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Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds

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Abstract. The object of the present paper is to study a 3-dimensional normal almost contact metric manifold admitting Ricci solitons and gradient Ricci solitons. At first we give an example of a 3-dimensional normal almost contact metric manifold with $\alpha, \beta =$ constant. We prove that a 3-dimensional normal almost contact metric manifold admitting a Ricci soliton with a potential vector field V collinear with the characteristic vector field ξ, is η-Einstein provided $\alpha, \beta =$ constant. Also we show that an η-Einstein 3-dimensional normal almost contact metric manifold with $\alpha, \beta =$ constant and $V = \xi$ admits a Ricci soliton. Finally we prove that if in a 3-dimensional normal almost contact metric manifold with constant scalar curvature, g is a gradient Ricci soliton, then the manifold is either α-Kenmotsu or an Einstein manifold provided $\alpha, \beta =$ constant.

1. Introduction

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton if [15]

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{1.1}$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a smooth vector field on M (called the potential vector field) and λ is a constant. Metrics satisfying (1.1) are interesting and useful in physics and are often referred as quasi-Einstein (e.g. [7], [8], [12]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$

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projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan who discusses some aspects of it [12].

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

A Ricci soliton on a compact manifold has constant curvature in dimension 2 [15] and also in dimension 3 [16]. For details we refer to Chow and Knopf [9]. We also recall the following significant result of Perelman [22]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

The roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory. For more details see [1], [13], [18] and [19].

In [23], Sharma started the study of Ricci solitons in K-contact manifolds. Also, in a subsequent paper [14] Ghosh, Sharma and Cho studied gradient Ricci soliton of a non-Sasakian (k, μ)-contact manifold. In a K-contact manifold the structure vector field ξ is Killing, that is, $\pounds_\xi g = 0$, which is not in general, in a normal almost contact metric manifold. Recently in [5] C. Calin and M. Crasmareanu have studied Ricci solitons in $f-$Kenmotsu manifolds.

Motivated by these circumtances, in this paper we study Ricci solitons and gradient Ricci solitons in 3-dimensional normal almost contact metric manifolds.

The paper is organized as follows: After preliminaries in section 3 among others we prove that in a 3-dimensional normal a.c.m. manifold if g is a Ricci soliton and the vector field V point-wise collinear with ξ, then V is a constant multiple of ξ and g is η-Einstein and also we show that an η-Einstein 3-dimensional normal almost contact metric manifold with α, β = constant and $V = \xi$ admits a Ricci soliton. Finally we prove that if a 3-dimensional normal a.c.m. manifold admits a gradient Ricci soliton, then the manifold is either α-Kenmotsu or Einstein manifold provided α, β = constant. We obtain some consequences of this result.

2. Preliminaries

Let M be an almost contact manifold and (ϕ, ξ, η) its almost contact structure. This means, M is an odd-dimensional differentiable manifold and ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$, $(0,1)$, respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Then also $\phi\xi = 0$, $\eta \circ \phi = 0$. Let $\mathbb{R}$ be the real line and t a coordinate on $\mathbb{R}$. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, \frac{fd}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where the pair $(X, fd/dt)$ denotes a tangent vector to $M \times \mathbb{R}$, f is a smooth function on $M \times \mathbb{R}$, X and fd/dt being tangent to M and $\mathbb{R}$, respectively.

M and (ϕ, ξ, η) are said to be normal if the structure J is integrable [2], [3]. The necessary and sufficient condition for (ϕ, ξ, η) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for any $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M.

A Riemannian metric g on M satisfying the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$, is said to be compatible with the structure (ϕ, ξ, η). If g is such a metric, then the quadruple (ϕ, ξ, η, g) is called an almost contact metric (shortly a.c.m.) structure on M and M is an almost contact metric manifold. On such a manifold we also have $\eta(X) = g(X, \xi)$ for any $X \in \chi(M)$ and we can always define the 2-form Φ by $\Phi(X, Y) = g(X, \phi Y)$, where $X, Y \in \chi(M)$.

It is no hard to see that if $\dim M = 3$, then two Riemannian metrics g and g' are compatible with the same almost contact structure (ϕ, ξ, η) on M if and only if $g' = \sigma g + (1 - \sigma)\eta \otimes \eta$, for a certain positive function σ on M.

A normal a.c.m. (ϕ, ξ, η, g) satisfying additionally the condition $d\eta = \Phi$ is called Sasakian. Also a normal a.c.m. structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [4].

For an a.c.m. structure (ϕ, ξ, η, g) on M, we have [20]

$$(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y)\xi - \eta(Y)\phi \nabla_X \xi, \tag{2.1}$$

$$\nabla_X \xi = \alpha\{X - \eta(X)\xi\} - \beta \phi X, \tag{2.2}$$

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) - \eta(X)\eta(Y)\} - \beta g(\phi X, Y), \tag{2.3}$$

where $2\alpha = \operatorname{div}\xi$ and $2\beta = \operatorname{tr}(\phi\nabla\xi)$, $\operatorname{div}\xi$ is the divergence of ξ defined by $\operatorname{div}\xi = \operatorname{trace}\{X \longrightarrow \nabla_X \xi\}$ and $\operatorname{tr}(\phi\nabla\xi) = \operatorname{trace}\{X \longrightarrow \phi\nabla_X \xi\}$.

$$\begin{aligned} R(X,Y)\xi = {} & \{Y\alpha + (\alpha^2 - \beta^2)\eta(Y)\}\phi^2 X - \{X\alpha + (\alpha^2 - \beta^2)\eta(X)\}\phi^2 Y \\ & + \{Y\beta + 2\alpha\beta\eta(Y)\}\phi X - \{X\beta + 2\alpha\beta\eta(X)\}\phi Y, \end{aligned} \tag{2.4}$$

$$S(Y,\xi) = -Y\alpha - (\phi Y)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\}\eta(Y), \tag{2.5}$$

$$\xi\beta + 2\alpha\beta = 0, \tag{2.6}$$

where R denotes the curvature tensor and S is the Ricci tensor.

On the other hand, the curvature tensor in dimension three always satisfies [24]

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = {} & g(X,W)S(Y,Z) - g(X,Z)S(Y,W) \\ & + g(Y,Z)S(X,W) - g(Y,W)S(X,Z) \\ & - \frac{r}{2}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)], \end{aligned} \tag{2.7}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and r is the scalar curvature.

From (2.4), we can derive that

$$\tilde{R}(\xi,Y,Z,\xi) = -(\xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z) - (\xi\beta + 2\alpha\beta)g(Y, \phi Z). \tag{2.8}$$

By (2.5), (2.7) and (2.8), we obtain for $\alpha, \beta = \text{constant}$,

$$S(Y,Z) = \left(\frac{r}{2} + \alpha^2 - \beta^2\right) g(\phi Y, \phi Z) - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z), \tag{2.9}$$

and

$$QY = \left(\frac{r}{2} + \alpha^2 - \beta^2\right) Y - \left(\frac{r}{2} - \alpha^2 + \beta^2\right) \eta(Y)\xi. \tag{2.10}$$

Applying (2.9) in (2.7), we get

$$R(X,Y)Z = \left[\frac{r}{2} + 2(\alpha^2 - \beta^2)\right][g(Y,Z)X - g(X,Z)Y]$$

$$
\begin{aligned}
&+ g(X,Z)\left[\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\eta(Y)\xi\right] \\
&- \left[\frac{r}{2}+3(\alpha^2-\beta^2)\right]\eta(Y)\eta(Z)X \\
&- g(Y,Z)\left[\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\eta(X)\xi\right] \\
&+ \left[\frac{r}{2}+3(\alpha^2-\beta^2)\right]\eta(X)\eta(Z)Y. \qquad (2.11)
\end{aligned}
$$

It is to be noted that the general formulas can be obtained by straightforward calculation.

From (2.6) it follows that if $\alpha, \beta =$ constant, then the manifold is either β-Sasakian, or α-Kenmotsu, or cosymplectic [17].

Proposition 2.1. *A three-dimensional normal a.c.m. manifold with* $\alpha, \beta =$ constant *is either β-Sasakian, or α-Kenmotsu or cosymplectic.*

We note that β-Sasakian manifolds are quasi Sasakian [4], [21].

Cosympletic manifolds provide a natural setting for time dependent mechanical systems as they are locally product of a Kaehler manifold and a real line or a circle [6].

3. Example of a 3-dimensional normal almost contact metric manifold

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are standard coordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0,$$

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,$$

that is, the form of the metric becomes

$$g = \frac{dx^2+dy^2+dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\begin{aligned} &\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \\ &g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W), \end{aligned}$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi–Civita connection with respect to metric g. Then we have

$$\begin{aligned} [e_1, e_3] = e_1e_3 - e_3e_1 &= z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) \\ &= z^2\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial z \partial x} - z\frac{\partial}{\partial x} = -e_1. \end{aligned}$$

Similarly

$$[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{3.1}$$

which is known as Koszul's formula.

Using (3.1) we have

$$2g(\nabla_{e_1} e_3, e_1) = -2g(e_1, e_1) = 2g(-e_1, e_1). \tag{3.2}$$

Again by (3.1)

$$2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2) \tag{3.3}$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3). \tag{3.4}$$

From (3.2), (3.3) and (3.4) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),$$

for all $X \in \chi(M)$. Thus

$$\nabla_{e_1} e_3 = -e_1.$$

Therefore, (3.1) further yields

$$\begin{aligned} &\nabla_{e_1} e_3 = -e_1, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = e_3, \\ &\nabla_{e_2} e_3 = -e_2, && \nabla_{e_2} e_2 = e_3, && \nabla_{e_2} e_1 = 0, \\ &\nabla_{e_3} e_3 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_1 = 0. \end{aligned} \tag{3.5}$$

(3.5) tells us that the manifold satisfies (2.2) for $\alpha = -1$ and $\beta = 0$ and $\xi = e_3$. Hence the manifold is a normal almost contact metric manifold with α, β = constant.

4. Ricci soliton

Suppose a 3-dimensional normal a.c.m. manifold admits a Ricci soliton defined by (1.1). It is well known that $\nabla g = 0$. Since λ in the Ricci soliton equation (1.1) is a constant, so $\nabla \lambda g = 0$. Thus $\pounds_V g + 2S$ is parallel. In [10] the authors prove that if a $3-$dimensional normal a.c.m. manifold admits a symmetric parallel $(0, 2)$ tensor, then the tensor is a constant multiple of the metric tensor. Hence $\pounds_V g + 2S$ is a constant multiple of the metric tensor g, i.e., $\pounds_V g + 2S = ag$, where a is non-zero constant. Hence $\pounds_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$. Using (1.1) we get $\lambda = -a/2$. So we have the following:

Proposition 4.1. *In a 3 dimensional normal a.c.m. manifold the Ricci soliton (g, λ, V) is shrinking or expanding according as a is positive or negative.*

In particular, let V be point-wise collinear with ξ i.e. $V = b\xi$, where b is a function on the 3-dimensional normal a.c.m. manifold. Then

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0,$$

which implies that

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

or,

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0.$$

Using (2.2), we obtain

$$bg(\alpha(X-\eta(X)\xi)-\beta\phi X, Y)+(Xb)\eta(Y)+bg(\alpha(Y-\eta(Y)\xi)-\beta\phi Y, X)$$
$$+(Yb)\eta(X)+2S(X,Y)+2\lambda g(X,Y)=0,$$

which yields

$$2b\alpha g(X,Y)-2\alpha b\eta(X)\eta(Y)+(Xb)\eta(Y)+(Yb)\eta(X)$$
$$+2S(X,Y)+2\lambda g(X,Y)=0. \quad (4.1)$$

In (4.1) replacing Y by ξ and using (2.9) it follows that

$$Xb+(\xi b)\eta(X)+2(-2(\alpha^2-\beta^2)\eta(X))+2\lambda\eta(X)=0. \quad (4.2)$$

Replacing X by ξ in (4.2) we get

$$\xi b=2(\alpha^2-\beta^2)-\lambda. \quad (4.3)$$

Putting this value in (4.2), we obtain

$$db=\{2(\alpha^2-\beta^2)-\lambda\}\eta. \quad (4.4)$$

Applying d on (4.4) we get

$$\{\lambda+2(\alpha^2-\beta^2)\}d\eta=0. \quad (4.5)$$

Since $d\eta\neq 0$ in a normal almost contact metric manifold, we have

$$2(\alpha^2-\beta^2)-\lambda=0. \quad (4.6)$$

Using (4.6) in (4.4) yields b is a constant. Therefore from (4.1) it follows

$$S(X,Y)=-(\lambda+\alpha b)g(X,Y)+\alpha b\eta(X)\eta(Y), \quad (4.7)$$

which implies M is an η-Einstein manifold. This leads to the following:

Theorem 4.1. *If in a 3-dimensional non-cosymplectic normal a.c.m. manifold the metric g is a Ricci soliton and V is point-wise collinear with ξ, then V is a constant multiple of ξ and g is η-Einstein provided* $\alpha,\beta=$ constant.

Conversely, let M be an 3-dimensional η-Einstein normal a.c.m. manifold with $\alpha, \beta = \text{constant}$ and $V = \xi$. Then

$$S(X,Y) = \gamma g(X,Y) + \delta \eta(X)\eta(Y), \tag{4.8}$$

where γ and δ are certain scalars.

Now using (2.2)

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2\alpha(g(X,Y) - \eta(X)\eta(Y)),$$

Therefore

$$\begin{aligned}(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 2(\alpha+\gamma+\lambda)g(X,Y)\\ - 2(\alpha-\delta)\eta(X)\eta(Y).\end{aligned} \tag{4.9}$$

From equation (4.9) it follows that M admits a Ricci soliton (g, ξ, λ) if $\alpha+\gamma+\lambda = 0$ and $\delta = \alpha = \text{constant}$. From (4.8) we have using (2.9), $-2(\alpha^2 - \beta^2) = \gamma + \delta$. Hence $\gamma = -2(\alpha^2 - \beta^2) - \alpha = \text{constant}$. So we have the following:

Theorem 4.2. *If a 3-dimensional non-cosymplectic normal a.c.m. manifold with $\alpha, \beta = \text{constant}$, is η-Einstein of the form $S = \gamma g + \delta \eta \otimes \eta$, then the manifold admits a Ricci soliton $(g, \xi, -(\gamma+\delta))$.*

Now let $V = \xi$. Then the equation (1.1) reduces to

$$\pounds_\xi g + 2S + 2\lambda g = 0. \tag{4.10}$$

Using (2.2), we get

$$(\pounds_\xi g)(X,Y) = 2\alpha\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{4.11}$$

Therefore

$$A(X,Y) = (\pounds_\xi g + 2S)(X,Y) = (\pounds_\xi g)(X,Y) + 2S(X,Y). \tag{4.12}$$

Now using (2.9) and (4.11) from (4.12) we obtain

$$\begin{aligned}A(X,Y) = \{&(\tau + 2(\alpha^2 - \beta^2 + \alpha))g(X,Y)\\ &- (\tau + 2(3(\alpha^2 - \beta^2) + \alpha))\eta(X)\eta(Y)\}.\end{aligned} \tag{4.13}$$

Applying (4.13) in (4.10) we get

$$\{\tau + 2(\alpha^2 - \beta^2) + \alpha) + \lambda\}g(X,Y) - \left\{\beta\frac{\tau}{2} + 3(\alpha^2 - \beta^2)\right\}\eta(X)\eta(Y) = 0. \quad (4.14)$$

Now taking $X = Y = \xi$ in (4.14) we obtain

$$\lambda = -\frac{1}{3}(\tau + 2\alpha). \quad (4.15)$$

Also

$$(\nabla_Z A)(X,Y) = d\tau(Z)(g(X,Y) - \eta(X)\eta(Y))$$
$$-(\tau + 6(\alpha^2 - \beta^2) + 2\alpha)\{(\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)\}. \quad (4.16)$$

From the proof of proposition 2 we find that the tensor A is parallel. Therefore putting $X = Y = e_i$ in (4.16), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and using (2.3) we obtain

$$d\tau = 0,$$

which implies that the scalar curvature τ is constant. Hence we have the followings:

Theorem 4.3. *If a 3-dimensional non-cosymplectic normal a.c.m. manifold admits a Ricci soliton (g, ξ, λ), then the manifold is of constant scalar curvature provided $\alpha, \beta =$ constant.*

Theorem 4.4. *If a 3-dimensional non-cosymplectic normal a.c.m. manifold admits a Ricci soliton (g, ξ, λ), then the Ricci soliton is shrinking, steady and expanding according as $\tau + 2\alpha > 0$, $\tau + 2\alpha = 0$ and $\tau + 2\alpha < 0$ respectively provided $\alpha, \beta =$ constant.*

In [23] Sharma proved that a compact Ricci soliton of constant scalar curvature is Einstein. Hence from Theorem 4.3 we state the following:

Corollary 4.1. *If a 3-dimensional normal non-cosymplectic a.c.m. manifold admits a compact Ricci soliton, then the manifold is Einstein provided $\alpha, \beta =$ constant.*

5. Gradient Ricci soliton

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and (1.1) assume the form

$$\nabla\nabla f = S + \lambda g. \quad (5.1)$$

This reduces to

$$\nabla_Y Df = QY + \lambda Y, \tag{5.2}$$

where D denotes the gradient operator of g. From (5.2) it is clear that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{5.3}$$

Differentiating (2.10) we have

$$\begin{aligned}(\nabla_W Q)(X) = {} & \frac{d\tau(W)}{2}(X - \eta(X)\xi) \\ & - \left(\frac{\tau}{2} + 3(\alpha^2 - \beta^2)\right)\{(\nabla_W \eta)(X)\xi - \eta(X)\nabla_W \xi\}.\end{aligned} \tag{5.4}$$

In (5.4) replacing W by ξ yields

$$\begin{aligned}(\nabla_\xi Q)(X) = {} & \frac{d\tau(\xi)}{2}(X - \eta(X)\xi) \\ & - \left(\frac{\tau}{2} + 3(\alpha^2 - \beta^2)\right)\{(\nabla_\xi \eta)(X)\xi - \eta(X)\nabla_\xi \xi\}.\end{aligned} \tag{5.5}$$

Then we have

$$\begin{aligned}g((\nabla_\xi Q)(X) - (\nabla_X Q)(\xi), \xi) = {} & -\left(\frac{\tau}{2} + 3(\alpha^2 - \beta^2)\right)\{(\nabla_\xi \eta)(X) \\ & - g((\nabla_X \eta)\xi, \xi) + g(\nabla_X \xi, \xi)\} = 0.\end{aligned} \tag{5.6}$$

Using (5.6) from (5.3), we obtain

$$g(R(\xi, X)Df, \xi) = 0. \tag{5.7}$$

From (2.8) we get

$$R(\xi, Y, Df, \xi) = (\alpha^2 - \beta^2)\{\eta(X)\eta(Df) - g(X, Df)\} + 2\alpha\beta g(\phi X, Df).$$

Using (5.7) in the above equation yields

$$(\alpha^2 - \beta^2)\{\eta(X)\eta(Df) - g(X, Df)\} + 2\alpha\beta g(\phi X, Df) = 0. \tag{5.8}$$

Replacing X by ϕX in (5.8) we get

$$-(\alpha^2 - \beta^2)g(\phi X, Df) - 2\alpha\beta g(X, Df) + 2\alpha\beta\eta(X)g(\xi, Df) = 0.$$

Replacing the value of $g(\phi X, Df)$ in(5.8), since $\alpha \neq \beta$ in a 3-dimensional normal almost contact metric manifold, we obtain

$$Df = (\xi f)\xi. \tag{5.9}$$

Using (5.9) in (5.2) we have

$$\begin{aligned} S(X,Y) + \lambda g(X,Y) &= g(\nabla_Y Df, X) = g(\nabla_Y (\xi f)\xi, X) \\ &= g(Y(\xi f)\xi + (\xi f)\nabla_Y \xi, X) \\ &= Y(\xi f)\eta(X) + (\xi f)\alpha\{g(X,Y) \\ &\quad - \eta(Y)\eta(X)\} - (\xi f)\beta g(\phi Y, X) \end{aligned} \tag{5.10}$$

Putting $X = \xi$ in (5.10) and using (2.5) we get

$$S(Y,\xi) + \lambda\eta(Y) = Y(\xi f) = \{\lambda - 2(\alpha^2 - \beta^2)\}\eta(Y). \tag{5.11}$$

Interchanging X and Y in (5.10) we obtain

$$\begin{aligned} S(X,Y) + \lambda g(X,Y) &= X(\xi f)\eta(Y) + (\xi f)\alpha\{g(X,Y) \\ &\quad - \eta(Y)\eta(X)\} - (\xi f)\beta g(\phi X, Y). \end{aligned} \tag{5.12}$$

Adding (5.10) and (5.12) we get

$$\begin{aligned} 2S(X,Y) + 2\lambda g(X,Y) &= 2\beta(\xi f)g(X,Y) - 2\beta(\xi f)\eta(X)\eta(Y) \\ &\quad + Y(\xi f)\eta(X) + X(\xi f)\eta(Y). \end{aligned} \tag{5.13}$$

Using (5.11) in (5.13) we have

$$S(X,Y)+\lambda g(X,Y) = (\xi f)\alpha g(X,Y)+\{\lambda-2(\alpha^2-\beta^2)-(\xi f)\alpha\}\eta(X)\eta(Y). \tag{5.14}$$

Then using (5.2) we have

$$\nabla_Y Df = \alpha(\xi f)Y + \{\lambda - 2(\alpha^2 - \beta^2)(\xi f)\alpha\}\eta(Y)\xi. \tag{5.15}$$

Using (5.15) we calculate

$$\begin{aligned} R(X,Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= \alpha X(\xi f)Y - \alpha X(\xi f)\eta(Y)\xi - \alpha Y(\xi f)X + \alpha Y(\xi f)\eta(X)\xi \\ &\quad + (\lambda - 2(\alpha^2 - \beta^2) - \alpha(\xi f))\{(\nabla_X \eta)(Y)\xi - (\nabla_Y \eta)(X)\xi\} \\ &\quad + (\lambda - 2(\alpha^2 - \beta^2) - \alpha(\xi f))\{\eta(Y)\nabla_X \xi - \eta(X)\nabla_Y \xi\}. \end{aligned} \tag{5.16}$$

Taking inner product with ξ in (5.16),we get

$$0 = g(R(X,Y)Df, \xi) = 2\beta(\lambda - 2(\alpha^2 - \beta^2) - \alpha(\xi f))g(\phi Y, X).$$

Thus we have

$$2\beta(\lambda - 2(\alpha^2 - \beta^2) - \alpha(\xi f)) = 0.$$

Now we consider the following cases:
Case i) $\beta = 0$,
Case ii) $\lambda + 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$,
Case iii) $\beta = 0$ and $\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f) = 0$.

Case i) If $\beta = 0$, then the manifold reduces to a $\alpha-$Kenmotsu manifold.

Case ii) Let $\lambda + 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$. Using this in (5.11) yields

$$Y(\xi f) = \alpha(\xi f)\eta(Y).$$

Substituting this value in (5.13) we obtain

$$S(X,Y) + \lambda g(X,Y) = \alpha(\xi f)g(X,Y).$$

Contracting this equation, we get

$$\tau + 3\lambda = 3\alpha(\xi f),$$

which implies that

$$(\xi f) = \frac{\tau}{3\alpha} + \frac{\lambda}{\alpha}.$$

If $\tau =$ constant, then $(\xi f) =$ constant $= c\,(say)$. Therefore from (5.9) we have

$$Df = (\xi f)\xi = c\xi.$$

Thus we can write from this equation

$$g(Df, X) = c\eta(X),$$

which means that

$$df(X) = c\eta(X).$$

Applying d on the above equation, we get

$$cd\eta = 0.$$

Since $d\eta \neq 0$ in a contact metric manifold, we have $c = 0$. Hence we get $Df = 0$. This means that $f =$ constant. Therefore equation (5.1) reduces to

$$S(X,Y) = 2(\alpha^2 - \beta^2)g(X,Y),$$

that is, M is an Einstein manifold.

Case iii) Using $\beta = 0$ and $\lambda + 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$ in (5.11) we obtain $Y(\xi f) = \alpha(\xi f)\eta(Y)$. Now as in *Case ii)* we conclude that the manifold is an Einstein manifold.

Thus we have the following:

Theorem 5.1. *If a 3-dimensional non-cosymplectic normal a.c.m. manifold with constant scalar curvature admits gradient Ricci soliton, then the manifold is either α-Kenmotsu or an Einstein manifold provided $\alpha, \beta =$ constant.*

In [11], De, Yildiz and Yaliniz proved that a 3-dimensional normal a.c.m. manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided $\alpha, \beta =$ constant. Hence from Theorem 5.1 we obtain the following:

Corollary 5.1. *If a locally ϕ-symmetric 3-dimensional non-cosymplectic normal a.c.m. manifold admits gradient Ricci soliton, then the manifold is either α-Kenmotsu or an Einstein manifold provided $\alpha, \beta =$ constant.*

Using the result of Perelman [22], we can state the following:

Corollary 5.2. *If the metric g of a compact 3-dimensional non-cosymplectic normal a.c.m. manifold with constant scalar curvature is a Ricci soliton, then the manifold is either α-Kenmotsu or an Einstein manifold provided $\alpha, \beta =$ constant.*

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