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Compactness of Riemann–Stieltjes operators between $F(p,q,s)$ spaces and α-Bloch spaces

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Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. In this paper we investigate the following integral operators

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t} \quad \text{and} \quad L_g(f)(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t},$$

$f \in H(B)$, $z \in B$, where $g \in H(B)$ and $\Re h(z) = \sum_{j=1}^n z_j \frac{\partial h}{\partial z_j}(z)$ is the radial derivative of h. The operator T_g can be considered as an extension of the Cesàro operator on the unit disk. The compactness of the operators T_g and L_g between the general function space $F(p,q,s)$, which includes the Hardy space, Bergman space, Bloch space, and Q_p space, and the α-Bloch space are discussed.

1. Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in the complex vector space $\mathbb{C}^n$, $\partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary, $B(a,r)$ be the ball centered at a with radius r. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{k=1}^n z_k \bar{w}_k$. Let $H(B)$ denote the class of all holomorphic functions on the unit ball. For $a, z \in B$, $a \neq 0$, let φ_a denote the Möbius transformation of

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B taking 0 to a defined by

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1-|a|^2}Q_a(z)}{1 - \langle z, a\rangle},$$

where $P_a(z)$ is the projection of z onto the one dimensional subspace of $\mathbb{C}^n$ spanned by a and $Q_a(z) = z - P_a(z)$. It is known that

$$\varphi_a \circ \varphi_a(z) = z, \quad \varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad 1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}.$$

φ_a belongs to the group of biholomorphic automorphisms, $\mathrm{Aut}(B)$, of B (see, for example, [26]).

For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let $\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f. It is well known that $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$.

Let $\alpha > 0$. The α-Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$ is the space of all holomorphic functions f on B such that

$$b_\alpha(f) = \sup_{z\in B}(1-|z|^2)^\alpha\, |\Re f(z)| < \infty.$$

The little α-Bloch space $\mathcal{B}_0^\alpha = \mathcal{B}_0^\alpha(B)$, consists of all $f \in H(B)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha\, |\Re f(z)| = 0.$$

It is clear that $\mathcal{B}^\alpha$ is a normed space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, and $\mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}$ for $\alpha_1 < \alpha_2$.

The Hardy space $H^p(B)$ $(0 < p < \infty)$ is defined on B by

$$H^p(B) = \{f \mid f \in H(B) \text{ and } \|f\|_{H^p(B)} = \sup_{0\leq r<1} M_p(f,r) < \infty\},$$

where

$$M_p(f,r) = \left(\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}$$

and $d\sigma$ is the normalized surface measure on ∂B.

For $p \in (0,\infty), \alpha > -1$, the weighted Bergman space $A_\alpha^p(B)$, is the space of all holomorphic functions f on B for which

$$\|f\|_{A_\alpha^p}^p = \int_B |f(z)|^p dv_\alpha(z) = c_\alpha \int_B |f(z)|^p (1-|z|^2)^\alpha dv(z) < \infty,$$

where $dv(\cdot)$ denotes the normalized Lebesgue measure of B, and c_α is the normalizing constant, $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$. When $\alpha = 0$, $A^p_0(B) = A^p(B)$ is the classical Bergman space.

For $a \in B$ and $0 < p < \infty$, the Q_p space is defined by ([13])

$$Q_p = Q_p(B) = \Big\{ f \in H(B) : \|f\|^2_{Q_p} = \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z) < \infty \Big\}.$$

Here $\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$ denotes the invariant gradient of f, and $G(z,a)$ is the invariant Green's function defined as $G(z,a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt,$$

$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$ is a Möbius invariant measure, that is, for any $\phi \in \mathrm{Aut}(B)$ and $f \in L^1(B)$, we have

$$\int_B f(z) d\lambda(z) = \int_B f \circ \phi(z)\, d\lambda(z).$$

For more details about Q_p spaces, see [13] and [22].

Let $0 < p,\ s < \infty$, $-n-1 < q < \infty$. A function $f \in H(B)$ is said to belong to $F(p,q,s) = F(p,q,s)(B)$ (see [24], [25]) if

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in B} \int_B |\nabla f(z)|^p (1-|z|^2)^q g^s(z,a) dv(z) < \infty, \tag{1}$$

where $g(z,a) = \log |\varphi_a(z)|^{-1}$ is the Green's function for B with logarithmic singularity at a. We call $F(p,q,s)$ a general function space because we can obtain many function spaces, such as $BMOA$ space, Q_p space, Bergman space, Hardy space, Bloch space, if we take special parameters of p,q,s. For example, $F(2,0,s) = Q_s$ and $F_0(2,0,s) = Q_{s,0}$; $F(2,0,1) = BMOA$ and $F_0(2,0,1) = VMOA$; $F(2,1,0) = H^2$, $F(p,p,0) = A^p$. If $q + s \le -1$, then $F(p,q,s)$ is the space of constant functions. For the unit disk setting, see [25].

For an analytic function $f(z)$ on the unit disk $\mathbb{D}$ with Taylor expansion $f(z) = \sum_{n=0}^\infty a_n z^n$, the Cesàro operator acting on f is

$$\mathcal{C} f(z) = \sum_{n=0}^\infty \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n.$$

The integral form of $\mathcal{C}$ is

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta}d\zeta = \frac{1}{z}\int_0^z f(\zeta)\left(\ln\frac{1}{1-\zeta}\right)' d\zeta,$$

taking simply as a path the segment joining 0 and z, we have that

$$\mathcal{C}(f)(z) = \int_0^1 f(tz)\left(\ln\frac{1}{1-\zeta}\right)'\Bigg|_{\zeta=tz} dt.$$

The following operator

$$z\mathcal{C}(f)(z) = \int_0^z \frac{f(\zeta)}{1-\zeta}d\zeta,$$

is closely related to the previous operator and on many spaces the boundedness of these two operators is equivalent. It is well known that the Cesàro operator acts as a bounded linear operator on various analytic function spaces (see, for example, [4], [12], [15], [16], [17], [20] and the references therein).

Suppose that $g \in H(\mathbb{D})$, the integral operator J_g and its companion operator I_g with symbol g are defined by

$$J_g f(z) = \int_0^z f dg = \int_0^1 f(tz)zg'(tz)dt = \int_0^z f(\xi)g'(\xi)d\xi$$

and

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

The operator J_g can be viewed as a generalization of the Cesàro operator. In [14] Pommerenke introduced J_g and showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. Alemann and Siskakis ([2]) showed that J_g is bounded (compact) on the Hardy space H^p, $1 \le p < \infty$, if and only if $g \in BMOA$ ($g \in VMOA$), and that J_g is bounded (compact) on the Bergman space A^p if and only if $g \in \mathcal{B}$ ($g \in \mathcal{B}_0$), see [3]. Recently, J_g acting on various function spaces, including the Dirichlet space, the Bloch space, the weighted Bergman space, $BMOA$ and $VMOA$ spaces, have been studied (see [1], [2], [3], [9], [18], [23] and the related references therein).

The operator J_g can be naturally extended to the unit ball. Suppose that $g \in H(B)$, define

$$T_g f(z) = \int_0^1 f(tz)\frac{dg(tz)}{dt} = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \tag{2}$$

for $z \in B$ and $f \in H(B)$. This operator is called Riemann–Stieltjes operator (or Extended Cesàro operator), it was introduced in [5], and studied in [5], [6], [7], [8], [10], [19], [21]. Similarly, the companion operator L_g with symbol g is defined as follows (see [8] and [10]):

$$L_g f(z) = \int_0^1 \Re f(tz) g(tz) \frac{dt}{t}, \qquad z \in B, \quad f \in H(B). \tag{3}$$

The purpose of this paper is to study the compactness of these two Riemann–Stieltjes operators, between $F(p,q,s)$ and the α-Bloch spaces as well as to the little α-Bloch spaces. These results can be seen as extensions of our earlier results on these operators (see [8], [19]), where we investigated the boundedness.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

In order to prove the main results of this paper, we need some auxiliary results which are incorporated in the following lemmas. The first one is an analog of the following one-dimensional results

$$\left(\int_0^z f(\zeta) g'(\zeta) d\zeta\right)' = f(z) g'(z), \quad \left(\int_0^z f'(\zeta) g(\zeta) d\zeta\right)' = f'(z) g(z).$$

Lemma 2.1 (see e.g. [5])**.** *For every $f, g \in H(B)$ it holds*

$$\Re[T_g(f)](z) = f(z) \Re g(z) \quad \text{and} \quad \Re[L_g(f)](z) = \Re f(z) g(z).$$

The following lemma can be found in [24].

Lemma 2.2. *Assume that $0 < p, s < \infty$, $-n-1 < q < \infty$, $q + s > -1$ and $f \in F(p,q,s)$. Then, $f \in \mathcal{B}^{\frac{n+1+q}{p}}$ and*

$$\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \le C \|f\|_{F(p,q,s)}, \tag{4}$$

for some positive constant C independent of f.

The following lemma is probably folklore. For a proof of the lemma see, for example, [19, Lemma 2.2] and [21].

Lemma 2.3. *If $f \in \mathcal{B}^\beta$, then*

$$|f(z)| \le C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^\beta} & 0 < \beta < 1, \\ |f(0)| + \|f\|_{\mathcal{B}^\beta} \ln \dfrac{1}{1-|z|^2} & \beta = 1, \\ |f(0)| + \dfrac{\|f\|_{\mathcal{B}^\beta}}{(1-|z|^2)^{\beta-1}} & \beta > 1. \end{cases}$$

for some C independent of f.

In order to investigate the compactness of operators T_g and L_g, which map a space into $\mathcal{B}_0^\alpha$, we also need the following lemma. For the case $\alpha = 1$ in the unit disk, the lemma was proved in [11]. For the general case the proof is similar, thus we omit the details.

Lemma 2.4. *A closed set K in $\mathcal{B}_0^\alpha(B)$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f \in K} (1-|z|^2)^\alpha |\Re f(z)| = 0.$$

The next lemma can be proved in a standard way.

Lemma 2.5. *The operator T_g (or L_g) $: F(p,q,s) \to \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) is compact if and only if for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $F(p,q,s)$ which converges to zero uniformly on compact subsets of B, $T_g(f_k) \to 0$ (or $L_g(f_k) \to 0$) in $\mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) as $k \to \infty$.*

Remark 2.1. If we interchange the position of the spaces $F(p,q,s)$ and $\mathcal{B}^\alpha$, Lemma 2.5 still holds.

3. Compactness of $T_g, L_g : F(p,q,s) \to \mathcal{B}^\alpha(\mathcal{B}_0^\alpha)$

In this section we study the compactness of the operators $T_g, L_g : F(p,q,s) \to \mathcal{B}^\alpha(\mathcal{B}_0^\alpha)$. We distinguish three cases: $p < q+n+1$, $p > q+n+1$ and $p = q+n+1$.

Before we formulate our main results, we introduce the following notation. We say that a sequence $(f_k)_{k\in\mathbb{N}}$, which belong to $F(p,q,s)$, is F-weakly convergent to zero if it is bounded in $F(p,q,s)$ and converges to 0 uniformly on compact subsets of B as $k \to \infty$. We will use the letter L for a bound of the sequence $(f_k)_{k\in\mathbb{N}}$ in $F(p,q,s)$ space.

3.1. Case $p < q+n+1$. In this subsection we consider the case $p < q+n+1$. Our first result is the following theorem.

Theorem 3.1. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p < q+n+1$. Then, $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)| = 0. \tag{5}$$

Proof. First assume that condition (5) holds. In order to prove that T_g is compact, by Lemma 2.5 it suffices to show that if $(f_k)_{k\in\mathbb{N}}$ is a sequence that is F-weakly convergent to zero, then $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Hence, assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $F(p,q,s)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{F(p,q,s)} \le L$ and $f_k \to 0$ uniformly on compact subsets of B as $k\to\infty$. From (5), we have that for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that $\delta < |z| < 1$ implies

$$(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)| < \varepsilon/2.$$

By Lemma 2.3 with $\beta = \frac{q+n+1}{p}$ and Lemma 2.2, we have

$$\begin{aligned}
\|T_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\Re(T_g f_k)(z)| = \sup_{z\in B}(1-|z|^2)^\alpha|\Re g(z)||f_k(z)| \\
&\le \sup_{z\in\overline{B(0,\delta)}}(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)|(1-|z|^2)^{\frac{q+n+1}{p}-1}|f_k(z)| \\
&\quad + C\sup_{z\in B\setminus\overline{B(0,\delta)}}(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)|\|f_k\|_{F(p,q,s)} \\
&\le M_1\sup_{z\in\overline{B(0,\delta)}}(1-|z|^2)^{\frac{q+n+1}{p}-1}|f_k(z)| + CL\varepsilon/2,
\end{aligned}$$

where $M_1 := \sup_{z\in B}(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)| < \infty$, by (5).

Using the fact that $(f_k)_{k\in\mathbb{N}}$ is a sequence F-weakly convergent to zero, we obtain

$$\limsup_{k\to\infty}\|T_g f_k\|_{\mathcal{B}^\alpha} \le CL\varepsilon/2.$$

Since ε is an arbitrary positive number we have that $\lim_{k\to\infty}\|T_g f_k\|_{\mathcal{B}^\alpha} = 0$, and therefore, $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k| \to 1$ as $k\to\infty$. Set

$$f_k(z) = \frac{1-|z_k|^2}{(1-\langle z, z_k\rangle)^{\frac{q+n+1}{p}}}, \quad k\in\mathbb{N}. \tag{6}$$

Then

$$|\Re f_k(z)| = \frac{q+n+1}{p}\frac{(1-|z_k|^2)|\langle z, z_k\rangle|}{|1-\langle z, z_k\rangle|^{1+\frac{q+n+1}{p}}} \le \frac{C(1-|z_k|^2)}{|1-\langle z, z_k\rangle|^{1+\frac{q+n+1}{p}}}.$$

Similar to the proof of Theorem C in [24] we see that $f_k \in F(p,q,s)$ for every $k \in \mathbb{N}$, moreover $\sup_{k\in\mathbb{N}} \|f_k\|_{F(p,q,s)} \le C$ and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since T_g is compact, we have $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}(1-|z_k|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z_k)| &= (1-|z_k|^2)^\alpha |\Re g(z_k)|\,|f_k(z_k)| \\ &\le \sup_{z\in B}(1-|z|^2)^\alpha |f_k(z)||\Re g(z)| \\ &= \sup_{z\in B}(1-|z|^2)^\alpha |\Re(T_g f_k)(z)| = \|T_g f_k\|_{\mathcal{B}^\alpha} \to 0\end{aligned}$$

as $k \to \infty$. Hence, we obtain (5), finishing the proof of the theorem. □

Theorem 3.2. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p < q+n+1$. Then, $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)| = 0. \tag{7}$$

Proof. Assume that (7) holds and that $(f_k)_{k\in\mathbb{N}}$ is a sequence that is F-weakly convergent to zero. By the assumption, for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that $\delta < |z| < 1$ implies

$$(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)| < \varepsilon/2.$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\|L_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |\Re(L_g f_k)(z)| = \sup_{z\in B}(1-|z|^2)^\alpha |g(z)||\Re f_k(z)| \\ &\le \sup_{z\in\overline{B(0,\delta)}}(1-|z|^2)^\alpha |g(z)\Re f_k(z)| + \sup_{z\in B\setminus\overline{B(0,\delta)}}(1-|z|^2)^\alpha |g(z)||\Re f_k(z)| \\ &\le \sup_{z\in\overline{B(0,\delta)}}(1-|z|^2)^\alpha |g(z)||\Re f_k(z)| \\ &\quad + C\sup_{z\in B\setminus\overline{B(0,\delta)}}(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)|\|f_k\|_{F(p,q,s)} \\ &\le M_2 \sup_{z\in\overline{B(0,\delta)}}(1-|z|^2)^{\frac{q+n+1}{p}}|\Re f_k(z)| + CL\varepsilon/2,\end{aligned} \tag{8}$$

where $M_2 := \sup_{z\in B}(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)| < \infty$ by (7).

By Cauchy's estimate the condition $f_k \to 0$ as $k \to \infty$ uniformly on compact subsets of B, implies that $\Re f_k \to 0$ as $k \to \infty$ uniformly on compact subsets of B. Hence, we have $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Therefore, $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Let the sequence $(f_k)_{k\in\mathbb{N}}$ be defined by (6). As we have previously mentioned $\sup_{k\in\mathbb{N}}\|f_k\|_{F(p,q,s)} \le C$ and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since L_g is compact, we have $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}\frac{q+n+1}{p}(1-|z_k|^2)^{\alpha-\frac{q+n+1}{p}}|g(z_k)||z_k|^2 &\le \sup_{z\in B}(1-|z|^2)^\alpha|g(z)|\Re f_k(z)| \\ &= \sup_{z\in B}(1-|z|^2)^\alpha|\Re(L_g f_k)(z)| = \|L_g f_k\|_{\mathcal{B}^\alpha} \to 0\end{aligned}$$

as $k \to \infty$, which implies that (7) holds. □

Now we characterize the compactness of the operators $T_g, L_g : F(p,q,s) \to \mathcal{B}_0^\alpha$.

Theorem 3.3. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p < q+n+1$. Then, $T_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)| = 0. \tag{9}$$

Proof. Sufficiency. From Lemma 2.4, we know that $T_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1}\sup_{\|f\|_{F(p,q,s)}\le 1}(1-|z|^2)^\alpha|\Re(T_g f)(z)| = 0. \tag{10}$$

On the other hand, by Lemma 2.3 with $\beta = \frac{q+n+1}{p}$ and Lemma 2.2, we have that

$$(1-|z|^2)^\alpha|\Re(T_g f)(z)| \le C(1-|z|^2)^{\alpha+1-\frac{q+n+1}{p}}|\Re g(z)|\,\|f\|_{F(p,q,s)}. \tag{11}$$

Taking the supremum in (11) over the the unit ball in the space $F(p,q,s)$, then letting $|z| \to 1$ and applying (10) the result follows.

Necessity. It is a consequence of the proof of Theorem 3.1. □

Theorem 3.4. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p < q+n+1$. Then, $L_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)| = 0. \tag{12}$$

Proof. Sufficiency. By Lemma 2.2 we have that

$$(1-|z|^2)^\alpha|\Re(L_g f)(z)| \le C(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)|\|f\|_{F(p,q,s)}.$$

From this, by Lemma 2.4 and (12), the result can be obtained similar to the proof of Theorem 3.3.

Necessity. It is a consequence of the proof of Theorem 3.2. □

Using some facts mentioned in the introduction, we have the following corollaries.

Corollary 3.1. *Assume that g is a holomorphic function on B and $\alpha > \frac{n+1}{p}$. Then the following statements are equivalent:*

(i) *$T_g : A^p \to \mathcal{B}^\alpha$ is compact;*

(ii) *$T_g : A^p \to \mathcal{B}_0^\alpha$ is compact;*

(iii) $$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-\frac{n+1}{p}}|\Re g(z)| = 0.$$

Corollary 3.2. *Assume that g is a holomorphic function on B and $\alpha > 1+\frac{n+1}{p}$. Then the following statements are equivalent:*

(i) *$L_g : A^p \to \mathcal{B}^\alpha$ is compact;*

(ii) *$L_g : A^p \to \mathcal{B}_0^\alpha$ is compact;*

(iii) $$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-1-\frac{n+1}{p}}|g(z)| = 0.$$

Corollary 3.3. *Assume that g is a holomorphic function on B and $\alpha > \frac{n}{2}$. Then the following statements are equivalent:*

(i) *$T_g : H^2 \to \mathcal{B}^\alpha$ is compact;*

(ii) *$T_g : H^2 \to \mathcal{B}^\alpha$ is compact;*

(iii) $$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-\frac{n}{2}}|\Re g(z)| = 0.$$

Corollary 3.4. *Assume that g is a holomorphic function on B and $\alpha > \frac{n+2}{2}$. Then the following statements are equivalent:*

(i) *$L_g : H^2 \to \mathcal{B}^\alpha$ is compact;*

(ii) *$L_g : H^2 \to \mathcal{B}_0^\alpha$ is compact;*

(iii) $$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-1-\frac{n}{2}}|g(z)| = 0.$$

3.2. Case $p > q+n+1$.

Theorem 3.5. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p > q+n+1$. Then, $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if $g \in \mathcal{B}^\alpha$.*

Proof. If $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact, then $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. By Theorem 2.8 of [8], we see that $g \in \mathcal{B}^\alpha$.

Assume that $g \in \mathcal{B}^\alpha$, and $(f_k)_{k\in\mathbb{N}}$ is a sequence that is F-weakly convergent to zero. Since $f_k \in F(p,q,s) \subset \mathcal{B}^{\frac{q+n+1}{p}} \subset A(B)$, $k \in \mathbb{N}$ (see [26, Theorem 7.9]), where $A(B)$ is the ball algebra, consisting of all functions in $H(B)$ that are continuous up to the boundary of the unit ball. It can be easily proved that $\lim_{k\to\infty} \sup_{z\in B} |f_k(z)| = 0$. Hence, we have that

$$\|T_g f_k\|_{\mathcal{B}^\alpha} = \sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)||f_k(z)| \le \|g\|_{\mathcal{B}^\alpha} \sup_{z\in B} |f_k(z)| \to 0, \tag{13}$$

as $k \to \infty$. Therefore, $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. $\square$

Theorem 3.6. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p > q+n+1$. Then, $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha - \frac{q+n+1}{p}} |g(z)| = 0. \tag{14}$$

The proof of Theorem 3.6 is identical to the proof of Theorem 3.2, hence will be omitted.

Now, we characterize the compactness of $T_g, L_g : F(p,q,s) \to \mathcal{B}_0^\alpha$.

Theorem 3.7. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p > q+n+1$. Then, $T_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if $g \in \mathcal{B}_0^\alpha$.*

Proof. Sufficiency. Since

$$(1-|z|^2)^\alpha |\Re(T_g f)(z)| \le C(1-|z|^2)^\alpha |\Re g(z)| \|f\|_{F(p,q,s)},$$

similar to the proof of Theorem 3.3, we obtain the desired result.

Necessity. Assume that $T_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact. Then $T_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is bounded. Taking $f(z) = 1$, then $T_g f \in \mathcal{B}_0^\alpha$, i.e. we obtain $g \in \mathcal{B}_0^\alpha$. $\square$

Theorem 3.8. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p > q+n+1$. Then, $L_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)| = 0.$$

Proof. Sufficiency. Since

$$(1-|z|^2)^\alpha|\Re(L_g f)(z)| \le C(1-|z|^2)^{\alpha-\frac{q+n+1}{p}}|g(z)|\|f\|_{F(p,q,s)}.$$

Similarly to the proof of Theorem 3.4 the result follows.

Necessity. It is a consequence of Theorem 3.6. □

3.3. Case $p = q+n+1$. Here we consider the case $p = q+n+1$.

Theorem 3.9. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p = q+n+1$. Then, $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{1}{1-|z|^2} = 0. \tag{15}$$

Proof. Assume (15) holds and that $(f_k)_{k\in\mathbb{N}}$ is a sequence that is F-weakly convergent to zero. We have that, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{1}{1-|z|^2} < \varepsilon/2,$$

when $\delta < |z| < 1$.

In addition

$$\begin{aligned}
\|T_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\Re(T_g f_k)(z)| = \sup_{z\in B}(1-|z|^2)^\alpha|\Re g(z)||f_k(z)| \\
&\le \sup_{z\in\overline{B(0,\delta)}}(1-|z|^2)^\alpha|\Re g(z) f_k(z)| \\
&\quad + C\|f_k\|_{F(p,q,s)}\sup_{z\in B\setminus\overline{B(0,\delta)}}(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{1}{1-|z|^2} \\
&\le M_4\sup_{z\in\overline{B(0,\delta)}}\frac{|f_k(z)|}{\ln\frac{1}{1-|z|^2}} + CL\varepsilon/2,
\end{aligned}$$

where

$$M_4 = \sup_{z\in B}(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{1}{1-|z|^2} < \infty,$$

by (15).

Similar to the proof of Theorem 3.1, we obtain $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Therefore, $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $T_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Set

$$f_k(z) = \left(\ln \frac{1}{1-|z_k|^2}\right)^{-1} \left(\ln \frac{1}{1-\langle z, z_k\rangle}\right)^2, \quad k \in \mathbb{N}. \tag{16}$$

Then

$$|\Re f_k(z)| = 2|\langle z, z_k\rangle| \left(\ln \frac{1}{1-|z_k|^2}\right)^{-1} \left|\ln \frac{1}{1-\langle z, z_k\rangle}\right| \left|\frac{1}{1-\langle z, z_k\rangle}\right| \le \frac{C}{|1-\langle z, z_k\rangle|}.$$

From this and after some calculations or from [24], we find that $\|f_k\|_{F(p,q,s)} \le C$ for some positive C independent of k, and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since T_g is compact, we have $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}(1-|z_k|^2)^\alpha |\Re g(z_k)| \ln \frac{1}{1-|z_k|^2} &\le \sup_{z\in B} (1-|z|^2)^\alpha |f_k(z)||\Re g(z)| \\ &= \sup_{z\in B} (1-|z|^2)^\alpha |\Re (T_g f_k)(z)| = \|T_g f_k\|_{\mathcal{B}^\alpha} \to 0\end{aligned}$$

as $k \to \infty$, which is equivalent to (15). □

Theorem 3.10. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p = q+n+1$. Then, $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha-1} |g(z)| = 0. \tag{17}$$

Proof. Assume that (17) holds and that $(f_k)_{k\in\mathbb{N}}$ is a sequence that is F-weakly convergent to zero. From (17) we obtain that for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that

$$(1-|z|^2)^{\alpha-1} |g(z)| < \varepsilon/2,$$

when $\delta < |z| < 1$.

From (8), by Lemmas 2.2 and 2.3, we have that

$$\begin{aligned}\|L_g f_k\|_{\mathcal{B}^\alpha} \le &\sup_{z\in \overline{B(0,\delta)}} (1-|z|^2)^{\alpha-1} |g(z)| (1-|z|^2) |\Re f_k(z)| \\ &+ C \sup_{z\in B\setminus \overline{B(0,\delta)}} (1-|z|^2)^{\alpha-1} |g(z)| \, \|f_k\|_{F(p,q,s)} \\ \le &\, M_5 \sup_{z\in \overline{B(0,\delta)}} (1-|z|^2) |\Re f_k(z)| + CL\varepsilon/2,\end{aligned}$$

where

$$M_5 := \sup_{z\in B}(1-|z|^2)^{\alpha-1}|g(z)| < \infty$$

by (17). Similar to the proof of Theorem 3.2, we obtain $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Therefore, $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $L_g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in B such that $|z_k| \to 1$ as $k \to \infty$. Let $(f_k)_{k\in\mathbb{N}}$ be defined by (16). Since the sequence is F-weakly convergent to zero and L_g is compact, we have that $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Thus

$$2(1-|z_k|^2)^{\alpha-1}|g(z_k)||z_k|^2 \le \sup_{z\in B}(1-|z|^2)^{\alpha}|\Re(L_g f_k)(z)| = \|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$$

as $k \to \infty$, form which (17) follows. □

Next, we characterize the compactness of $T_g, L_g : F(p,q,s) \to \mathcal{B}_0^\alpha$. Similar to the proof of Theorems 3.3 and 3.4, we obtain the following theorems.

Theorem 3.11. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p = q+n+1$. Then, $T_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if* (15) *holds.*

Theorem 3.12. *Assume that g is a holomorphic function on B, $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$ and $p = q+n+1$. Then, $L_g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if* (17) *holds.*

Remark 3.1. Note that if $\alpha \le \frac{q+n+1}{p}$ in Theorems 3.2, 3.4, 3.6, 3.8, 3.10, 3.12 respectively, by using the maximum modulus theorem, we see that L_g is compact if and only if $g \equiv 0$.

4. Compactness of $T_g : \mathcal{B}^\alpha \to F(p,q,s)$

In this section we investigate the compactness of the operator $T_g : \mathcal{B}^\alpha \to F(p,q,s)$. Before we formulate the main results of this section, we note that

$$\|f\|_{F(p,q,s)} \asymp |f(0)|^p + \sup_{a\in B}\int_B |\Re f(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dv(z),$$

which can be proved similar to [5, Lemma 2], and by using the following asymptotic relation $\ln\frac{1}{x} \sim 1-x$ as $x \to 1^+$.

Theorem 4.1. *Assume that g is a holomorphic function on B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$. Then the following statements hold.*

(i) *If* $T_g : \mathcal{B}^\alpha \to F(p,q,s)$ *is compact, then*

$$\lim_{|z|\to 1}(1-|z|^2)^{(n+1+q)/p}|\Re g(z)| \ln \frac{2}{1-|z|^2} = 0, \ \textit{when } \alpha = 1; \tag{18}$$

and

$$\lim_{|z|\to 1}(1-|z|^2)^{(n+1+q)/p+1-\alpha}|\Re g(z)| = 0, \ \textit{when } \alpha \neq 1. \tag{19}$$

(ii) *If* $T_g : \mathcal{B} \to F(p,q,s)$ *is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^{(q+t)/p}|\Re g(z)| \ln \frac{2}{1-|z|^2} = 0, \tag{20}$$

for some $t < 1$, *then* $T_g : \mathcal{B} \to F(p,q,s)$ *is compact.*

(iii) *Assume that* $\alpha < 1$, $T_g : \mathcal{B}^\alpha \to F(p,q,s)$ *is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^{(q+t)/p}|\Re g(z)| = 0, \tag{21}$$

for some $t < 1$. *Then* $T_g : \mathcal{B}^\alpha \to F(p,q,s)$ *is compact.*

(vi) *Assume that* $\alpha > 1$, $T_g : \mathcal{B}^\alpha \to F(p,q,s)$ *is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^{(q+t)/p+1-\alpha}|\Re g(z)| = 0, \tag{22}$$

for some $t < 1$, *then* $T_g : \mathcal{B}^\alpha \to F(p,q,s)$ *is compact.*

Proof. (i) First, we consider the case of $\alpha = 1$. Suppose that $T_g : \mathcal{B} \to F(p,q,s)$ is compact. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in B such that $\lim_{k\to\infty} |z_k| = 1$. Let

$$f_k(z) = \left(\ln \frac{2}{1-|z_k|^2}\right)^{-1}\left(\ln \frac{2}{1-\langle z, z_k\rangle}\right)^2, \quad k \in \mathbb{N}.$$

Then for any $z \in B$,

$$(1-|z|^2)|\Re f_k(z)| \le 2(1-|z|^2)\left|\frac{\ln \frac{2}{1-\langle z,z_k\rangle}}{\ln \frac{2}{1-|z_k|^2}}\right| \frac{1}{1-|z|} \le C.$$

On the other hand, $|f_k(0)| \le \left(\ln \frac{2}{1-|z_k|^2}\right)^{-1}(\ln 2)^2 \le \ln 2$. Thus $\|f_k\|_{\mathcal{B}} \le M$, where M is a constant independent of k. For $|z| \le r < 1$, we have

$$|f_k(z)| = \frac{\left|\ln \frac{2}{1-\langle z,z_k\rangle}\right|^2}{\ln \frac{2}{1-|z_k|^2}} \le \frac{\left(\ln \frac{2}{1-r} + C\right)^2}{\ln \frac{2}{1-|z_k|^2}} \to 0 \quad (k \to \infty),$$

that is, $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By Lemma 2.5, it follows that $\lim_{k\to\infty} \|T_g f_k\|_{F(p,q,s)} = 0$. Since $T_g f_k \in F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$, $k \in \mathbb{N}$, by Lemma 2.2 we have

$$|\Re(T_g f_k)(z)| \leq \frac{\|T_g f_k\|_{\mathcal{B}^{(n+1+q)/p}}}{(1-|z|^2)^{(n+1+q)/p}} \leq \frac{C\|T_g f_k\|_{F(p,q,s)}}{(1-|z|^2)^{(n+1+q)/p}}.$$

Hence

$$|\Re g(z)||f_k(z)|(1-|z|^2)^{(n+1+q)/p} \leq C\|T_g f_k\|_{F(p,q,s)},$$

and consequently

$$(1-|z_k|^2)^{(n+1+q)/p}|\Re g(z_k)| \ln \frac{2}{1-|z_k|^2} \leq \|T_g f_k\|_{F(p,q,s)} \to 0$$

as $k \to \infty$. Therefore we obtain that (18) holds.

When $\alpha \neq 1$, we choose the following family of functions

$$f_k(z) = \frac{1-|z_k|^2}{(1-\langle z, z_k\rangle)^\alpha}, \quad k \in \mathbb{N}.$$

It is easy to see that $f_k \in \mathcal{B}_0^\alpha$, $k \in \mathbb{N}$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} \leq C$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. Since T_g is compact, we have $\lim_{k\to\infty} \|T_g f_k\|_{F(p,q,s)} = 0$. The rest of the proof is similar to the proof in the case $\alpha = 1$ and will be omitted.

(ii) From (20), for any $\varepsilon > 0$, there exist an r, $0 < r < 1$, such that

$$(1-|z|^2)^{\frac{q+t}{p}}|\Re g(z)| \ln \frac{2}{1-|z|^2} < \varepsilon, \tag{23}$$

when $|z| > r$, and that there exist $C > 0$ such that

$$\sup_{|z|\leq r} (1-|z|^2)^{q/p}|\Re g(z)| < C. \tag{24}$$

From the boundedness of T_g, by taking the function $f \equiv 1$, we have that $g \in F(p,q,s)$. Let $(f_k)_{k\in\mathbb{N}}$ be any sequence in the unit ball of $\mathcal{B}$ converging to 0 uniformly on compact subsets of B. For the above ε, there exists a $k_0 \in \mathbb{N}$ such that $\sup_{|z|\leq r} |f_k(z)| < \varepsilon$ for $k \geq k_0$. Hence, by (23), (24), the fact that $g \in F(p,q,s)$ and Lemma 2.3, we have

$$\begin{aligned}
&\|T_g f_k\|^p_{F(p,q,s)} \\
&\quad = \sup_{a\in B} \Big(\int_{|z|\leq r} + \int_{|z|>r} \Big) |\Re(T_g f_k)(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dv(z) \leq C\varepsilon^p
\end{aligned}$$

$$+\|f_k\|_{\mathcal{B}}^p \sup_{a\in B}\int_{|z|>r}|\Re g(z)|^p\left(\ln\frac{2}{1-|z|^2}\right)^p(1-|z|^2)^{q+t}\frac{(1-|\varphi_a(z)|^2)^s}{(1-|z|^2)^t}dv(z)$$
$$\le C\varepsilon^p+\varepsilon^p\|f_k\|_{\mathcal{B}}^p\sup_{a\in B}\int_B\frac{dv(z)}{(1-|z|^2)^t}\le C\varepsilon^p+C\varepsilon^p\|f_k\|_{\mathcal{B}}^p,$$

for $k\ge k_0$. From which we obtain the desired result.

(iii)–(iv) The proofs of these two statements are similar to the proof of statement *(ii)*, hence they will be omitted. □

Remark 4.1. We are not able, at the moment, to obtain significant results regarding the compactness of the operator $L_g:\mathcal{B}^\alpha\to F(p,q,s)$. Hence, we leave the problem to the readers interested in this research area.

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