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On the monotonicity of an additive representation function

By SÁNDOR Z. KISS (Budapest)

Abstract. Let $\mathcal{A} = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ be an infinite sequence of positive integers, and let $k \geq 2$ be a fixed integer. Let $r_1(\mathcal{A}, n, k)$ denote the number of solutions of $a_{i_1} + a_{i_2} + \dots + a_{i_k} = n$, $a_{i_1} \in \mathcal{A}, \dots, a_{i_k} \in \mathcal{A}$. For $k = 2$, P. Erdős, A. Sárközy and V. T. Sós studied the monotonicity of $r_1(\mathcal{A}, n, k)$. In this paper I extend one of their results to any $k > 2$.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, and let $k \geq 2$ be a fixed integer. Let $\mathcal{A} = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ be an infinite sequence of positive integers, and put

$$A(n) = \sum_{\substack{a \in \mathcal{A} \\ a \leq n}} 1.$$

For $k \geq 2$ integer and $\mathcal{A} \subset \mathbb{N}$, let $r_1(\mathcal{A}, n, k)$, $r_2(\mathcal{A}, n, k)$, $r_3(\mathcal{A}, n, k)$ denote the number of solutions of the equation

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \in \mathcal{A}, \dots, a_{i_k} \in \mathcal{A},$$

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \in \mathcal{A}, \dots a_{i_k} \in \mathcal{A}, \qquad a_{i_1} < a_{i_2} < \dots < a_{i_k},$$

and

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \in \mathcal{A}, \dots a_{i_k} \in \mathcal{A}, \qquad a_{i_1} \leq a_{i_2} \leq \dots \leq a_{i_k},$$

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respectively. For $i = 1, 2, 3$ we say $r_i(\mathcal{A}, n, k)$ is monotonous increasing in n from a certain point on, if there exists an integer n_0 with

$$r_i(\mathcal{A}, n+1, k) \geq r_i(\mathcal{A}, n, k) \quad \text{for } n \geq n_0.$$

In the special case $k = 2$ we write $r_i(n) = r_i(\mathcal{A}, n, 2)$ for $i = 1, 2, 3$. In a series of papers P. Erdős, A. Sárközy and V. T. Sós studied the monotonicity properties of the three representation functions $r_1(n)$, $r_2(n)$, $r_3(n)$. In [1] they proved the following theorems:

Theorem 1. *The function $r_1(n)$ is monotonous increasing from a certain point on, if and only if the sequence $\mathcal{A}$ contains all the integers from a certain point on, i.e., there exists an integer n_1 with*

$$\mathcal{A} \cap \{n_1, n_1+1, n_1+2, \dots\} = \{n_1, n_1+1, n_1+2, \dots\}.$$

Theorem 2. *If*

$$A(n) = o\Big(\frac{n}{\log n}\Big)$$

then the functions $r_2(n)$ and $r_3(n)$ cannot be monotonous increasing from a certain point on, i.e., for $j = 2$ or 3, there does not exist an integer n_0 such that

$$r_j(n+1) \geq r_j(n) \quad \text{for } n \geq n_0.$$

A. Sárközy proposed the study of the monotonicity of the functions $r_i(\mathcal{A}, n, k)$ for $k > 2$ [2, Problem 5]. He conjectured [3, p. 337] that for any $k \geq 2$ integer, if $r_i(\mathcal{A}, n, k)$ $(i = 1, 2, 3)$ is monotonous increasing in n from a certain point on, then $A(n) = O(n^{2/k-\varepsilon})$ cannot hold. In this paper I will prove the following slightly stronger result on $r_1(\mathcal{A}, n, k)$ by using similar methods as in [1]:

Theorem 3. *If $k \in \mathbb{N}$, $k \geq 2$, $\mathcal{A} \subset \mathbb{N}$ and $r_1(\mathcal{A}, n, k)$ is monotonous increasing in n from a certain point on, then*

$$A(n) = o\bigg(\frac{n^{2/k}}{(\log n)^{2/k}}\bigg)$$

cannot hold.

Unfortunately I have not been able to prove the conjecture for $r_2(\mathcal{A}, n, k)$ and $r_3(\mathcal{A}, n, k)$, thus the conjecture remains open in these cases.

2. Proof of Theorem 3

We write $r_1(\mathcal{A}, n, k) = R_k(n)$. We prove the result by contradiction. Assume that $R_k(n)$ is monotonous increasing from a certain point on and $A(n) = o\big(\frac{n^{2/k}}{(\log n)^{2/k}}\big)$. First we show that there exist infinitely many integers N satisfying

$$A(N+j) < A(N)\left(\frac{N+j}{N}\right)^2 \quad \text{for } j = 1, 2, \dots. \tag{1}$$

If (1) holds only for finitely many N, then there exists an integer N_0 such that

$$A(N_0) > 1$$

and for $N \geq N_0$, there exists an integer $N' = N'(N)$ satisfying $N' > N$ and

$$A(N') \geq A(N)\left(\frac{N'}{N}\right)^2.$$

Then we get by induction that there exist integers $N_1 < N_2 < \dots < N_j < \dots$ such that

$$A(N_{j+1}) \geq A(N_j)\left(\frac{N_{j+1}}{N_j}\right)^2 \quad \text{for } j = 0, 1, 2, \dots,$$

hence

$$\begin{aligned} A(N_{l+1}) &= A(N_0)\prod_{j=0}^{l}\frac{A(N_{j+1})}{A(N_j)} \geq A(N_0)\prod_{j=0}^{l}\left(\frac{N_{j+1}}{N_j}\right)^2 \\ &= A(N_0)\left(\frac{N_{l+1}}{N_0}\right)^2 > \left(\frac{N_{l+1}}{N_0}\right)^2 > N_{l+1}^{3/2} \end{aligned} \tag{2}$$

for large enough l. On the other hand, clearly we have

$$A(N_{l+1}) = \sum_{\substack{a\in\mathcal{A}\\ a\leq N_{l+1}}} 1 \leq \sum_{a\leq N_{l+1}} 1 = N_{l+1} \tag{3}$$

(2) and (3) cannot hold simultaneously and this contradiction proves the existence of infinitely many integers N satisfying (1).

Throughout the remaining part of the proof of Theorem 3 we use the following notations: N denotes a large integer satisfying (1). We write $e^{2i\pi\alpha} = e(\alpha)$ and we put $r = e^{-1/N}$, $z = re(\alpha)$ where α is a real variable (so that a function of form $p(z)$ is a function of the real variable $\alpha : p(z) = p(re(\alpha)) = P(\alpha)$). We write

$$f(z) = \sum_{j=1}^{+\infty} z^{a_j}.$$

(Since $r < 1$, this infinite series and all the other infinite series in the remaining part of the proof are absolutely convergent.) Then we have

$$f^k(z) = \sum_{n=1}^{+\infty} R_k(n) z^n.$$

Let I denote

$$I = \int_0^1 |f(z)|^k d\alpha.$$

We will give lower and upper bound for I. The lower bound will be greater then the upper bound, and this contradiction will prove that our indirect assumption cannot hold which will complete the proof of Theorem 3.

First we will give lower bound for I. Using Hölder's inequality and Parseval's formula we have

$$\begin{aligned} I^{2/k} &= \left(\int_0^1 |f(z)|^k d\alpha \right)^{2/k} \left(\int_0^1 1 d\alpha \right)^{1-2/k} \geq \int_0^1 |f(z)|^2 d\alpha \\ &= \sum_{a \in \mathcal{A}} r^{2a} \geq \sum_{\substack{a \in \mathcal{A} \\ a \leq N}} r^{2N} = e^{-2} \sum_{\substack{a \in \mathcal{A} \\ a \leq N}} 1 = e^{-2} A(N) \end{aligned}$$

hence

$$I \geq e^{-k} (A(N))^{k/2}. \tag{4}$$

Now we will give upper bound for I. First we will estimate $R_k(n)$ in terms of $A(2n)$. Since $R_k(n)$ is monotonous increasing from a certain point on, i.e., there exists an integer n_0 such that $R_k(n+1) \geq R_k(n)$ for $n \geq n_0$, we have

$$\begin{aligned} (A(2n))^k &= \Bigg(\sum_{\substack{a \in \mathcal{A} \\ a \leq 2n}} 1 \Bigg)^k = \sum_{\substack{a_1 \in \mathcal{A}, a_2 \in \mathcal{A}, \dots, a_k \in \mathcal{A} \\ a_1 \leq 2n, a_2 \leq 2n, \dots, a_k \leq 2n}} 1 \geq \sum_{\substack{a_1 + a_2 + \dots + a_k \leq 2n \\ a_1 \in \mathcal{A}, \dots, a_k \in \mathcal{A}}} 1 \\ &\geq \sum_{i=1}^{2n} R_k(i) \geq \sum_{i=n+1}^{2n} R_k(i) \geq \sum_{i=n+1}^{2n} R_k(n) = n R_k(n) \end{aligned}$$

hence

$$\frac{(A(2n))^k}{n} \geq R_k(n) \tag{5}$$

for $n \geq n_0$. In view of the monotonicity of $R_k(n)$, and since $\mathcal{A}$ is infinite, we have $R_k(n) \geq 1$ for n large enough. Thus we obtain from (5) that

$$(A(2n))^k \geq n \tag{6}$$

for n large enough. We have

$$I = \int_0^1 |f(z)|^k d\alpha = \int_0^1 |f^k(z)| d\alpha = \int_0^1 \left| \sum_{n=1}^{+\infty} R_k(n) z^n \right| d\alpha$$
$$= \int_0^1 |(1-z) \sum_{n=1}^{+\infty} R_k(n) z^n| |1-z|^{-1} d\alpha. \tag{7}$$

By the monotonicity, and if N and n_0 large enough we have

$$\left|(1-z)\sum_{n=1}^{+\infty} R_k(n) z^n\right| = \left|\sum_{n=1}^{+\infty}(R_k(n) - R_k(n-1)) z^n\right|$$
$$\leq \sum_{n=1}^{n_0} |R_k(n) - R_k(n-1)| r^n + \sum_{n=n_0+1}^{+\infty} |R_k(n) - R_k(n-1)| r^n$$
$$< \sum_{n=1}^{n_0} |R_k(n) - R_k(n-1)| + \sum_{n=n_0+1}^{+\infty} |R_k(n) - R_k(n-1)| r^n$$
$$= \sum_{n=1}^{n_0} |R_k(n) - R_k(n-1)| + \sum_{n=n_0+1}^{+\infty} (R_k(n) - R_k(n-1)) r^n$$
$$< 2\sum_{n=1}^{n_0} |R_k(n) - R_k(n-1)| + \sum_{n=1}^{+\infty}(R_k(n) - R_k(n-1)) r^n$$
$$= c_1 + \sum_{n=1}^{+\infty} R_k(n)(r^n - r^{n+1})$$
$$= c_1 + (1-r)\sum_{n=1}^{+\infty} R_k(n) r^n < c_1 + \sum_{n=1}^{n_0-1} R_k(n) + (1-r)\sum_{n=n_0}^{+\infty} R_k(n) r^n$$
$$< c_2 + (1-e^{-1/N})\Bigg(\sum_{n=n_0}^{N} R_k(N) + \sum_{n=N+1}^{+\infty} R_k(n) r^n\Bigg).$$

Thus by (1), (5) and (6) we have

$$\left|(1-z)\sum_{n=1}^{+\infty} R_k(n) z^n\right| < c_2 + N^{-1}\left(N\frac{(A(2N))^k}{N} + \sum_{n=N+1}^{+\infty} \frac{(A(2n))^k}{n} r^n\right)$$
$$< c_2 + N^{-1}\Bigg((A(N))^k \left(\frac{2N}{N}\right)^{2k} + \sum_{n=N+1}^{+\infty} \left(A(N)\left(\frac{2n}{N}\right)^2\right)^k \frac{1}{n} r^n\Bigg)$$

$$
\begin{aligned}
&< c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}}{N^{2k+1}} \sum_{n=1}^{+\infty} n^{2k-1} r^n \right) \\
&< c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}}{N^{2k+1}} \sum_{n=1}^{+\infty} (n+1)(n+2)\dots(n+2k-1) r^n \right) \\
&= c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}}{N^{2k+1}} \sum_{m=2k}^{+\infty} m(m-1)\dots(m-2k+2) r^{m-2k+1} \right) \\
&< c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}}{N^{2k+1}} (\sum_{m=0}^{+\infty} r^m)^{(2k-1)} \right) \\
&= c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}}{N^{2k+1}} \left(\frac{1}{1-r} \right)^{(2k-1)} \right) \\
&= c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}}{N^{2k+1}} (2k-1)!(1-r)^{-2k} \right) \\
&= c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}(2k-1)!}{N^{2k+1}} (1-e^{-1/N})^{-2k} \right).
\end{aligned}
$$

Since

$$
1 - e^{-x} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots > x - \frac{x^2}{2!} = x\left(1 - \frac{x}{2}\right) > \frac{x}{2}
$$

for $0 < x < 1$, it follows by (5) that

$$
\begin{aligned}
\left| (1-z) \sum_{n=1}^{+\infty} R_k(n) z^n \right| &< c_2 + (A(N))^k \left(2^{2k} N^{-1} + \frac{2^{2k}(2k-1)!}{N^{2k+1}} (2N)^{2k} \right) \\
&= c_2 + (A(N))^k N^{-1} (2^{2k} + 2^{4k}(2k-1)!) < c_3 (A(N))^k N^{-1}. \qquad (8)
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
|1-z| &= ((1-z)(1-\bar{z}))^{1/2} = (1 + |z|^2 - 2Rez)^{1/2} \\
&= (1 + r^2 - 2r\cos 2\pi\alpha)^{1/2} = ((1-r)^2 + 2r(1-\cos 2\pi\alpha))^{1/2} \\
&> (2r(1-\cos 2\pi\alpha))^{1/2} = (2e^{-1/N} 2\sin^2 \pi\alpha)^{1/2} \geq (2(2\alpha)^2)^{1/2} \geq 2\alpha \qquad (9)
\end{aligned}
$$

for $0 \leq \alpha \leq \frac{1}{2}$ and for large N, and

$$
\begin{aligned}
|1-z| &= ((1-r)^2 + 2r(1-\cos 2\pi\alpha))^{1/2} \geq ((1-r)^2)^{1/2} \\
&= 1 - r = 1 - e^{-1/N} > 1/2N \qquad (10)
\end{aligned}
$$

for all α. It follows from (7), (8), (9) and (10) that

$$\begin{aligned}
I &\le \int_0^1 c_3(A(N))^k N^{-1}|1-z|^{-1}d\alpha = 2c_3(A(N))^k N^{-1}\int_0^{1/2}|1-z|^{-1}d\alpha \\
&= c_4(A(N))^k N^{-1}\left(\int_0^{1/N}|1-z|^{-1}d\alpha + \int_{1/N}^{1/2}|1-z|^{-1}d\alpha\right) \\
&< c_4(A(N))^k N^{-1}\left(\int_0^{1/N}2N d\alpha + \int_{1/N}^{1/2}(2\alpha)^{-1}d\alpha\right) \\
&< c_4(A(N))^k N^{-1}(2+\frac{1}{2}\log N) < c_5(A(N))^k N^{-1}\log N. \qquad (11)
\end{aligned}$$

In view of (4), (11) and our indirect assumption we have

$$e^{-k}(A(N))^{k/2} \le I < c_5(A(N))^k N^{-1}\log N,$$

$$N < c_6(A(N))^{k/2}\log N = o\left(\left(\frac{N^{2/k}}{(\log N)^{2/k}}\right)^{k/2}\log N\right) = o(N).$$

This contradiction completes the proof of Theorem 3.

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SÁNDOR Z. KISS
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C
H-1117 BUDAPEST
HUNGARY

E-mail: kisspest@cs.elte.hu

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