

A semi-analytic approach to calculating the Strehl ratio for a circularly symmetric system. Part 1: static wavefront

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Close-form expression for the Strehl ratio calculated in the spatial frequency domain of the optical transfer function (SOTF) is considered for the case of an optical system that has circular symmetry. First, it is proved that the SOTF for the aberration-free diffraction limited optical system is equal to one. Further, a semi-analytic solution for the SOTF for a system described by the second (defocus) and the fourth (spherical) order aberrations is provided. It is shown that the proposed semi-analytical solution is of an order of a magnitude computationally more efficient than the commonly used approach based on the discrete Fourier transformation.

Keywords: image quality assessment, optical transfer function, mathematical methods in physics, finite analogs of Fourier transforms.

1. Introduction

The Strehl ratio is one of the fundamental measures of image quality of an optical system. It can be calculated in either spatial domain of the point spread function or in the spatial frequency domain of the optical transfer function (OTF)

$$\text{SOTF} = \frac{\int_{-\infty}^{\infty} df_x \int_{-\infty}^{\infty} df_y \text{OTF}(f_x, f_y)}{\int_{-\infty}^{\infty} df_x \int_{-\infty}^{\infty} df_y \text{OTF}_{\text{DL}}(f_x, f_y)} \quad (1)$$

where the subscript DL refers to the case of an aberration-free diffraction limited system. The Strehl ratio has gained particular interest within the community of vision scientists where, when combined with the contrast sensitivity function [1], it is often referred to as the visual Strehl ratio [2]. In particular, the visual Strehl ratio calculated

in the spatial frequency domain and based on the optical transfer function has been claimed to be the best descriptor of visual performance that can be calculated from wavefront aberrations data [3–5].

There have been several analytic approaches to calculating Strehl ratio resulting in close-form expressions valid for a particular type of primary aberration [6] or such that relate the Strehl ratio to the statistical characterization of wavefront aberrations [7, 8]. Nevertheless, for a given specific wavefront, the Strehl ratio is usually calculated using discrete Fourier transformation [9–11], where several computational limitations arise [12].

In this work, we aim to arrive at an integral expression for the Strehl ratio of a circularly symmetric optical system, which unlike its original definition, does not involve oscillatory integrals. This allows for its more efficient and accurate evaluation, and we analyze its computational performance in comparison to that derived using discrete Fourier transformation. This part (Part 1) of the discourse focuses on static wavefronts containing the second and the fourth order aberrations while Part 2 considers dynamic wavefronts in which temporal averaging occurs. That kind of temporal changes, referred to as longitudinal vibrations, could have a positive effect on an optical system when a certain amount of defocus is present [13].

2. Preliminaries

For an optical system with the wavefront error $W(\rho, \theta)$ we define its pupil function by

$$P(\rho, \theta) = A(\rho, \theta) \exp\left(-i \frac{2\pi n}{\lambda} W(\rho, \theta)\right) \quad (2)$$

where $A(\rho, \theta)$ is the aperture or amplitude transmittance function of the system, n is the refractive index and λ stands for the wavelength. In the following, without loss of generality, we consider the case of a monochromatic wavefront external to the eye with the refractive index $n = 1$ and $\lambda = 0.555 \mu\text{m}$. Finally, the wavefront $W(\rho, \theta)$ in (2) is also measured in micrometers, so that $(2\pi n/\lambda)W(\rho, \theta)$ is in *wavelengths* (dimensionless units), such as observed in [14].

In the simplest case of a uniform circular pupil, which we assume here, the aperture is the characteristic function of the unit disc

$$A(\rho, \theta) = \chi_{[0, 1]}(\rho) = \begin{cases} 1 & \text{if } 0 \leq \rho \leq 1 \\ 0 & \text{if } \rho > 1 \end{cases} \quad (3)$$

From the pupil function we calculate the diffraction integral

$$\begin{aligned} U(r, \varphi) &= \frac{1}{\pi} \int_0^1 d\rho \int_0^{2\pi} d\theta P(\rho, \theta) \exp\left(2\pi i r \rho \cos(\theta - \varphi)\right) \rho \\ &= \frac{1}{\pi} \mathcal{F}_2^{-1}[P](r, \varphi) \end{aligned} \quad (4)$$

and the point spread function (PSF),

$$\text{PSF}(r, \varphi) = |U(r, \varphi)|^2 = \frac{1}{\pi^2} \left| \mathcal{F}_2^{-1}[P](r, \varphi) \right|^2 \quad (5)$$

Here and in what follows, \mathcal{F}_2 and \mathcal{F}_2^{-1} stand for the 2D direct and inverse Fourier transforms, respectively.

The OTF is, in the Cartesian coordinates (u, v) ,

$$\begin{aligned} \text{OTF}(u, v) &= \mathcal{F}_2[\text{PSF}](u, v) \\ &= \int_0^\infty dr \int_0^{2\pi} d\varphi \text{PSF}(r, \varphi) \exp\left(-2\pi i \left(ur \cos(\varphi) + vr \sin(\varphi) \right)\right) r \end{aligned} \quad (6)$$

Using (5) and the autocorrelation theorem from the Fourier analysis (see, for example, (2-16) in [15]), we can also write

$$\begin{aligned} \text{OTF}(u, v) &= \frac{1}{\pi^2} \mathcal{F}_2 \left[\mathcal{F}_2^{-1}[P] \overline{\mathcal{F}_2^{-1}[P]} \right](u, v) = \frac{1}{\pi^2} (P \star P)(u, v) \\ &= \frac{1}{\pi^2} \iint_{\mathbb{R}^2} dx dy P(x, y) \overline{P(x-u, y-v)} \end{aligned} \quad (7)$$

where \star denotes the convolution operation and the bar stands for the complex conjugation. Finally, the Strehl ratio based on the OTF (*the SOTF metric* [2, 4, 12]) is given by the ratio of the averages in the frequency domain of the calculated OTF and the diffraction-limited OTF_{DL}, obtained in the same way as OTF but setting $W \equiv 0$, leading to

$$\text{SOTF} = \frac{\iint_{\mathbb{R}^2} du dv \text{OTF}(u, v)}{\iint_{\mathbb{R}^2} du dv \text{OTF}_{\text{DL}}(u, v)} \quad (8)$$

3. SOTF for a circularly symmetric system

If in polar coordinates, $W(\rho, \theta) = W(\rho)$ and $A(\rho, \theta) = A(\rho) = \chi_{[0, 1]}(\rho)$, then P in (2) reduces to

$$P(\rho) = \chi_{[0, 1]}(\rho) \exp\left(-i \frac{2\pi}{\lambda} W(\rho)\right) \quad (9)$$

and by the circular symmetry,

$$U(r, \varphi) = U(r) = \frac{1}{\pi} \mathcal{F}_2^{-1}[P](r, \varphi) = \frac{1}{\pi} \mathcal{H}_0[P](r) \quad (10)$$

$$\text{PSF}(r) = |U(r)|^2 = \frac{1}{\pi^2} \left| \mathcal{H}_0[P](r) \right|^2 \quad (11)$$

where $\mathcal{H}_0[\cdot](r)$ is the Hankel transform of order 0 [16, 17]. Since the PSF is also circularly-symmetric, we obtain that in polar coordinates,

$$\text{OTF}(\rho, \theta) = \text{OTF}(\rho) = \mathcal{F}_2[\text{PSF}](\rho, \theta) = \mathcal{H}_0[\text{PSF}](\rho) \quad (12)$$

Finally, in the case of a uniform circular pupil (7) reduces to

$$\text{OTF}(\rho) = \text{OTF}(\rho, 0) = \frac{1}{\pi^2} \iint_{D_\rho} dx dy P(x, y) \overline{P(x - \rho, y)} \quad (13)$$

where

$$D_\rho = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leq 1\} \cap \{(x, y) \in \mathbb{R}^2: (x - \rho)^2 + y^2 \leq 1\} \quad (14)$$

is the intersection of two unit disks centered at $(0, 0)$ and $(\rho, 0)$, respectively. The OTF can be written in a more symmetric form,

$$\text{OTF}(\rho) = \frac{1}{\pi^2} \iint_{S_\rho} dx dy P\left(x + \frac{\rho}{2}, y\right) \overline{P\left(x - \frac{\rho}{2}, y\right)} \quad (15)$$

where

$$S_\rho = \left\{ (x, y) \in \mathbb{R}^2: \left(x + \frac{\rho}{2}\right)^2 + y^2 \leq 1 \right\} \cap \left\{ (x, y) \in \mathbb{R}^2: \left(x - \frac{\rho}{2}\right)^2 + y^2 \leq 1 \right\} \quad (16)$$

Hence, taking into account the specific form of P , we can write

$$\begin{aligned} \text{OTF}(\rho, \theta) &= \frac{1}{\pi^2} \left(\begin{array}{cc} \int_0^{1-\rho/2} dx \int_{-\sqrt{1-(x+\rho/2)^2}}^{\sqrt{1-(x+\rho/2)^2}} dy V(x, y, \rho) & + \int_{\rho/2-1}^0 dx \int_{-\sqrt{1-(x-\rho/2)^2}}^{\sqrt{1-(x-\rho/2)^2}} dy V(x, y, \rho) \end{array} \right) \\ &= \frac{1}{\pi^2} \int_0^{1-\rho/2} dx \int_{-\sqrt{1-(x+\rho/2)^2}}^{\sqrt{1-(x+\rho/2)^2}} dy \left(V(x, y, \rho) + V(-x, y, \rho) \right) \end{aligned} \quad (17)$$

with

$$V(x, y, \rho) := \exp\left(i \frac{2\pi}{\lambda} \left(W\left(\sqrt{\left(x - \frac{\rho}{2}\right)^2 + y^2}\right) - W\left(\sqrt{\left(x + \frac{\rho}{2}\right)^2 + y^2}\right) \right) \right) \quad (18)$$

Observe that

$$V(x, y, \rho) + V(-x, y, \rho) = 2\text{Re}\left(V(x, y, \rho)\right) \quad (19)$$

and

$$V(x, -y, \rho) = V(x, y, \rho) \quad (20)$$

so we conclude that in this case

$$\text{OTF}(\rho, \theta) = \frac{4}{\pi^2} \text{Re} \left(\int_0^{1-\rho/2} dx \int_0^{\sqrt{1-(x+\rho/2)^2}} dy V(x, y, \rho) \right) \quad (21)$$

We can also use polar coordinates in this integral, $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$, to obtain an alternative but equivalent expression

$$\text{OTF}(\rho, \theta) = \frac{4}{\pi^2} \text{Re} \left(\int_0^{\pi/2} d\varphi \int_0^{(-\rho\cos(\varphi) + \sqrt{\rho^2\cos^2(\varphi) + 4 - \rho^2})/2} dr r V(r\cos(\varphi), r\sin(\varphi), \rho) \right) \quad (22)$$

3.1. The diffraction-limited case for a circular pupil

Let us consider particularly the diffraction-limited case for a circular pupil, with $W \equiv 0$, *i.e.*, $P_{\text{DL}}(\rho, \theta) = \chi_{[0, 1]}(\rho)$. From (10),

$$U_{\text{DL}}(r, \varphi) = \frac{1}{\pi} \mathcal{H}_0[P_{\text{DL}}](r) = \frac{J_1(2\pi r)}{\pi r} \quad (23)$$

and from (11)

$$\text{PSF}_{\text{DL}}(r, \varphi) = |U_{\text{DL}}(r, \varphi)|^2 = \frac{J_1^2(2\pi r)}{\pi^2 r^2} \quad (24)$$

where $J_n(\cdot)$ is the Bessel function of the first kind of order n .

In polar coordinates,

$$\begin{aligned} \text{OTF}_{\text{DL}}(\rho, \theta) &= \mathcal{H}_0[\text{PSF}_{\text{DL}}](\rho) = \frac{2}{\pi} \int_0^{+\infty} dr \frac{J_1^2(2\pi r) J_0(2\pi r \rho)}{r} \\ &= \frac{1}{\pi^2} \left(2\arccos\left(\frac{\rho}{2}\right) - \frac{\rho}{2} \sqrt{4 - \rho^2} \right) \end{aligned} \quad (25)$$

with $0 \leq \rho \leq 2$, while $\text{OTF}_{\text{DL}}(\rho, \theta) \equiv 0$ for $\rho > 2$. This formula can be obtained either using the identities for integrals of Bessel functions or their Hankel transforms, or by

the geometric interpretation (15), finding the area of S_ρ in (16). Alternatively, from (21) we obtain,

$$\begin{aligned} \text{OTF}_{\text{DL}}(\rho, \theta) &= \frac{4}{\pi^2} \int_0^{1-\rho/2} dx \int_0^{\sqrt{1-(x+\rho/2)^2}} dy \\ &= \frac{4}{\pi^2} \int_0^{1-\rho/2} dx \sqrt{1-\left(x+\frac{\rho}{2}\right)^2} \end{aligned} \quad (26)$$

which again yields (25).

Finally, the dominator in (8) is given by

$$\begin{aligned} \iint_{\mathbb{R}^2} dudv \text{OTF}_{\text{DL}}(u, v) &= 2\pi \int_0^2 d\rho \text{OTF}_{\text{DL}}(\rho)\rho \\ &= \frac{1}{\pi} \int_0^2 d\rho \left(4\arccos(\rho/2) - \rho\sqrt{4-\rho^2}\right)\rho = 1 \end{aligned} \quad (27)$$

We conclude that, for the case of a circular pupil,

$$\text{SOTF} = \iint_{\mathbb{R}^2} dudv \text{OTF}(u, v) \quad (28)$$

and if the OTF is circularly symmetric,

$$\text{SOTF} = 2\pi \int_0^{+\infty} d\rho \text{OTF}(\rho)\rho \quad (29)$$

3.2. The case of aberrated circularly symmetric system

In this part of the work, we assume that the wavefront is given only by the second (de-focus) and the fourth (spherical) aberration terms, so that

$$W(\rho) = f_0 \rho^2 + s_0 \rho^4 \quad (30)$$

where both f_0 and s_0 are constants. We use (18) and (21) to calculate $\text{OTF}(\rho)$. So, for the case of the pupil function P in (9),

$$\begin{aligned} &W\left(\sqrt{\left(x-\frac{\rho}{2}\right)^2 + y^2}\right) - W\left(\sqrt{\left(x+\frac{\rho}{2}\right)^2 + y^2}\right) \\ &= -\rho x \left(2f_0 + s_0 \left(\rho^2 + 4(x^2 + y^2)\right)\right) \end{aligned} \quad (31)$$

so that

$$\begin{aligned}
 V(x, y, \rho) &:= \exp\left(i \frac{2\pi}{\lambda} \left(W\left(\sqrt{\left(x - \frac{\rho}{2}\right)^2 + y^2}\right) - W\left(\sqrt{\left(x + \frac{\rho}{2}\right)^2 + y^2}\right) \right)\right) \\
 &= \exp\left(-i \frac{2\pi}{\lambda} \left(2f_0 + s_0 \left(\rho^2 + 4(x^2 + y^2) \right) \right)\right)
 \end{aligned} \tag{32}$$

In consequence,

$$\begin{aligned}
 \text{OTF}(\rho, \theta) &= \frac{4}{\pi^2} \text{Re} \left(\int_0^{1-\rho/2} dx \int_0^{\sqrt{1-(x+\rho/2)^2}} dy \exp\left(-i \frac{2\pi}{\lambda} \rho x \left(2f_0 + s_0 \left(\rho^2 + 4(x^2 + y^2) \right) \right)\right) \right) \\
 &= \frac{4}{\pi^2} \int_0^{1-\rho/2} dx \int_0^{\sqrt{1-(x+\rho/2)^2}} dy \cos\left(\frac{2\pi}{\lambda} \rho x \left(2f_0 + s_0 \left(\rho^2 + 4(x^2 + y^2) \right) \right)\right)
 \end{aligned} \tag{33}$$

Using this expression and (29) we obtain

$$\begin{aligned}
 \text{SOTF} &= \frac{8}{\pi} \int_0^2 d\rho \int_0^{1-\rho/2} dx \int_0^{\sqrt{1-(x+\rho/2)^2}} dy \cos\left(\frac{2\pi}{\lambda} \rho x \left(2f_0 + s_0 \left(\rho^2 + 4(x^2 + y^2) \right) \right)\right) \rho \\
 &= \frac{8}{\pi} \int_0^1 dx \int_0^{2(1-x)} d\rho \int_0^{\sqrt{1-(x+\rho/2)^2}} dy \cos\left(\frac{2\pi}{\lambda} \rho x \left(2f_0 + s_0 \left(\rho^2 + 4(x^2 + y^2) \right) \right)\right) \rho
 \end{aligned} \tag{34}$$

4. Simulation results

The expression for SOTF given in (34) provides a semi-analytic solution alternative to commonly used approach of discrete Fourier transformation. In fact, to calculate SOTF with that method one needs to apply it twice, first to calculate the PSF, by taking the Fourier transform squared of the pupil function P and then repeating the algorithm to calculate the OTF. Also, since such a calculation does not normally guarantee that the diffraction limited OTF integrates to one as shown in (27), the procedure has to be repeated for the denominator of (1). This makes the presented semi-analytic approach for calculating SOTF particularly useful.

Figure 1 shows the results of SOTF calculated by evaluating (34) by standard quadratures (blue crosses) and using the discrete Fourier transform (red dotted line) for a particular case, where the optical system has the defocus term f_0 only. For the purpose of simulation, the defocus is given in diopters. A transformation from the wavefront domain to the refractive power domain [18] provides the means to readily switch between the units of micrometers and the units of diopters and *vice versa*.

Since both plots are visually indistinguishable, we present in Fig. 2 also the absolute errors (absolute value of the difference between the calculations). All computations have been performed in Matlab (Mathworks, Natick, MI) using the standard IEEE machine precision.

The execution times are 4.35 and 45.16 seconds for the semi-analytic approach and the discrete Fourier transform based method, respectively. In other words, the direct

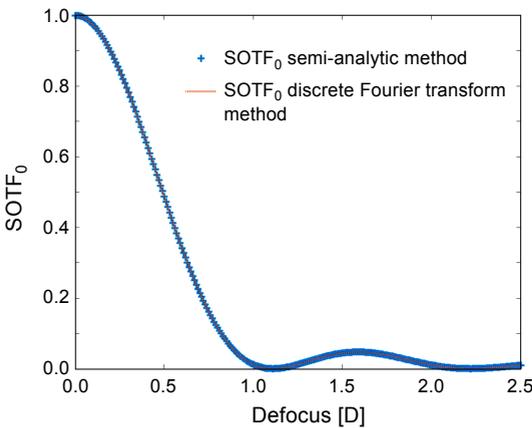


Fig. 1. SOTF₀ for (30) with $s_0 = 0$, as a function of f_0 in diopters, calculated by (34) and using the discrete Fourier transformation.

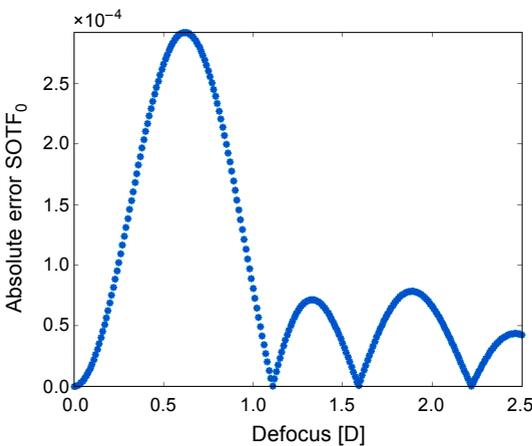


Fig. 2. Absolute error of SOTF₀ for (30) with $s_0 = 0$, as a function of f_0 , calculated by (34) and using the discrete Fourier transformation.

Fourier transform based method is about 10 times slower than the semi-analytic method. Computationally, the result for the case where $s_0 \neq 0$ is similar to that given above.

5. Conclusions

Assessing image quality with the Strehl ratio is typically performed using discrete Fourier transformation in which issues such as the size of the matrix describing the pupil function and the amount of zero-padding, or in general the spatial sampling frequency, need to be considered. Also, a number of limitation has been identified when calculating Strehl ratio using optical transfer function [12]. They include the complexity (numbers containing imaginary components) of the Strehl ratio and lack of normalization when an aberration-free diffraction limited system is considered.

The semi-analytical approach presented here, although currently restricted to circularly symmetric optical systems, resolves some of those limitations. It is not only more efficient than traditionally used method of discrete Fourier transform, but also, as shown in Part 2 of this discourse [19], leads itself to a number of further analytical solutions available for dynamic wavefronts, where temporal averaging of wavefront aberrations is of essence [13].

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