THE COEFFICIENT OF REFLECTION OF ELECTRICAL WAVES AT A TRANSITION POINT.

By Louis Cohen.

The subject of the propagation of electrical waves along conductors is becoming of great practical importance, problems of this nature presenting themselves in nearly all branches of electrical engineering. A discussion, therefore, of any phase of the many problems in electrical wave propagation which may arise in practice will, I trust, be of some interest.

The discussion of the problems as they are usually treated in text-books assumes a uniform conductor; this, however, is rarely the case. In practice we always have to deal with a complex network of conductors, and an interesting question arises as to what happens at a transition point when an electric wave passes over from one conductor to another of different electrical constants. We know, of course, that the velocity of propagation depends on the constants of the electrical circuits, and so for different circuits the velocity will be different; hence we may expect that, at a transition point, the wave will be partly reflected and partly transmitted. It is not always clear, however, as to what are likely to be the relative magnitudes of the reflected and transmitted waves as compared with the incoming wave.

In this brief communication I wish to develop an expression for the ratios of the reflected and transmitted waves to the incoming wave at a transition point.

Dr. O. Heaviside, who has contributed so much to our knowledge of electricity, has also indicated briefly a method for the solution of the problems under discussion. Heaviside makes use of the differential operator, the theory of which he has extensively developed and which has proven to be a powerful tool for the solution of this class of problems. In this particular problem, however, we can obtain our result in a very simple way without the aid of the differential operator.

Let us denote by V_1 the potential on a line of the incoming wave, by V_2 and V_3 the reflected and transmitted waves respectively. We assume that the potential on the line consists of a single wave, which is not usually the case, as the potential will generally be the resultant of the superposition of several waves, but as far as the transition point is concerned what is true of one wave will be true of any number of waves; hence we may limit our discussion to the case of a single wave.

The general expression for a single potential wave developed on a line of uniform inductance and capacity is as follows:

$$V = Ae^{-\alpha s} \cos (pt - \beta s)$$

where α is the damping factor, β the velocity constant, and A is the amplitude. In this particular problem, however, we are mainly concerned with the potentials and currents at points close to the transition point, the amount of damping therefore in passing from a point on one side of the transition point to a point on the other side will be very small; hence we may neglect the damping factor. If we denote by β_1 and β_2 the velocity constants of the two sections of the line on the different sides of the transition point we have the following expressions for the potentials:

$$\left.\begin{aligned} V_1 &= A_1 \cos (pt - \beta_1 s) \\ V_2 &= A_2 \cos (pt - \beta_1 s) \\ V_3 &= A_3 \cos (pt - \beta_2 s) \end{aligned}\right\} \qquad (1)$$

The currents corresponding to these potentials can be derived of course by the aid of the well-known relation:

$$C\frac{dV}{dt} = -\frac{dI}{ds}$$

hence we get,

$$\left.\begin{aligned} I_1 &= \frac{C_1 p}{\beta_1} A_1 \cos (pt - \beta_1 s) \\ I_2 &= \frac{C_1 p}{\beta_1} A_2 \cos (pt - \beta_1 s) \\ I_3 &= \frac{C_2 p}{\beta_2} A_3 \cos (pt - \beta_2 s) \end{aligned}\right\} \qquad (2)$$

At the transition point, the potential will be the sum of the incoming and reflected waves, while the current will be represented by the difference of two waves, since they travel in opposite directions. Since there is no discontinuity of any sort at the transition point we must have the following relations satisfied:

$$\left.\begin{aligned}(A_1+A_2)\cos(pt-\beta_1 s) &= A_3\cos(pt-\beta_2 s)\\ \frac{C_1 p}{\beta_1}(A_1-A_2)\cos(pt-\beta_1 s) &= \frac{C_2 p}{\beta_2}A_3\cos(pt-\beta_2 s)\end{aligned}\right\}\qquad(3)$$

If we take the origin at the transition point, s will be zero in the above equations, and we have by dividing the first equation by the second,

$$\frac{A_1+A_2}{A_1-A_2}=\frac{C_1\beta_2}{C_2\beta_1}\qquad(4)$$

and

$$\frac{A_1+A_2}{A_1-A_2}+1=\frac{2A_1}{A_1-A_2}=\frac{C_1\beta_2+C_2\beta_1}{C_2\beta_1}$$

$$\frac{A_1-A_2}{2A_1}=\frac{1}{2}-\frac{A_2}{2A_1}=\frac{C_2\beta_1}{C_1\beta_2+C_2\beta_1}$$

$$\frac{A_2}{A_1}=\frac{C_1\beta_2-C_2\beta_1}{C_1\beta_2+C_2\beta_1}\qquad(5)$$

$\frac{A_2}{A_1}$ is the ratio of the amplitudes of the reflected to the incoming wave. In case of very high frequencies we may neglect the resistance as compared with the reactance and the expressions for β_1 and β_2 will be $p\sqrt{L_1C_1}$ and $p\sqrt{L_2C_2}$; replacing β_1 and β_2 in equation (5) by their corresponding values we get:

$$\frac{A_2}{A_1}=\frac{\sqrt{\frac{L_2}{C_2}}-\sqrt{\frac{L_1}{C_1}}}{\sqrt{\frac{L_2}{C_2}}+\sqrt{\frac{L_1}{C_1}}}\qquad(6)$$

The ratio of the transmitted wave to the incoming wave can be obtained in the following way:

We have by equation (3),

$$A_1 + A_2 = A_3$$

Replacing A_2 by its value in terms of A_1 from equation (5) we get,

$$A_1\left\{1 + \frac{C_1\beta_2 - C_2\beta_1}{C_1\beta_2 + C_2\beta_1}\right\} = A_3$$

Therefore,

$$\frac{A_3}{A_1} = \frac{2C_1\beta_2}{C_1\beta_2 + C_2\beta_1} = \frac{2\sqrt{\frac{L_2}{C_2}}}{\sqrt{\frac{L_2}{C_2}} + \sqrt{\frac{L_1}{C_1}}} \tag{7}$$

We see by equation (7) that if we pass from a line of low inductance and high capacity into a line of high inductance and low capacity the voltage will be increased. As an example, we may consider the following case: Suppose we connect an underground cable, whose constants are $L = 0.4 \times 10^{-3}$ henry and $C = 0.6 \times 10^{-6}$ farads, with that of an overhead line whose constants are $L = 1.95 \times 10^{-3}$ henry and $C = 0.0162 \times 10^{-6}$ farads, in passing from the underground cable to the overhead line we shall have,

$$\frac{A_3}{A_1} = \frac{2\sqrt{\frac{1.95}{0.0162} \times 10^3}}{\sqrt{\frac{1.95}{0.0162} \times 10^3} + \sqrt{\frac{0.4}{0.6} \times 10^3}} = 1.9$$

The potential will rise to nearly double its value.

The results as given by equations (6) and (7) are in agreement with some of the results obtained by Professor Steinmetz [1] in his discussion of the "General Equation of the Electric Circuit."

The relations between the reflected and transmitted waves to the incoming wave as given by equations (6) and (7) are only applicable to the case where the wave in passing the transition point continues its travel in the form of a wave; that is, we have

[1] C. P. Steinmetz: Proc. Am. Inst. of Elect. Eng., **27**, p. 1190.

distributed inductance and capacity on both sides of the transition point. In the case, however, where at the transition point we have a lumped inductance and capacity the conditions are different.

Let us denote the impedance of the apparatus at the transition point by Z, and as before we will denote the potential of the incoming wave by $V_1 = A_1 \cos(pt - \beta s)$; that of the reflected wave by $V_2 = A_2 \cos(pt - \beta s)$. The currents corresponding to these two waves will be

$$\frac{Cp}{\beta} A_1 \cos(pt - \beta s) \text{ and } \frac{Cp}{\beta} A_2 \cos(pt - \beta s).$$

Now, at the transition point, the potential is the sum of the incoming and reflected waves, and this must be equal to the drop of potential at the terminal apparatus.

We have, therefore,

$$V_1 + V_2 = ZI$$

But the current at the transition point is the difference of the currents of the incoming and reflected waves, since they travel in opposite directions; hence we have,

$$V_1 + V_2 = Z(I_1 - I_2)$$

Introducing the values of V_1, V_2, I_1 and I_2, we get,

$$(A_1 + A_2) \cos(pt - \beta s) = \frac{pC}{\beta} Z(A_1 - A_2) \cos(pt - \beta s)$$

From which we obtain,

$$\frac{A_1 + A_2}{A_1 - A_2} = \frac{pCZ}{\beta}$$

Proceeding in the same way as in deriving equation (5), we obtain,

$$\frac{A_2}{A_1} = \frac{pCZ - \beta}{pCZ + \beta} \tag{8}$$

If we denote the inductance and capacity of the terminal apparatus by L_1, C_1 and neglect the resistance, we have $Z = pL_1 - \frac{1}{pC_1}$.

Replacing also β by its corresponding value $p\sqrt{LC}$, equation (8) will become

$$\frac{A_2}{A_1}=\frac{C\left(pL_1-\frac{1}{p_1C}\right)-\sqrt{LC}}{C\left(pL_1-\frac{1}{p_1C}\right)+\sqrt{LC}} \tag{9}$$

In the case of a free oscillation, we have $p=\frac{1}{\sqrt{LC}}$ and therefore,

$$\frac{A_2}{A_1}=\frac{\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)-\sqrt{\frac{L}{C}}}{\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)+\sqrt{\frac{L}{C}}} \tag{10}$$

The rise of potential at the terminal, or the ratio of the potential at the transition point to that of line will be,

$$\frac{A_2+A_1}{A_1}=\frac{2\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)}{\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)+\sqrt{\frac{L}{C}}} \tag{11}$$

If the inductance of the terminal apparatus is very high, we may write approximately equation (11) in the following form:

$$\frac{A_2+A_1}{A_1}=\frac{2\,L_1}{L_1+L}$$

The voltage may therefore rise to nearly twice the voltage of the line

WASHINGTON, February 27, 1909.