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On certain Schur-convex functions

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Dedicated to the 60^{th} birthday of Professor Zsolt Páles

Abstract. In this paper, we examine the properties, possible characterizations and lower Hermite–Hadamard inequalities of certain Schur-convex functions.

1. Introduction

The concept of Schur-convexity is almost one hundred years old now. It was introduced in 1923 by Issai Schur, in [Sch23]. Since then, a vast number of publications has been printed using Schur-convexity from different areas of mathematics, including real function theory (e.g. [Olb15b]), inequalities (e.g. [Ste07]), financial mathematics (e.g. [GZ12]), and optimization (e.g. [YL11], [HR05]).

The original definition was the following: a function $f\colon I^n \to \mathbb{R}$ is Schur-convex if

$$f(Sx) \leq f(x)$$

for all doubly stochastic matrix S and for all $x = (x_1, \dots, x_n) \in I^n$, where I is a non-empty interval.

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Here we examine a special case of the previous inequality, namely, when the matrix S is a certain fixed one. The advantage of this choice is a unification of earlier well-known concepts.

The remaining parts of the paper are organized as follows. In Section 2, we fix the notations and show some preliminary results. Some characterization theorems can be found in the third Section. At last, in the fourth Section, there are lower Hermite–Hadamard type inequalities (for more information of this type of inequalities, see [HP09] or [HM17], and the references therein).

2. Preliminary results

Here and hereafter, unless stated otherwise, D denotes a non-empty, convex subset of a linear space X. We say that a function $f : D \times D \to \mathbb{R}$ is *Schur-convex* if

$$f(tx + (1-t)y, (1-t)x + ty) \le f(x, y) \tag{1}$$

for all $x, y \in D$ and $t \in [0, 1]$. If the above inequality stands only for one fixed $t \in]0, 1[$ and f is symmetric, we say that f is *t-Schur-convex.*

With special choices of f, we get known convexity notions:

- $f(x, y) = g(x) + g(y)$ gives Wright-convexity; (for more information about Wright-convexity, see e.g. [Wri54], [Ng87], [MNP91], [Olb06], [Olb11], [Olb15b], and the references therein);
- $f(x, y) = \max\{g(x), g(y)\}$ gives quasi-convexity. This concept has a great importance in optimization theory, game theory and others (see e.g. [Lue68], [ADSZ88], and the references therein).

Let $\Phi : D+D \to \mathbb{R}$ be an arbitrary function, then the function $f : D\times D \to \mathbb{R}$ defined by $f(x, y) = \Phi(x + y)$ satisfies (1) with equality. So, the class of Schur-affine functions is quite rich.

If a function is t-Wright-convex in the usual sense, then it is also t-Wright-convex for all t from a dense subset of $[0, 1]$. This was proved in [MNP91] by Maksa, Nikodem and Páles. The same question in connection with t-Schur-convexity is an open problem at this moment, however, we can state the following partial result.

Proposition 1.
- *If $f : D \times D \to \mathbb{R}$ is t-Schur-convex, then it is also $(1-t)$-Schur-convex.*
- *If $f : D \times D \to \mathbb{R}$ is t-Schur-convex, then it is also t_n-Schur-convex, where $t_1 = t$ and $t_{n+1} = (1-t)t_n + t(1-t_n)$.*

- *If $f : D \times D \to \mathbb{R}$ is t-Schur-convex and s-Schur-convex, then it is also $st + (1-s)(1-t)$-Schur-convex.*

Proof. The first one is a trivial consequence of inequality (1).

For the second one, substitute $x = tx + (1-t)y$ and $y = (1-t)x + ty$, using inequality (1) we have the statement by induction.

For the last one, $x = sx + (1-s)y$ and $y = (1-s)x + sy$, using inequality (1) we have the statement. □

Corollary 2. *If X is a normed linear space, $f : D \times D \to \mathbb{R}$ is lower semi-continuous, and t-Schur-convex for some t, then it is also $\frac{1}{2}$-Schur-convex.*

Proof. It is easy to see that the sequence t_n defined in the previous proposition tends to $\frac{1}{2}$ as n tends to ∞. From this and the lower semi-continuity assumption, we have our statement. □

Let us define the following auxiliary function.

$$\varphi_{x,y} \colon [0,1] \to \mathbb{R}, \qquad \varphi_{x,y}(t) := f(tx + (1-t)y, (1-t)x + ty)$$

for every $x, y \in D$.

Proposition 3. *A function $f : D \times D \to \mathbb{R}$ is $\frac{1}{2}$-Schur-convex if and only if the function $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$ for every fixed $x, y \in D$.*

Proof. Assume that f is $\frac{1}{2}$-Schur-convex. Take the inequality (1) at $t = \frac{1}{2}$, substitute x by $sx + (1-s)y$, and y by $(1-s)x + sy$, we get

$$\varphi_{x,y}\left(\tfrac{1}{2}\right) \le \varphi_{x,y}(s)$$

for every $s \in [0,1]$. So, the first part is ready.

Assume now that $\varphi_{x,y}(t)$ has a global minimum at $t = \frac{1}{2}$. Then

$$f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \le f\left(tx + (1-t)y, (1-t)x + ty\right) \qquad x, y \in D,\ t \in [0,1].$$

From this, we get the desired result with $t = 1$. □

The above statement motivates the following definition. With the previous notations, a function $f : D \times D \to \mathbb{R}$ is called *weakly $\frac{1}{2}$-Schur-convex* if $\varphi_{x,y}$ has a local minimum at $\frac{1}{2}$ for every $(x, y) \in D \times D$.

We can state more, if f is Schur-convex.

Theorem 4. *Let $f\colon D\times D\to\mathbb{R}$ be a symmetric function. Then f is Schur-convex if and only if for all arbitrarily fixed $x,y\in D$ the function $\varphi_{x,y}$ is monotone decreasing on $[0,\frac{1}{2}]$, monotone increasing on $[\frac{1}{2},1]$, and $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$.*

Proof. For the necessity, assume that f is Schur-convex.

Let $x,y\in D$ are arbitrarily fixed. Firstly, we intend to prove that $\varphi_{x,y}$ is monotone decreasing on $[0,\frac{1}{2}[$.

The Schur-convexity of f implies that

$$f(tu+(1-t)v),(1-t)u+tv)\le f(u,v)\qquad (u,v\in D,\ t\in[0,1]).$$

Let $0\le r<s<\frac{1}{2}$, then substitute u by $rx+(1-r)y$, v by $(1-r)x+ry$ and $t=\frac{s-r}{1-2r}$,

$$\begin{aligned}\varphi_{x,y}(r)&=f(rx+(1-r)y,(1-r)x+ry)\\&\ge f\Big\{\tfrac{s-r}{1-2r}(rx+(1-r)y)+\tfrac{1-s-r}{1-2r}((1-r)x+ry),\\&\quad\tfrac{1-s-r}{1-2r}(rx+(1-r)y)+\tfrac{s-r}{1-2r}((1-r)x+ry)\Big\}\\&=f(sx+(1-s)y,(1-s)x+sy)=\varphi_{x,y}(s).\end{aligned}$$

The proof of that $\varphi_{x,y}$ is monotone increasing on $]\frac{1}{2},1]$ comes from the fact that $\varphi_{x,y}(t)=\varphi_{x,y}(1-t)$.

Since f is also $\frac{1}{2}$-Schur-convex, we have that $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$.

For the sufficiency, assume that $\varphi_{x,y}$ is monotone decreasing on $[0,\frac{1}{2}[$, monotone increasing on $]\frac{1}{2},1]$, and it has a global minimum at $\frac{1}{2}$. Then

$$f(tx+(1-t)y,(1-t)x+ty)=\varphi_{x,y}(t)\le\varphi_{x,y}(0)=f(y,x)=f(x,y)$$

for all $t\in[0,\frac{1}{2}[$, and, similarly,

$$f(tx+(1-t)y,(1-t)x+ty)=\varphi_{x,y}(t)\le\varphi_{x,y}(1)=f(x,y)$$

for all $t\in]\frac{1}{2},1]$. The case $t=\frac{1}{2}$ is clear. So, the proof is complete. □

In [Olb15b, Remark 1], the author mentioned that the corresponding sufficiency part of the previous theorem is true in the case of Wright-convexity.

3. Characterization of regular Schur-convex functions

In this section, we examine the properties of directionally differentiable and differentiable Schur-convex functions. As a special case, we get classical characterization theorems of convex functions. We say that a function $g : D \to \mathbb{R}$ is *directionally differentiable* at a point $x_0 \in D$ in a direction $h \in X$ if there exists the limit

$$g'(x_0, h) := \lim_{t \to 0+} \frac{g(x_0 + th) - g(x_0)}{t}.$$

The function g is *directionally differentiable* if it is directionally differentiable at every point of D in any direction of X.

Theorem 5. *Assume that $f : D \times D \to \mathbb{R}$ is directionally differentiable and symmetric on $D \times D$. Then f is Schur-convex if and only if for $(x, y) \in D \times D$,*

$$\partial_{(x-y,y-x)} f(y, x) \leq 0. \tag{2}$$

Proof. First assume that f is Schur-convex, then by (1) we get that

$$\frac{f(y + t(x - y), x + t(y - x)) - f(y, x)}{t} \leq 0.$$

Taking the limit $t \to 0+$, we get (2).

Assume now that $\partial_{(x-y,y-x)} f(y, x) \leq 0$. Using the substitutions $x \leftrightarrow (1 - t)x + ty$ and $y \leftrightarrow tx + (1 - t)y$, we have

$$0 \geq \partial_{(1-2t)(x-y,y-x)} f((1 - t)x + ty, tx + (1 - t)y).$$

Assume that $t \in [0, \frac{1}{2}[$, because of the positive homogeneity of the directional derivative we have

$$0 \geq \partial_{(x-y,y-x)} f((1 - t)x + ty, tx + (1 - t)y) = \varphi'_{x,y}(t).$$

This proves that $\varphi_{x,y}$ is monotone decreasing on the interval $[0, \frac{1}{2}[$. Because of the identity $\varphi_{x,y}(t) = \varphi_{x,y}(1 - t)$, we get the monotone increasing property on the interval $]\frac{1}{2}, 1]$. Moreover, the directional differentiability of f entails differentiability of $\varphi_{x,y}$. So, $\varphi_{x,y}$ is continuous, which together with the previously mentioned monotonicity implies that $\varphi_{x,y}$ has a global minimum at $\frac{1}{2}$. Applying Theorem 4, we get that f is Schur-convex. $\square$

The next corollary (see [Sch23]) is an easy consequence of the previous theorem. Let I be a nonempty open real interval.

Corollary 6. *Assume that $f : I \times I \to \mathbb{R}$ is differentiable and symmetric on $I \times I$. Then f is Schur-convex if and only if for $(x, y) \in I \times I$,*

$$(\partial_1 f(x, y) - \partial_2 f(x, y))(x - y) \geq 0. \tag{3}$$

Proof. Let $f : I \times I \to \mathbb{R}$ be a function, then

$$\begin{aligned}\partial_{(x-y,y-x)} f(y, x) &= f'(y, x)(x - y, y - x)\\ &= (\partial_1 f(x, y), \partial_2 f(x, y))^T (x - y, y - x)\\ &= (\partial_1 f(x, y) - \partial_2 f(x, y))(x - y),\end{aligned}$$

which proves the statement. □

Example 7. As it has been mentioned earlier, if $f(x, y) = g(x) + g(y)$, we get the notion of Wright-convexity. It comes from the theorem of Ng [Ng87] that g is a sum of a convex and an additive function. If g is additionally differentiable, we have that g is convex in the ordinary sense. Using the previous theorem, we have that

$$(g'(x) - g'(y))(x - y) \geq 0, \qquad x, y \in I,$$

which is the well-known characterization inequality of differentiable, convex functions.

We will use the following simple formula for the following theorem. If f is symmetric and differentiable, it is easy to see that

$$\partial_1 f(u, u) = \partial_2 f(u, u) \qquad \text{for all} \qquad u \in D.$$

Theorem 8. *Assume that $f : I \times I \to \mathbb{R}$ is twice differentiable and Schur-convex on $I \times I$, then for all $y \in I$ and $h \in \mathbb{R} \setminus \{0\}$,*

$$(\partial_1^2 f(y, y) - \partial_1 \partial_2 f(y, y))(h, h) \geq 0. \tag{4}$$

Proof. Assume that f is Schur-convex, then using Theorem 5, f also satisfies (3). Substituting $x - y$ by th, where $h \in \mathbb{R}$ and t is arbitrary small positive number such that $th \in I$, then dividing by t^2,

$$\frac{(\partial_1 f(y + th, y) - \partial_2 f(y + th, y))h}{t} \geq 0.$$

Taking the limit $t \to 0$, then computing,

$$\begin{aligned}
&\lim_{t\to 0}\frac{(\partial_1 f(y+th,y)-\partial_2 f(y+th,y))h}{t}\\
&\quad=\lim_{t\to 0}\frac{\partial_1 f(y+th,y)-\partial_1 f(y,y)}{t}h-\lim_{t\to 0}\frac{\partial_2 f(y+th,y)-\partial_2 f(y,y)}{t}h\\
&\quad=\Big(\partial_1\Big(\lim_{t\to 0}\frac{f(y+th,y)-f(y,y)}{t}\Big)-\partial_2\Big(\lim_{t\to 0}\frac{f(y+th,y)-f(y,y)}{t}\Big)\Big)h\\
&\quad=(\partial_1^2 f(y,y)-\partial_1\partial_2 f(y,y))(h,h),
\end{aligned}$$

which proves the statement. □

Unfortunately, the reverse statement is not true, so we have not got a second-order characterization of smooth Schur-convex functions, however, we can prove a weaker statement.

Theorem 9. *Let $f : I \times I \to \mathbb{R}$ be twice continuously differentiable, symmetric function on $I \times I$, and assume that f satisfies* (4) *with strict inequality. Then f is weakly $\frac{1}{2}$-Schur-convex on $I \times I$.*

Proof. Let $x, y \in D$ be arbitrary fixed. Let us consider the function $\varphi_{x,y}(t) := f((1-t)x+ty, tx+(1-t)y)$ on $[0,1]$. Then

$$\begin{aligned}
\varphi'_{x,y}(t) &= \partial_1 f((1-t)x+ty, tx+(1-t)y)(y-x)\\
&\quad+\partial_2 f((1-t)x+ty, tx+(1-t)y)(x-y)\\
&=(\partial_1 f((1-t)x+ty, tx+(1-t)y)\\
&\quad-\partial_2 f((1-t)x+ty, tx+(1-t)y))(y-x).
\end{aligned}$$

Then, substituting $t=\frac{1}{2}$, and using the symmetry of the partial derivative, we have that

$$F'_{x,y}(\tfrac{1}{2})=\partial_1 f(\tfrac{x+y}{2},\tfrac{x+y}{2})(y-x)+\partial_2 f(\tfrac{x+y}{2},\tfrac{x+y}{2})(x-y)=0$$

Computing the second derivative of $\varphi_{x,y}$, we get that

$$\begin{aligned}
\varphi''_{x,y}(t) &= (\partial_1^2 f((1-t)x+ty, tx+(1-t)y)(y-x)\\
&\quad+\partial_1\partial_2 f((1-t)x+ty, tx+(1-t)y)(x-y))\\
&\quad-(\partial_1\partial_2 f((1-t)x+ty, tx+(1-t)y)(y-x)\\
&\quad+\partial_2^2 f((1-t)x+ty, tx+(1-t)y)(x-y))(y-x)\\
&=2\Big(\partial_1^2 f((1-t)x+ty, tx+(1-t)y)\\
&\quad-\partial_1\partial_2 f((1-t)x+ty, tx+(1-t)y)\Big)(y-x,y-x).
\end{aligned}$$

Then substituting $t = \frac{1}{2}$, and using (4) with strict inequality, we have that $\varphi''_{x,y}\left(\frac{1}{2}\right)$ is positive, thus $\varphi_{x,y}$ has a local minimum at $\frac{1}{2}$, which means that f is weakly $\frac{1}{2}$-Schur-convex on $I \times I$. $\square$

Example 10. Now, we give an example for smooth weakly $\frac{1}{2}$-Schur-convex function, which is not Schur-convex.

$$f(x,y) := -\sin(x)\sin(y) \qquad x, y \in \mathbb{R}$$

It can be seen that it is a symmetric and it is also continuously differentiable twice on $\mathbb{R} \times \mathbb{R}$. Computing the derivatives of f, we can get that, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} \partial_1 f(x,y) &= -\cos x \sin y \qquad & \partial_2 f(x,y) &= -\sin x \cos y \\ \partial_1^2 f(x,y) &= \sin x \sin y \qquad & \partial_1\partial_2 f(x,y) &= -\cos x \cos y \end{aligned}$$

Hence,

$$(\partial_1^2 f(y,y) - \partial_1\partial_2 f(y,y))(h,h) = (\sin^2(y) + \cos^2(y))(h,h) = 1 > 0,$$

which means that by Corollary 17, f is weakly $\frac{1}{2}$-Schur-convex. On the other hand, there exists $x, y \in \mathbb{R}$ (namely, $x = \frac{\pi}{2}, y = \frac{\pi}{4}$), such that

$$-\sin^2\left(\frac{x+y}{2}\right) > -\sin x \sin y,$$

which means that f is not $\frac{1}{2}$-Schur-convex.

4. Lower Hermite–Hadamard type inequalities for Schur-convex functions

For a function $f : D \times D \to \mathbb{R}$, we say that f is *hemi-P*, if, for all $x, y \in D$, the function

$$\varphi_{x,y}(t) = f((1-t)x + ty, tx + (1-t)y) \qquad (t \in [0,1]) \tag{5}$$

has property P. For example, f is hemi-integrable, if for all $x, y \in D$ the mapping defined by (5) is integrable.

In the sequel, denote by $C([0,1])$ and $B([0,1])$ the space of continuous and bounded Borel measurable real valued functions defined on the interval $[0,1]$

equipped with the usual supremum norm. Denote by $p_i\colon [0,1]\to\mathbb{R}$ the following polynomials:

$$p_i(u) := u^i, \qquad (i = 0,1,2).$$

Let μ be a Borel probability measure on $[0,1]$, and denote by μ_1 the first moment of μ, namely $\int_{[0,1]} t d\mu(t)$.

First, we recall a Korovkin type theorem, which will play an important role in the proof of the main result Theorem 11. For the historical background of these theorems, see the classical Korovkin theorem ([Kor53], [AC94], [MP12]), which has a great importance in functional analysis.

We say that a linear operator $\mathcal{T}\colon B([0,1])\to B([0,1])$ is *positive* if for all $g\geq 0$ from $B([0,1])$, $\mathcal{T}g\geq 0$. It is easy to see that if $\mathcal{T}$ is positive, then it is also monotone.

Let μ be a Borel probability measure on $[0,1]$, and define a sequence of linear operators $\mathcal{T}_n^\mu\colon B([a,b])\to B([a,b])$ by the following formula:

$$(\mathcal{T}_n^\mu\varphi)(u) := \int\limits_{[0,1]}\dots\int\limits_{[0,1]} \varphi\big(\tfrac12+\tfrac12(2t_1-1)\dots(2t_n-1)\big)d\mu(t_1)\dots d\mu(t_n)p_0(u). \quad (6)$$

The following theorem is the main goal of this section and gives a connection between a lower Hermite–Hadamard type inequality and $\frac12$-Schur-convexity.

Theorem 11. *Let $f\colon D\times D\to\mathbb{R}$ be hemi-bounded, lower hemi-continuous and symmetric. Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu\notin\{\alpha\delta_0+(1-\alpha)\delta_1 \mid \alpha\in[0,1]\}$. Moreover, assume that, for all $x,y\in D$, the function f satisfies the following Hermite–Hadamard type inequality:*

$$\int_{[0,1]} f(tx+(1-t)y,(1-t)x+ty)d\mu(t)\leq f(x,y). \quad (7)$$

Then f is $\frac12$-Schur-convex.

The proof of Theorem 11 is similar to the proof of the main theorem of [HM17], and it is based on the following lemmas.

Lemma 12. *If $f\colon D\times D\to\mathbb{R}$ is lower hemi-continuous and fulfills the approximate Hermite–Hadamard inequality (7), then, for all $n\in\mathbb{N}$, the function f also satisfies the Hermite–Hadamard inequality*

$$\int\limits_{[0,1]}\dots\int\limits_{[0,1]} f\big(T_nx+(1-T_n)y, f(1-T_n)x+T_ny\big)d\mu(t_1)\dots d\mu(t_n)\leq f(x,y) \quad (8)$$

for all $x,y\in D$, whenever $n\in\mathbb{N}$, where

$$T_1 = t_1, \quad \textit{and} \quad T_{n+1} = t_{n+1}(1-T_n)+(1-t_{n+1})T_n. \quad (9)$$

Proof. We prove by induction on n. If $n=1$, (8) holds. Let $x, y \in D$, and assume that (8) holds for some $n \in \mathbb{N}$. Write x by $(1-t_{n+1})x + t_{n+1}y$, and y by $t_{n+1}x + (1-t_{n+1})y$ in (8). Using the definition of T_{n+1}, we obtain:

$$\int\limits_{[0,1]} \dots \int\limits_{[0,1]} f(T_{n+1}x + (1-T_{n+1})y, (1-T_{n+1})x + T_{n+1}y\big) d\mu(t_1) \dots d\mu(t_n)$$
$$\leq f((1-t_{n+1})x + t_{n+1}y, t_{n+1}x + (1-t_{n+1})y)$$

Integrating with respect to t_{n+1}, and applying the inductive assumption and (7), we obtain that

$$\int\limits_{[0,1]} \dots \int\limits_{[0,1]} f\big(T_{n+1}x + (1-T_{n+1})y, (1-T_{n+1})x + T_{n+1}y\big) d\mu(t_1) \dots d\mu(t_{n+1})$$
$$\leq \int\limits_{[0,1]} f((1-t_{n+1})x + t_{n+1}y, t_{n+1}x + (1-t_{n+1})y) d\mu(t_{n+1}) \leq f(x,y),$$

which proves the statement. □

Now, we recall the following lemma from Házy–Makó [HM17].

Lemma 13. *Let T_n be definied by* (9), *then*

$$T_n = \tfrac{1}{2} - \tfrac{1}{2}(2t_1 - 1) \cdot \dots \cdot (2t_n - 1). \tag{10}$$

In the proof of the following, we also use the following Korovkin type theorem from [HM17].

Proposition 14. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$, and for all $n \in \mathbb{N}$ define $\mathcal{T}_n^\mu$ by* (11). *Then, for all lower semicontinuous $h \in B([0,1])$,*

$$h\big(\tfrac{1}{2}\big) \leq \liminf_{n\to\infty} (\mathcal{T}_n^\mu h)(u) \qquad (u \in [0,1]). \tag{11}$$

Lemma 15. *Let μ be a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$. If $f\colon D \times D \to \mathbb{R}$ is a symmetric lower hemicontinuous function, then*

$$\liminf_{n\to\infty} \int\limits_{[0,1]} \dots \int\limits_{[0,1]} f\big(T_n x + (1-T_n)y, (1-T_n)x + T_n y\big) d\mu(t_1) \dots d\mu(t_n)$$
$$\geq f\big(\tfrac{x+y}{2}, \tfrac{x+y}{2}\big). \tag{12}$$

Proof. Let $x, y \in D$ be fixed, and define $\varphi_{x,y}\colon [0,1] \to \mathbb{R}$ by

$$\varphi_{x,y}(t) := f\big((1-t)x + ty, tx + (1-t)y\big).$$

Since f is lower hemi-continuous, $\varphi_{x,y}$ is lower semi-continuous. Using Lemma 13, we have that the operator $\mathcal{T}_n^\mu$ defined by (11) can be expressed as

$$\mathcal{T}_n^\mu \varphi_{x,y} = \int\limits_{[0,1]} \dots \int\limits_{[0,1]} \varphi_{x,y}(T_n) d\mu(t_1) \dots d\mu(t_n).$$

By Proposition 14, we get that (12) also holds. $\square$

Proof of Theorem 11. Assume that the conditions of Theorem 11 hold, and $f\colon D \times D \to \mathbb{R}$ is an upper hemi-continuous solution of (7). Using Lemma 12, we obtain (8). Then taking the limit $\liminf$ in (8), then applying Lemma 15, we obtain that the function f is $\frac{1}{2}$-Schur-convex. $\square$

In the sequel, let D be a nonempty open convex subset of the normed space X (see [HM17]).

Corollary 16. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$. Let $\lambda \in \mathbb{R}$ and assume that $g\colon D \to \mathbb{R}$ is lower semi-continuous and, for all $x, y \in D$, satisfies the following Hermite–Hadamard type inequality*

$$\int_{[0,1]} g(tx + (1-t)y) d\mu(t) \le \lambda g(x) + (1-\lambda) g(y) \tag{13}$$

then g is Jensen-convex.

Proof. Assume that $g\colon D \to \mathbb{R}$ satisfies the inequality (13). Changing the role of x and y, and adding the inequalities, we obtained that

$$\int\limits_{[0,1]} g(tx + (1-t)y) + g((1-t)x + ty) d\mu(t) \le g(x) + g(y).$$

Applying Theorem 11 for the function

$$f(x, y) := g(x) + g(y),$$

we obtained that f is $\frac{1}{2}$-Schur-convex, which means that g is Jensen-convex. $\square$

The following corollary provides connection between a lower Hermite–Hadamard inequality and quasi-convexity.

Corollary 17. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$. Let $\lambda \in \mathbb{R}$, and assume $g\colon D \to \mathbb{R}$ is lower semi-continuous, and, for all $x, y \in D$, satisfies the following Hermite–Hadamard type inequality*

$$\int\limits_{[0,1]} \max\big(g(tx+(1-t)y), g((1-t)x+ty)\big)d\mu(t) \leq \max(g(x), g(y)),$$

then g is Jensen-quasi-convex, i.e. it satisfies the inequality:

$$g\Big(\frac{x+y}{2}\Big) \leq \max(g(x), g(y)) \qquad (x, y \in D). \tag{14}$$

Proof. Applying Theorem 11 for the function

$$f(x,y) = \max(g(x), g(y)),$$

we get the required inequality (14). □

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