	4
.584	.634	25	.173	21	.263	1	.357	16	.7933	54	13	4
1.585	6.609	25	11.152	22	4.262	1	7.341	16	2.7879	54	.6309	4
.586	.584	24	.130	22	.261	1	.325	16	.7825	55	05	4
.587	.560	25	.108	21	.260	1	.309	15	.7770	54	01	4
.588	.535	24	.087	22	.259	0	.294	16	.7716	53	.6297	4
.589	.511	25	.065	21	.259	1	.278	15	.7663	54	93	4
1.590	6.486	24	11.044	22	4.258	1	7.263	16	2.7609	53	.6289	4
.591	.462	24	.022	21	.257	1	.247	15	.7556	54	85	4
.592	.438	24	.001	21	.256	1	.232	16	.7502	53	81	4
.593	.414	24	10.980	21	.255	0	.216	15	.7449	52	77	3
.594	.390	23	.959	21	.255	1	.201	15	.7397	53	74	4
1.595	6.367	24	10.938	21	4.254	1	7.186	15	2.7344	53	.6270	4
.596	.343	23	.917	21	.253	1	.171	15	.7291	52	66	4
.597	.320	24	.896	21	.252	0	.156	15	.7239	52	62	4
.598	.296	23	.875	21	.252	1	.141	15	.7187	52	58	4
.599	.273	23	.854	21	.251	1	.126	15	.7135	52	54	4
.600	.250		.833		.250		.111		.7083		50	

P. P.	29	28	27	26	25		19	18	17	62	60	58	5	4	3
1	3	3	3	3	2	----------	2	2	2	6	6	6	0	0	0
2	6	6	5	5	5	----------	4	4	3	12	12	12	1	1	1
3	9	8	8	8	8	----------	6	5	5	19	18	17	2	1	1
4	12	11	11	10	10	----------	8	7	7	25	24	23	2	2	1
5	14	14	14	13	12	----------	10	9	8	31	30	29	2	2	2
6	17	17	16	16	15	----------	11	11	10	37	36	35	3	2	2
7	20	20	19	18	18	----------	13	13	12	43	42	41	4	3	2
8	23	22	22	21	20	----------	15	14	14	50	48	46	4	3	2
9	26	25	24	23	22	----------	17	16	15	56	54	52	4	4	3

P. P.	24	23	22	21		16	15	56	54	52	
1	2	2	2	2	----------	2	2	6	5	5	----------
2	5	5	4	4	----------	3	3	11	11	10	----------
3	7	7	7	6	----------	5	4	17	16	16	----------
4	10	9	9	8	----------	6	6	22	22	21	----------
5	12	12	11	10	----------	8	8	28	27	26	----------
6	14	14	13	13	----------	10	9	34	32	31	----------
7	17	16	15	15	----------	11	10	39	38	36	----------
8	19	18	18	17	----------	13	12	45	43	42	----------
9	22	21	20	19	----------	14	14	50	49	47	----------

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.600	6.250	23	10.833	20	4.250	1	7.111	15	2.7083	51	0.6250	4
.601	.227	23	.813	21	.249	1	.096	14	.7032	52	46	4
.602	.204	22	.792	20	.248	0	.082	15	.6980	51	42	4
.603	.182	23	.772	21	.248	1	.067	15	.6929	51	38	4
.604	.159	23	.751	20	.247	1	.052	14	.6878	51	34	3
1.605	6.136	22	10.731	20	4.246	1	7.038	15	2.6827	50	.6231	4
.606	.114	22	.711	21	.245	0	.023	14	.6777	51	27	4
.607	.092	22	.690	20	.245	1	.009	14	.6726	50	23	4
.608	.070	22	.670	20	.244	1	6.995	15	.6676	50	19	4
.609	.048	22	.650	20	.243	1	.980	14	.6626	50	15	4
1.610	6.026	22	10.630	20	4.242	1	6.966	14	2.6576	50	.6211	4
.611	.004	22	.610	19	.241	0	.952	14	.6526	50	07	4
.612	5.982	21	.591	20	.241	1	.938	14	.6476	49	03	3
.613	.961	22	.571	20	.240	1	.924	14	.6427	49	.6200	4
.614	.939	21	.551	20	.239	1	.910	14	.6378	50	.6196	4
1.615	5.918	21	10.531	19	4.238	0	6.896	14	2.6328	49	.6192	4
.616	.897	21	.512	20	.238	1	.882	14	.6279	48	88	4
.617	.876	21	.492	19	.237	1	.868	13	.6231	49	84	4
.618	.855	21	.473	20	.236	1	.855	14	.6182	48	80	3
.619	.834	21	.453	19	.235	0	.841	14	.6134	49	77	4
1.620	5.813	21	10.434	19	4.235	1	6.827	13	2.6085	48	.6173	4
.621	.792	20	.415	19	.234	1	.814	14	.6037	48	69	4
.622	.772	21	.396	19	.233	1	.800	13	.5989	48	65	4
.623	.751	20	.377	20	.232	0	.787	14	.5941	47	61	3
.624	.731	20	.357	19	.232	1	.773	13	.5894	48	58	4
1.625	5.711	20	10.338	18	4.231	1	6.760	13	2.5846	47	.6154	4
.626	.691	20	.320	19	.230	1	.747	14	.5799	47	50	4
.627	.671	20	.301	19	.229	0	.733	13	.5752	47	46	3
.628	.651	20	.282	19	.229	1	.720	13	.5705	47	43	4
.629	.631	20	.263	19	.228	1	.707	13	.5658	47	39	4
1.630	5.611	20	10.244	18	4.227	1	6.694	13	2.5611	46	.6135	4
.631	.591	19	.226	19	.226	1	.681	13	.5565	47	31	4
.632	.572	20	.207	18	.225	0	.668	13	.5518	46	27	3
.633	.552	19	.189	19	.225	1	.655	13	.5472	46	24	4
.634	.533	19	.170	18	.224	1	.642	12	.5426	46	20	4
1.635	5.514	20	10.152	18	4.223	1	6.630	13	2.5380	46	.6116	4
.636	.494	19	.134	19	.222	0	.617	13	.5334	46	12	3
.637	.475	19	.115	18	.222	1	.604	12	.5288	45	09	4
.638	.456	18	.097	18	.221	1	.592	13	.5243	45	05	4
.639	.438	19	.079	18	.220	0	.579	13	.5198	46	01	3
1.640	5.419	19	10.061	18	4.220	1	6.566	12	2.5152	45	.6098	4
.641	.400	19	.043	18	.219	1	.554	13	.5107	44	94	4
.642	.381	18	.025	18	.218	1	.541	12	.5063	45	90	4
.643	.363	19	.007	18	.217	0	.529	12	.5018	45	86	3
.644	.344	18	9.989	18	.217	1	.517	13	.4973	44	83	4
1.645	5.326	18	9.971	17	4.216	1	6.504	12	2.4929	45	.6079	4
.646	.308	18	.954	18	.215	1	.492	12	.4884	44	75	3
.647	.290	18	.936	18	.214	0	.480	12	.4840	44	72	4
.648	.272	18	.918	17	.214	1	.468	12	.4796	44	68	4
.649	.254	18	.901	18	.213	1	.456	12	.4752	43	64	3
.650	.236		.883		.212		.444		.4709		61	

P. P.	23	22	21	20		15	14	51	49	47	4	3
1	2	2	2	2	—	2	1	5	5	5	0	0
2	5	4	4	4	—	3	3	10	10	9	1	1
3	7	7	6	6	—	3	4	15	15	14	1	1
4	9	9	8	8	—	6	6	20	20	19	2	1
5	12	11	10	10	—	8	7	26	24	24	2	2
6	14	13	13	12	—	9	8	31	29	28	2	2
7	16	15	15	14	—	10	10	36	34	33	3	2
8	18	18	17	16	—	12	11	41	39	38	3	2
9	21	20	19	18	—	14	13	46	44	42	4	3

P. P.	19	18	17			13	12	46	45	44	
1	2	2	2	—	—	1	1	5	4	4	—
2	4	4	3	—	—	3	2	9	9	9	—
3	6	5	5	—	—	4	4	14	14	13	—
4	8	7	7	—	—	5	5	18	18	18	—
5	10	9	8	—	—	6	6	23	22	22	—
6	11	11	10	—	—	8	7	28	27	26	—
7	13	13	12	—	—	9	8	32	32	31	—
8	15	14	14	—	—	10	10	37	36	35	—
9	17	16	15	—	—	12	11	41	40	40	—

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.650	5.236	18	9.883	17	4.212	1	6.444	12	2.4709	44	0.6061	4
.651	.218	18	.866	17	.211	0	.432	12	.4665	43	57	4
.652	.200	17	.849	18	.211	1	.420	12	.4622	44	53	3
.653	.183	18	.831	17	.210	1	.408	12	.4578	43	50	4
.654	.165	17	.814	17	.209	1	.396	12	.4535	43	46	4
1.655	5.148	18	9.797	17	4.208	0	6.384	11	2.4492	43	.6042	3
.656	.130	17	.780	17	.208	1	.373	12	.4449	43	39	4
.657	.113	17	.763	17	.207	1	.361	12	.4406	42	35	4
.658	.096	17	.746	17	.206	0	.349	11	.4364	43	31	3
.659	.079	17	.729	17	.206	1	.338	12	.4321	42	28	4
1.660	5.062	17	9.712	17	4.205	1	6.326	12	2.4279	42	.6024	4
.661	.045	17	.695	17	.204	1	.314	11	.4237	42	20	3
.662	.028	17	.678	17	.203	0	.303	12	.4195	42	17	4
.663	.011	17	.661	17	.203	1	.291	11	.4153	42	13	3
.664	4.994	16	.644	16	.202	1	.280	11	.4111	42	10	4
1.665	4.978	17	9.628	17	4.201	1	6.269	11	2.4069	41	.6006	4
.666	.961	16	.611	17	.200	0	.258	12	.4028	42	02	3
.667	.945	17	.594	16	.200	1	.246	11	.3986	41	.5999	4
.668	.928	16	.578	16	.199	1	.235	11	.3945	41	95	3
.669	.912	16	.562	17	.198	0	.224	11	.3904	41	92	4
1.670	4.896	17	9.545	16	4.198	1	6.213	11	2.3863	41	.5988	4
.671	.879	16	.529	17	.197	1	.202	11	.3822	41	84	3
.672	.863	16	.512	16	.196	1	.191	11	.3781	41	81	4
.673	.847	16	.496	16	.195	0	.180	11	.3740	40	77	3
.674	.831	16	.480	16	.195	1	.169	11	.3700	41	74	4
1.675	4.815	15	9.464	16	4.194	1	6.158	11	2.3659	40	.5970	3
.676	.800	16	.448	16	.193	0	.147	11	.3619	40	67	4
.677	.784	16	.432	16	.193	1	.136	11	.3579	40	63	4
.678	.768	15	.416	16	.192	1	.125	10	.3539	40	59	3
.679	.753	16	.400	16	.191	1	.115	11	.3499	40	56	4
1.680	4.737	15	9.384	16	4.190	0	6.104	11	2.3459	39	.5952	3
.681	.722	16	.368	16	.190	1	.093	10	.3420	40	49	4
.682	.706	15	.352	16	.189	1	.083	11	.3380	39	45	3
.683	.691	15	.336	15	.188	0	.072	11	.3341	39	42	4
.684	.676	15	.321	16	.188	1	.061	10	.3302	40	38	3
1.685	4.661	15	9.305	16	4.187	1	6.051	11	2.3262	39	.5935	4
.686	.646	15	.289	15	.186	0	.040	10	.3223	39	31	3
.687	.631	15	.274	16	.186	1	.030	10	.3184	38	28	4
.688	.616	15	.258	15	.185	1	.020	11	.3146	39	24	3
.689	.601	15	.243	16	.184	1	.009	10	.3107	39	21	4
1.690	4.586	15	9.227	15	4.183	0	5.999	10	2.3068	38	.5917	3
.691	.571	14	.212	15	.183	1	.989	11	.3030	38	14	4
.692	.557	15	.197	16	.182	1	.978	10	.2992	39	10	3
.693	.542	14	.181	15	.181	0	.968	10	.2953	38	07	4
.694	.528	15	.166	15	.181	1	.958	10	.2915	38	03	3
1.695	4.513	14	9.151	15	4.180	1	5.948	10	2.2877	38	.5900	4
.696	.499	15	.136	15	.179	0	.938	10	.2839	37	.5896	3
.697	.484	14	.121	15	.179	1	.928	10	.2802	38	93	4
.698	.470	14	.106	15	.178	1	.918	10	.2764	38	89	3
.699	.456	14	.091	15	.177	1	.908	10	.2726	37	86	4
.700	.442		.076		.176		.898		.2689		82	

P.P.	18	17	16	15	14		12	11	10	43	42	41	4	3
1	2	2	2	2	1		1	1	1	4	4	4	0	0
2	4	3	3	3	3		2	2	2	9	8	8	1	1
3	5	5	5	4	4		4	3	3	13	13	12	1	1
4	7	7	6	6	6		5	4	4	17	17	16	2	1
5	9	8	8	8	7		6	6	5	22	21	20	2	2
6	11	10	10	9	8		7	7	6	26	25	25	2	2
7	13	12	11	10	10		8	8	7	30	29	29	3	2
8	14	14	13	12	11		10	9	8	34	34	33	3	2
9	16	15	14	14	13		11	10	9	39	38	37	4	3

P.P.					40	39	38	
1					4	4	4	
2					8	8	8	
3					12	12	11	
4					16	16	15	
5					20	20	19	
6					24	23	23	
7					28	27	27	
8					32	31	30	
9					36	35	34	

n	$\frac{n+2}{n(n-1)^2}$		$\frac{4(n+1)}{n(n-1)}$		$\frac{3n+2}{n}$		$\frac{n^2}{(n-1)^2}$		$\frac{n+1}{n(n-1)}$		$\frac{1}{n}$	
1.700	4.442	14	9.076	15	4.176	0	5.898	10	2.2689	37	0.5882	3
.701	.428	14	.061	15	.176	1	.888	10	.2652	37	.79	4
.702	.414	14	.046	15	.175	1	.878	10	.2615	37	75	3
.703	.400	14	.031	15	.174	0	.868	9	.2578	37	72	3
.704	.386	14	.016	15	.174	1	.859	10	.2541	37	69	4
1.705	4.372	14	9.001	14	4.173	1	5.849	10	2.2504	37	.5865	3
.706	.358	13	8.987	15	.172	0	.839	10	.2467	37	62	4
.707	.345	14	.972	14	.172	1	.829	9	.2430	36	58	3
.708	.331	14	.958	15	.171	1	.820	10	.2394	37	55	4
.709	.317	13	.943	15	.170	0	.810	9	.2357	36	51	3
1.710	4.304	14	8.928	14	4.170	1	5.801	10	2.2321	36	.5848	3
.711	.290	13	.914	14	.169	1	.791	9	.2285	36	45	4
.712	.277	13	.900	15	.168	0	.782	10	.2249	36	41	3
.713	.264	14	.885	14	.168	1	.772	9	.2213	36	38	4
.714	.250	13	.871	15	.167	1	.763	10	.2177	36	34	3
1.715	4.237	13	8.856	14	4.166	0	5.753	9	2.2141	36	.5831	3
716	224	13	842	14	166	1	744	9	2105	35	28	4
717	211	13	828	14	165	1	.735	10	.2070	36	24	3
.718	.198	13	.814	14	.164	1	.725	9	.2034	35	21	4
.719	.185	13	.800	14	.163	0	.716	9	.1999	35	17	3
1.720	4.172	13	8.786	15	4.163	1	5.707	9	2.1964	35	.5814	3
.721	.159	13	.771	14	.162	1	.698	10	.1929	35	11	4
.722	.146	12	.757	14	.161	0	.688	9	.1894	35	07	3
.723	.134	13	.743	13	.161	1	.679	9	.1859	35	04	4
.724	.121	13	.730	14	.160	1	.670	9	.1824	35	00	3
1.725	4.108	12	8.716	14	4.159	0	5.661	9	2.1739	35	.5797	3
.726	.096	13	.702	14	.159	1	.652	9	.1754	34	94	4
.727	.083	12	.688	14	.158	1	.643	9	.1720	35	90	3
.728	.071	13	.674	14	.157	0	.634	9	.1685	34	87	3
.729	.058	12	.660	13	.157	1	.625	9	.1651	34	84	4
1.730	4.046	12	8.647	14	4.156	1	5.616	9	2.1617	34	.5780	3
.731	.034	13	.633	14	.155	0	.607	8	.1583	34	77	3
.732	.021	12	.619	13	.155	1	.599	9	.1549	34	74	4
.733	.009	12	.606	14	.154	1	.590	9	.1515	34	70	3
.734	3.997	12	.592	13	.153	0	.581	9	.1481	34	67	3
1.735	3.985	12	8.579	14	4.153	1	5.572	9	2.1447	33	.5764	4
.736	.973	12	.565	13	.152	1	.563	8	.1414	34	60	3
.737	.961	12	.552	13	.151	0	.555	9	.1380	34	57	3
.738	.949	12	.539	14	.151	1	.546	9	.1346	33	54	4
.739	.937	12	.525	13	.150	1	.537	8	.1313	33	50	3
1.740	3.925	12	8.512	13	4.149	0	5.529	9	2.1280	33	.5747	3
.741	.913	11	.499	14	.149	1	.520	8	.1247	33	44	3
.742	.902	12	.485	13	.148	1	.512	9	.1214	33	41	4
.743	.890	12	.472	13	.147	0	.503	8	.1181	33	37	3
.744	.878	11	.459	13	.147	1	.495	9	.1148	33	34	3
1.745	3.867	12	8.446	13	4.146	1	5.486	8	2.1115	33	.5731	4
.746	.855	11	.433	13	.145	0	.478	9	.1082	32	27	3
.747	.844	12	.420	13	.145	1	.469	8	.1050	33	24	3
.748	.832	11	.407	13	.144	0	.461	8	.1017	32	21	3
.749	.821	11	.394	13	.144	1	.453	9	.0985	33	18	4
.750	.810		.381		.143		.444		.0952		14	

P. P.	15	14	13	12	11		10	9	8	37	36	35	4	3
1	2	1	1	1	1		1	1	1	4	4		0	0
2	3	3	3	2	2		2	2	2	7	7		1	1
3	4	4	4	4	3		3	3	2	11	11		1	1
4	6	6	5	5	4		4	4	3	15	14		2	1
5	8	7	6	6	6		5	4	4	18	18		2	2
6	9	8	8	7	7		6	5	5	22	22		2	2
7	10	10	9	8	8		7	6	6	26	25		3	2
8	12	11	10	10	9		8	7	6	30	29		3	2
9	14	13	12	11	10		9	8	7	33	32		4	3

P. P.					34	33	32	
1					3	3	3	
2					7	7	6	
3					10	10	10	
4					14	13	13	
5					17	16	16	
6					20	20	19	
7					24	23	22	
8					27	26	26	
9					31	30	29	

WASHINGTON, October 9, 1926.