Appropriate Starter for Solving the Kepler's Equation

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ABSTRACT
This article, focuses on the methods that have been used for solving the Kepler's equation for thirty years, then Kepler's equation will be solved by Newton-Raphson's method. For increasing the stability of Newton's method, various guesses studied and the best of them introduced base on minimum number repetition of algorithm. At the end, after studying various guesses base on time of Implementation, one appropriate choice first guesses that increase the isotropy and decrease the time of Implementation of solving is introduced.

Keywords
Kepler's equation; initial guesses; iterative solution; Newton - Raphson method

1. INTRODUCTION
Various problems are solved by Kepler's equation. This equation is used for describing movement of a body under central gravity. Kepler's equation looks simple and shows as below [1]:

$$M = E - esinE \tag{1}$$

$$e \in [0,1]$$

$$M \in [0,2\pi]$$

Where E designates eccentric anomaly, M designates mean anomaly, and e designates eccentricity. This equation can be used to determine the relationship of time and angle place of the body in the orbit. In some cases, E is given and also e is determined and M is unknown, in this case, equation (1) can be directly used. But most of the time, M and e are determined and E must be calculated. In this case, equation (1) cannot be directly used.

During years, for solving Kepler's equation, many methods are introduced. Colwell [2] provides an in-depth survey of solution methods. In following of Colwell's work, a brief survey of solution methods of Kepler's Equation that are published from 1979 until now is explained. Then, choice Newton's method for solving Kepler's equation and scrutiny convergence and time of Implementation of this method for several first guesses and choose the best of them. Finally, use guesses that reduce the time of Implementation of Newton-Raphson's method and have desirable convergence rate for solving equation.

2. METHOD FOR SOLVING KEPLER'S EQUATION
Solving Kepler's Equation attracts many scholars. Although, this equation looks sample, cannot be solved analytically and other methods must to be used. The methods that suggested for solving Kepler's equations can be classified in three categories: classic methods, non-iteration method and iteration methods.

2.1 Classic Method
These methods depend on using power series. E power series is one kind of them that known as Lagrange series [3]:

$$E = M + \sum_{n=1}^{\infty} a_n e^n \tag{2}$$

If e is small enough, Lagrange series is convergent and the E with a good accuracy will be found. But this series is divergent for $e > 0.66$ and the more parameters the worse result, the parameters of Lagrange series must be reduced, too. Another method is Fourier series that is shown as below [3]:

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne)\sin(nM) \tag{3}$$

j_n is known as the Bessel function and defined as follows:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} (\frac{x}{2})^{n+2k} \tag{4}$$

Series solution of Bessel function is convergent for all values $e < 1$. Sentences of series to be increased, more accurate solutions are closer to Kepler's equation. Another method is using Taylor series expansion to approximate the Kepler's equation.

$$f(x) = \sum_{n=0}^{\infty} \frac{f(x)^n x^n}{n!} \tag{5}$$

If the series is convergent, approximation of the $f(x)$ function will result for small values of x. Depending on what the function of $f(x)$ is considered, different solutions can be represented. According to what was mentioned, classical methods of solving Kepler's equation based on the direct use of the series expansion. There are other methods, some of them use series expansion, but as a part of the whole procedure, these methods are called direct or non-iterative methods.

2.2 Non-Iterative Methods
Such methods are a non-iterative solution of Kepler's equation and like the classic methods directly provide estimate of Kepler's equation. In 1987, Mikkola [4] presented a non-iterative two-step solution; initially provided below approximation by using an auxiliary variable and $Arcsine$ function expansion.

$$E = M + e(3s - 4s^3) \tag{6}$$

Mikkola's method is a direct solution of Kepler's equation with a desirable precision.

In 1995, Markly [5] published a solution that is based on Pedé estimating of $sine$ function. In his method, he tried to minimize the using of trigonometric function. In 2006, Feinstein [6] provide a non-iterative solution with using dynamical discretization techniques combined with dynamic program that was superior of all published methods. In 2007, Mortari and Clocchiatti [7] provided a non-iterative solution for Kepler's equation with the Bézier curves. Compare with dynamic discretization method, this method does not need any pre-computed information.

2.3 Iterative Methods

Although non-iterative methods for solving Kepler's equation worked, a number of researchers have devised iterative numerical methods that base on Newton-Raphson method. The idea of Newton-Raphson's method is to approximate the nonlinear function $f(x)$ by the first two terms in a Taylor series expansion around the point x. Newton-Raphson's method is defined as follows [1]:

$$f(E) = E - M - e\ sinE$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{7}$$

For Newton's method, the rate of convergence is said to be quadratic that is a very desirable property for an algorithm to possess. But, if the initial guess is not sufficiently close to the solution, i.e., within the region of convergence, Newton's method may diverge. In fact, Newton's method behaves well near the solution (locally) but lacks something permitting it to converge globally. The second problem occurs when the derivative of the function is zero. In fact, Newton's method loses its quadratic convergence property if the slope is zero at the solution. Therefore, this method requires that the slop of the function is computed on each iteration and mustn't be zero [8]. Halley generalized the Newton's method by applying the second derivative that resulted from Taylor series expansion [6]:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \tag{8}$$

Additional terms of Halley method include additional calculation on each iteration. His method is more dependent on the initial guess but have a strong convergence property [9].

Proper choice of the initial guess can greatly reduce the computation and guarantee the convergence of method. In 1978, Smith [10] found that the root of Kepler's equation is between two values M and $M + e$ by replacement of with E. Then, by using the equation of a straight line between M and M+e, introduced this initial guess for Kepler's equation [10]:

$$E_0 = M + e\frac{sinM}{1 - \sin(M + e) + sinM} \tag{9}$$

In fact, his initial guess is a linear approximation of the root of Kepler's equation between two points M and $M + e$. He compared this initial guess with Newton's method in two regions with some other value. Regions deemed by him as follows:

Region 1: $0.05 \leq \mathrm{M} \leq \pi$ and $0.01 \leq e \leq 0.99$.
Region 2: $0.005 \leq M \leq 0.4$ and $0.95 \leq e \leq 0.999$.

His guesses are in the Table 1. Smith's criterion for a good initial guess was the average number of iterations to reach a solution by Newton's method in each region. He considered tolerance 5×10^{-8} to stop the algorithm. Finally, after comparing the initial guess concluded that without initial guess $M + e$, initial guess (9) is the best of the number of iterations. Difference between the Smith's initial guess and $M + e$ in the region 2 is not big. So, the initial guess introduced as the best one. By using it the Newton's method doesn't need to add correction clauses to avoid divergence and the number of iteration of this approach is also desirable.

In 1979, Edward NG [9] used a method like Halley. He considered for distinct areas in space (M, e), and used a different value for each area. The first three areas had current calculations, but found that the Kepler's equation treats like a third degree function near this point $(M, e) = (0,1)$; so, used third degree root for this area.

Another iterative method was introduced by Danby [11] in 1983, he argued that the degree of convergence goes upper, the sensitivity of initial value and risk of diverging reduced. Danby method shows as below:

$$x_{n+1} = x_n + \delta_n$$

$$\delta_{n1} = -\frac{f}{f'} \tag{10}$$

$$\delta_{n2} = -\frac{f}{f' - \frac{1}{2}\delta_{n1}f''} \tag{11}$$

$$\delta_{n3} = -\frac{f}{f' - \frac{1}{2}\delta_{n2}f'' + \frac{1}{6}\delta_{n2}^2 f'''} \tag{12}$$

By using equation (10), the Newton's method with second degree convergence will be resulted. If equation (11) used, the Halley's method with third degree convergence will be resulted and using equation (12) will result fourth degree convergence. Danby used initial guess $E = M$ in his article, but later in 1987 [12], understood that division of computing is useful in two regions. So declared below guesses:

$$E_0 = \begin{cases} M + \left((6M)^{1/3} - M\right)e^2 & 0 \leq M < 0.1 \\ M + 0.85e & 0.1 \leq M \leq \pi \end{cases} \tag{13}$$

In 1986, Serafin [13] stated that a good choosing for initial guess E, needed rang the root of E belong to it. He defined intervals that include root of Kepler's equation by using the property of $sine$ function. Table 2 shows his results. In the same year, Conway [14] stated a method base on Leguerre's method that using for finding the root of a polynomial.

$$x_{i+1} = x_i - \frac{nf(x_i)}{f'(x_i) \pm \sqrt{(n-1)^2(f'(x_i))^2 - n(n-1)f''(x_i)}} \tag{14}$$

Table 1. Smith's initial guesses

Initial guesses
$E_0 = M$
$E_0 = M + e$
$E_0 = M + e\ sin\ M$
$E_0 = M + e\frac{sinM}{1 - \sin(M + e) + sinM}$
$E_0 = M + e\sin M + e^2 \sin M \cos M$
$E_0 = M + \alpha\left(1 - \frac{\alpha^2}{2}\right), \quad \alpha = \frac{e\sin M}{1 - e\cos M}$

He selected $n = 5$. The convergence of this method, regardless of the initial guess is guaranteed. Odell and Gooding [15], in a part of their article studied twelve different initial guesses. They believe that rapid convergence in small M, and big e, only can be possible when initial value shows a good status of E. Then, they stated their method. In 1989, Taff [16] evaluated thirteen different initial guesses and finally the best and simplest initial guess and solution stated in order $E_0 = M + e$ and Wegstein's method. Nijenhuis in 1991 [17] in his article, divided (M, e) space to four areas and using a different initial guess for each area. His work was like Edward regardless stated different areas. His initial guesses are:

1- Area of A that includes big M:

$$E = \frac{M + e\pi}{1 + e} \tag{15}$$

2- Area of B for middle M:

$$E = \frac{M}{1 - e} \tag{16}$$

3- Area of C for small M:

$$E = \frac{M}{1 - e} \tag{17}$$

4- Area of D includes a space near the point $(M, e) = (0,1)$ that uses Mikkola's method:

$$E = M + e(3s - 4s^3) \tag{18}$$

Outline of his method contain three steps. First, space of (M, e) is divided into four separate areas and defines the initial guess for each area. Second, refine initial guesses of three areas by one use of Halley's method and for area of four, by one use of Newton's method. Third, modified Newton's method used for solving Kepler's equation. His modification of Newton's method is different from what Danby used. Chobotov [18] compared Newton's method with Conway's method. He shows although convergence of Conway's method is guarantee, Newton's method for computing the execution time is preferred. In 1997, Vallado [19] solved the Kepler's equation for elliptic, parabolic and hyperbola orbits. For elliptical and hyperbola orbits, studied the number of iteration of Newton's method for three below initial guess:

$$M \tag{19}$$

$$M + e \tag{20}$$

$$M + e\,sinM + \frac{e^2}{2} sin2M \tag{21}$$

Table 2. Intervals of E_0

$M \in$	E_0
$[0, 1 - e\alpha]$	$\frac{M}{1 - 2\frac{e}{\pi}} \le E \le \frac{M}{1 - e}$
$[1 - e\alpha_0, \frac{\pi}{2} - e]$	$\frac{M}{1 - 2\frac{e}{\pi}} \le E \le \mathrm{M + e}$
$[\frac{\pi}{2} - e, \pi - (1 - e\alpha_0)]$	$\frac{M + 2e}{1 + 2\frac{e}{\pi}} \le E \le \mathrm{M + e}$
$[\pi - (1 - e\alpha_0), \pi]$	$\frac{M + 2e}{1 + 2\frac{e}{\pi}} \le E \le \frac{M + e\pi}{1 + e}$

He concluded that Newton's method with initial guess (21) compare with two others is convergent with less number of iteration. But due to transcendental function, amount of computing time is more than two others. He understood initial guess (20) is 15% faster than guess (19), while the initial guess of (21) is only 2% faster. He choose initial guess (20) for elliptic and hyperbola orbits base on the overall computation time to avoid the divergence of Newton's method and reduce the computation time. In 1998, Charles [20] stated the Newton's chaotic behavior and examined it for these guesses: $E_0 = M$, $E_0 = \pi$. He stated that Newton's method all the time is convergent for $E_0 = \pi$, but for $\mathrm{E_0 = M}$ there is a possibility of divergence. Bellow equation is provided to obtain a better initial guess:

$$E_0 = M + e[(\pi^2 M)^{1/3} - \frac{\pi}{15} sinM - M] \tag{22}$$

Curtis, in 2010 [3], used following initial guesses for the solution of Kepler's equation:

$$E_0 = \begin{cases} M + \frac{e}{2}, & M < \pi \\ M - \frac{e}{2}, & M > \pi \end{cases} \tag{23}$$

Among the presented paper, these that have focused on iterative methods are considered and after collecting used initial guesses, examine them with suggested method and choose the best of them in term of the number of iterations.

Choosing a good initial guess can significantly reduce the number of required iterations. In general, two clear demand of an initial guess for solving Kepler's equation must be fast and accurate enough. Being a fast initial guess can be determined by counting the number of iteration do reach a solution or by measuring time of computation.

Table 3 shows the initial guesses that will be tested. First guess from Table 3 is the simplest that can be considered. The root of Kepler's equation is between two values, these values as guesses 2 and 3 that belong to Smith's work are considered. The conjectures of 4, 5 and 6 from Table 3, are obtained from E power series expansion that has one, two and three sentences. The conjecture of 7 from Table 3 results of this unequal $|\sin E| \le |E|$ and $M - E = e \sin E$ [13]. Conjectures of 8 and 9 belong to Smith's work. Conjecture 11, is resulted of one using Newton- Raphson's method on initial π value. Conjectures 12, 20, 21, 22 from Table 3 are the solution of $\mathrm{S_9}$, $\mathrm{S_8}$, $\mathrm{S_{10}}$‘ $\mathrm{S_{12}}$ Odell and Gooding.

Conjecture 13 from Table 3 belongs to Danby in 1987. The conjecture 14 is the result of linear interpolation between 2 and 16 conjectures. Conjecture 16, introduced by Edward Ng. conjectures 17 and 18 from Table 3 are the above and below roots of Kepler's equation that introduce by Serafin, and conjecture 19 belongs to Charles paper.

3. THE PROPOSED ALGORITHM

Two important factors are considered in the compassion of the iterative methods are: Number of iterations and how much work must the considered method do for each iteration. The ideal solution is a balance between these two indicators. Since Newton's method known standard method for solving Kepler's equation and among the method's that have been proposed so far, has the lowest calculations, this method will be chosen and try to minimize the number of iterations by selecting a good initial guess. The only problem with this method is the possibility of divergence in some areas that can be solved by defining of different initial guesses for some areas first of all. First, rough initial guesses are refined with one use of Newton's method.

$$f(E) = E - M - e\ sinE$$

$$E_1 = E_0 - \frac{f(E_0)}{f'(E_0)} \tag{24}$$

Then these refined guesses are used as the starter in the solution of Kepler's equation. The using algorithm has two steps: first, using Newton's method on the initial guesses, second, solving Kepler's equation by Newton's method and refined guesses. To stop this algorithm, this tolerance (10^{-10}) has been considered.

4. RESULTS

By using any of the initial guess in Table 3, the Kepler's equation is solved in this range $0 \le e \le 1$, $0 \le M \le \pi$ with semi-major axis with this length $a = 5850.6753\ km$ by MATLAB software. Then results introduce in $M - e$ diagrams. Some of them are showed in below.

In Figure 1, the lowest iteration occurs at small M. Important property of this is no divergence near the singularity point $(M, e) = (0,1)$. The number of iterations of this guess in small M is less than guesses number 8 and 12 from Table 3. In Figure 2, almost three repeats for middle M is observed. According to Figures 3 and 4, Newton's method in a wide range of space (M, e) has the lowest iteration but near the point $(M, e) = (0,1)$ number of iterations increased. In Figure 5, best performance for $M > 2$ can be observed. Figure 6 shows the five repeats, but this guess shows increasing the number of iteration for small M, $e > 0.99$ and going to become divergence. According to the Figure 7, guess number 13 from table 3 has better convergence than initial guess number 10 from Table 3. According to Figure 8, guess number 16 from Table 3, has three iterations in wide space has the best performance in small M.

Among the initial guesses were examined, two conjectures 8 and 12 from Table 3, have the best performance of number of iteration in the whole space. On the other hand, these are the fastest guesses of convergence. So they will be chosen because they supply an overall convergence for Newton's method with minimum number of iteration. For reaching the minimum iteration in the whole space (M, e), it is divided into three area and have been considered an initial guess for each area according to the results.

$$E_0 = \begin{cases} M + \left((6M)^{1/3} - M\right)e^2\ , & 0 \le M \le 0.25 \\ M + e\dfrac{sinM}{1 - \sin(M+e) + sinM}\ , & 0.25 \le M \le 2 \\ M + \dfrac{esinM}{\sqrt{1 - 2ecosM + e^2}}\ , & 2 \le M \le \pi \end{cases} \tag{25}$$

Table 3. Initial guesses

	E_0		E_0
1	π	12	$M + \frac{esinM}{\sqrt{1 - 2ecosM + e^2}}$
2	M	13	$M + 0.85e$
3	$M + e$	14	$M + \left((6M)^{1/3} - M\right)e^2$
4	$M + e\ sinM$	15	$M - e$
5	$M + e\ sinM + \frac{e^2}{2}sin2M$	16	$(6M)^{1/3}$
6	$M + e\ sinM + \frac{e^2}{2}sin2M + \frac{e^3}{8}(3sin3M - sinM)$	17	$\frac{M + 2e}{1 + 2\frac{e}{\pi}}$
7	$\frac{M}{1 + e}$	18	$\frac{M + e\pi}{1 + e}$
8	$M + e\frac{sinM}{1 - \sin(M+e) + sinM}$	19	$M + e[(\pi^2 M)^{1/3}\frac{\pi}{15}sinM - M]$
9	$M + \alpha\left(-\frac{\alpha^2}{2}\right)$, $\alpha = \frac{e\ sinM}{1 - e\ cosM}$	20	$s + \frac{\pi}{20}e^4(\pi - s), s = M + esin + e^2 sinMcosM$
10	$M + \frac{e}{2}$	21	$s - \frac{q}{s}\ , s = [(r^2 + q^3)^{\frac{1}{2}} + r]^{\frac{1}{3}}, q = \frac{2(1-e)}{e}\ , r = \frac{3M}{e}$
11	$M + \frac{e(\pi - M)}{1 + e}$	22	$eE_{01} + (1 - e)M, E_{01} = \pi - \frac{(\pi - 1)^2(\pi - M)}{2\left(\pi - \frac{1}{6}\right)^2 - (\pi - M)(\pi - \frac{2}{3})}$

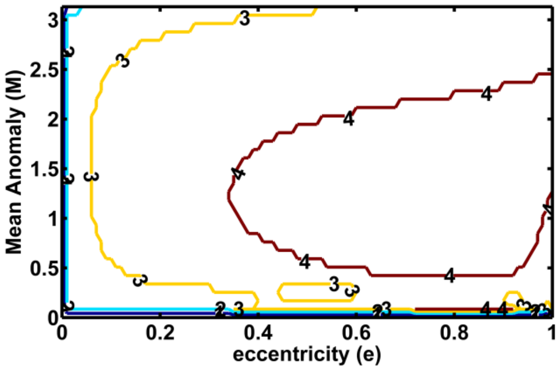

Fig 1: Number of iterations with initial guess

$$M+\left((6M)^{1/3}-M\right)e^2$$

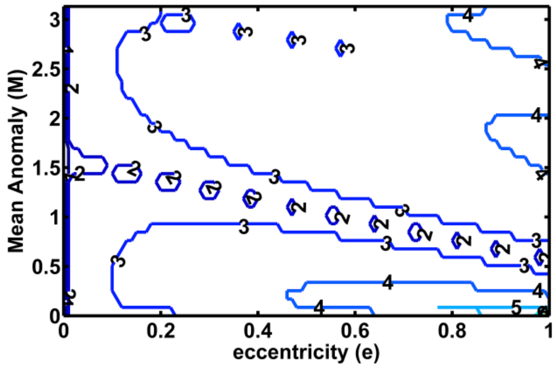

Fig 2: Number of iterations with initial guess

$$M+e$$

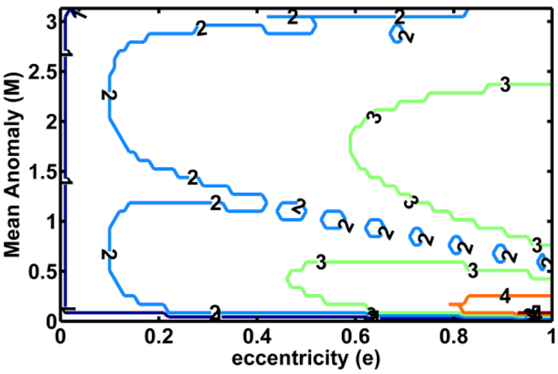

Fig 3: Number of iterations with initial guess

$$M+e\frac{\sin M}{1-\sin(M+e)+\sin M}$$

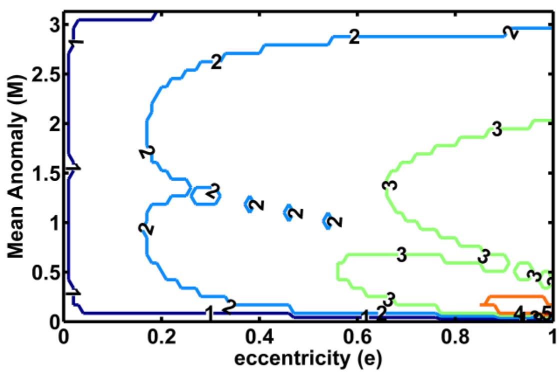

Fig 4: Number of iterations with initial guess

$$M+\frac{e\sin M}{\sqrt{1-2e\cos M+e^2}}$$

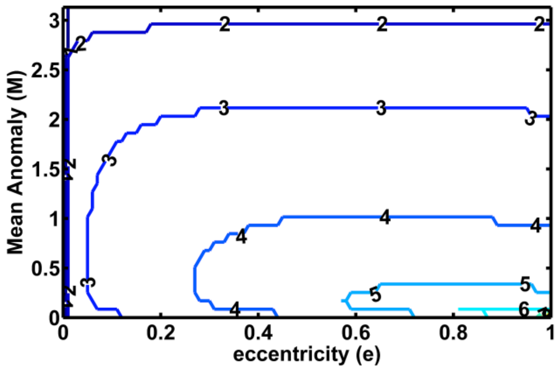

Fig 5: Number of iterations with initial guess

$$M+\frac{e(\pi-M)}{1+e}$$

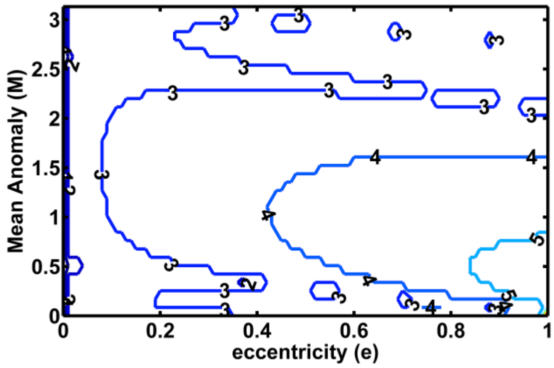

Fig 6: Number of iterations with initial guess

$$M+\frac{e}{2}$$

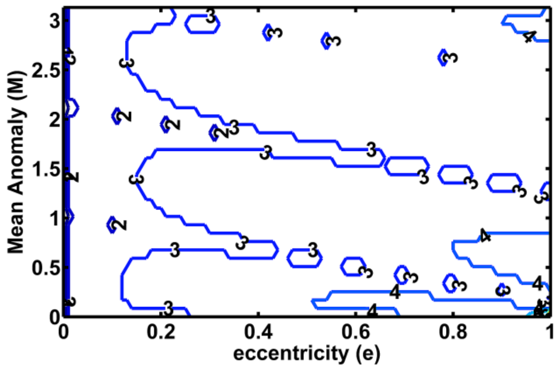

Fig 7: Number of iterations with initial guess

$$M+0.85e$$

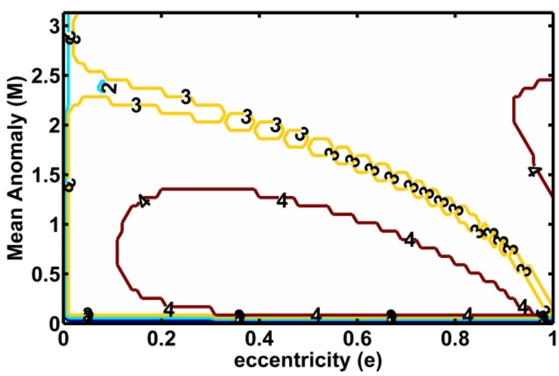

Fig 8: Number of iterations with initial guess

$$(6M)^{1/3}$$

The minimum number of iteration means that the speed of convergence is more [13]. Then, rate convergence Newton's method increase by these initial values. But, a smaller number of iterations does not necessarily imply a shorter computation time. For any given algorithm, the associated computing time will vary with the computer used to perform the calculation. The computational efficiency will also depend on the manner in which the algorithm is implemented. Computation time (or runtime) reduces whatever number of multiplications, divisions, square roots, additions, subtractions and trigonometry function is minor. Then, inexpensive initial guesses can affect strongly on computation time. Hereinafter, the runtime of various conjectures will be surveyed briefly and finally the initial values will be chosen that increase convergence rate and decrease computation time of Newton-Raphson's method at the same time.

Style implementation of Newton-Raphson's method equal Tewari's approach [21]. Used Intel Core i7-2630QM and DDR III 8G (4GB×2) RAM. The e value with step 0.01 between $0 \leq \mathrm{e} \leq 1$ and the M value with the time step 60 second (0.875 radiant) are changed that the M's range is defined in the range of equation (25) and the run time of algorithm for each initial guess and range are measured.

According to Figures of 9, 10, 11 the difference between time of calculating of initial conjectures are low near the π point (millisecond), and also the suggested algorithm for initial guess 10 from table 3 has the least runtime for $0 < m < 0.25$ and $0.25 < m < 2$, in other word, has the maximum speed in calculations.

Initial guess 11 from Table 3 for $2 < m < \pi$ has the least runtime. And after it the guess 18 is, but these values are not as well as values of equation (25) in convergence.

Guess 15 from Table 3 has the lower calculation time in Comparison of guess 21 from Table 3 for $0 < m < 2$ it's because of nonlinear functions in guess 21; so, for having a good choice of speed of convergence and calculation, the values of equation (25) are corrected as follow:

$$E_0 = \begin{cases} M + e\dfrac{sinM}{1 - \sin(M+e) + sinM} & 0 < M < 0.25 \\ M + e\ , & 0.25 < M < 2 \\ M + \dfrac{e(\pi - M)}{1 + e}\ , & 2 < M < \pi \end{cases} \quad (26)$$

Equation (26) is similar to Odell and Gooding with this difference that they used $\frac{M}{1-e}$ for small M. The fault of it is, in the small M and e near number one become divergent, because of it, Nijenhuis used Mikkola's solution for this range, but increase the time of calculation. In follow, although, there is no demand for using Mikkola's solution by suggested conjectures, the time of calculation will be reduced and also the speed of convergence will be maintained.

Figures 12 and 13 show Comparison of suggested method with Nijenhuis method. Nijenhuis's solution has minimum repetition because uses Halley's method for refining the conjectures and also using Newton's method of degree of three for solving Kepler's equation and reaches to maximum three repetition in one small area.

However, the suggested method use Newton's method for refining conjectures and solving Kepler's equation has maximum four repetitions.

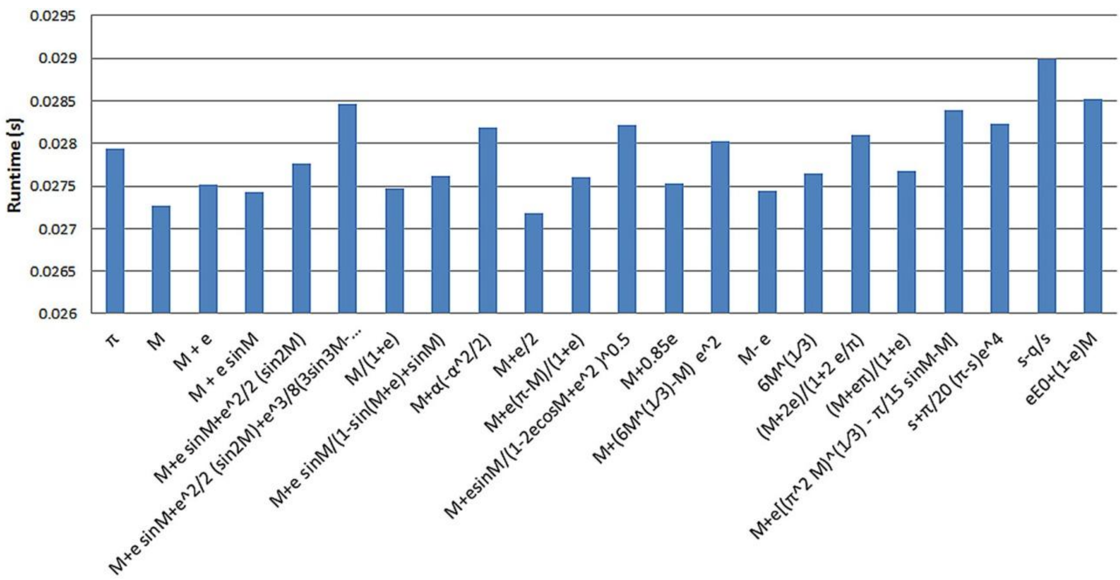

Fig 9: Runtime for each initial guess within $\mathbf{0} < e < 1,\ \mathbf{0} < M < 0.25$

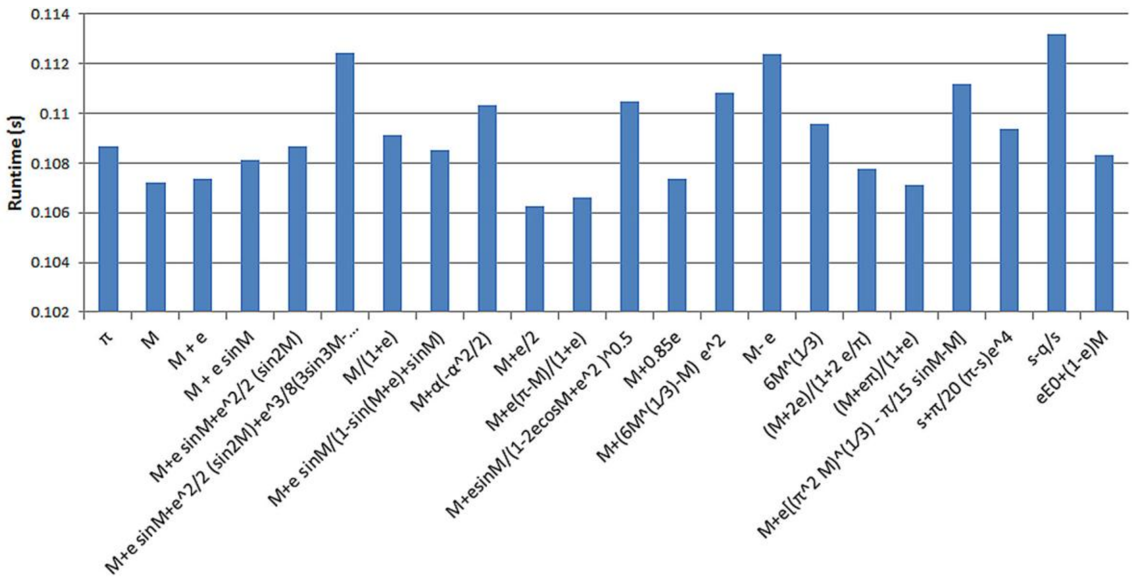

Fig 10: Runtime for each initial guess within 0 $< e < 1$, $0.25 < M < 2$

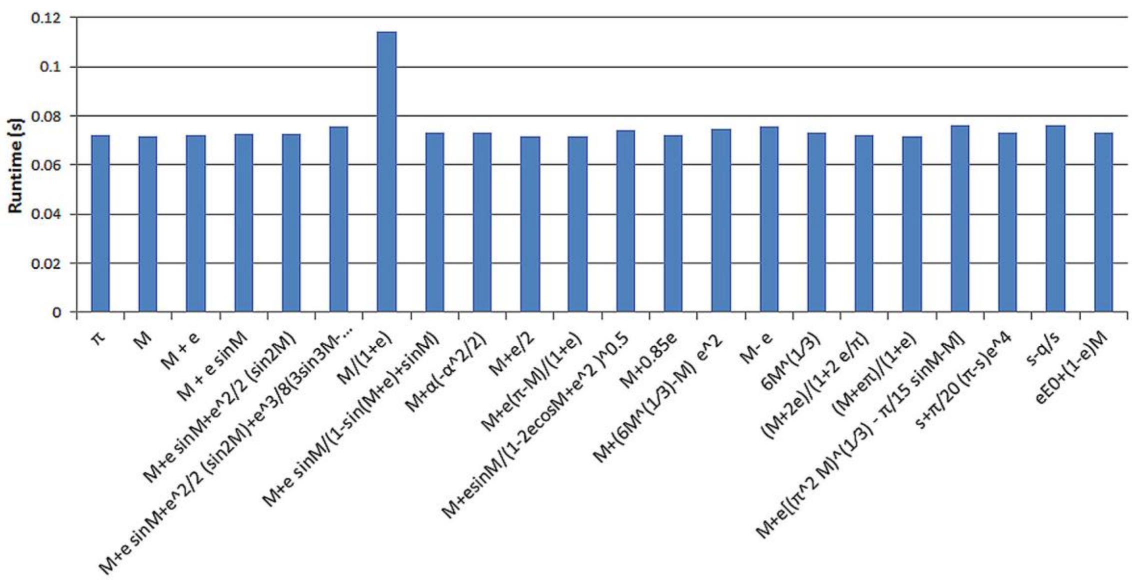

Fig 11: Runtime for each initial guess within 0 $< e < 1$,$2 < M < \pi$

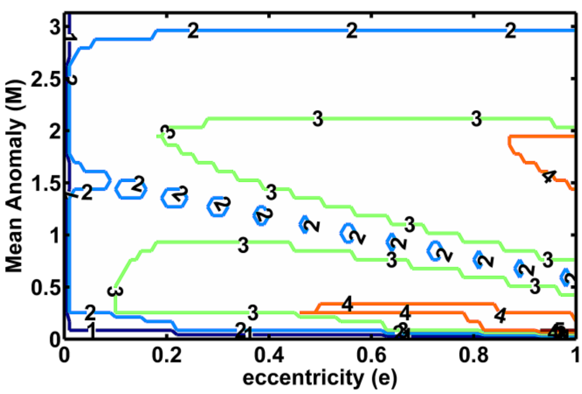

Fig 12: Number of iteration of newton's method with conjectures of equation (26)

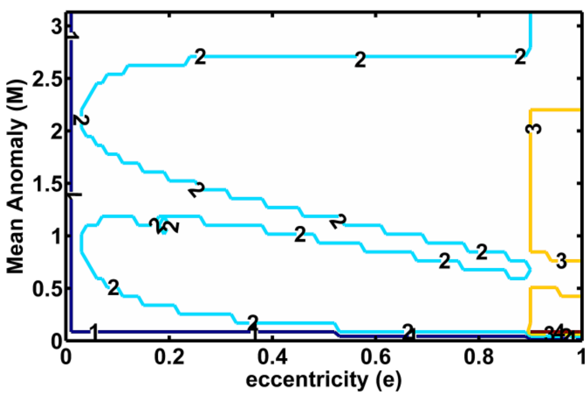

Fig 13: Number of iteration of Nijenhuis 's method

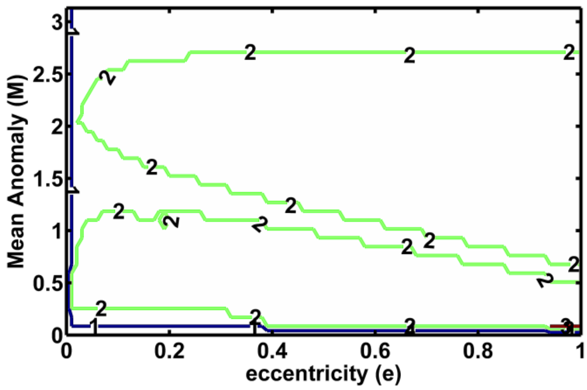

Fig 14: Number of repetition of Nijenhuis's method with conjectures of equation (26)

If suggested conjectures (26) combine with Nijenhuis's method, a speed of convergence like Nijenhuis's solution will be obtained, and there is no need using Mikkola's solution for avoiding divergence. It is showed in Figure 14, also the time of calculation reduce by deleting Mikkola's solution.

In the Table 4 the runtime of these three approaches is compared.

According to Table 4, Newton's method with the new guesses has the lowest runtime, and time of calculation of Nijenhuis's method with the new guesses reduce almost 4 millisecond, so, Conjectures of equation (26) are better than others in speed of convergence and runtime. Combining Conjectures of equation (26) with better methods (like Nijenhuis's method) is better than Newton's method.

Table 4. comparison of time of whole run

method	Time of whole run of algorithm (millisecond)
Equation (26) with Newton's method	176.791
Equation (26) with Nijenhuis's method	188.843
Nijenhuis's method	192.681

5. CONCLUSION

In this article, Newton's method has been chosen as a known standard method for solving Kepler's equation and tested different initial guesses. The used method has two steps: first, refining guesses with one use of Newton's method; second, solving Kepler's equation with this method. The speed of convergence of initial guesses were studied and chosen the best of them. Then examined the calculation time of them and to reach these two properties in the whole space (M, e), divided this space into three spaces. That used different initial guesses for each area according to equation (26). The initial conjectures of equation (26) are not the best but have two properties in the same time: speed in convergence and calculation. For this reason, they are considered as optimized initial conjectures for suggested algorithm. By them, the solving method reached a good speed in convergence and time of calculation.

6. REFERENCES

[1] R. H. Battin. 2001. An Introduction to the Mathematics and Methods of Astrodynamics. Revised Edition, AIAA education series, pp. 160.

[2] P. Colwell. 1993. Solving Kepler equation over three centuries. Willmann-Bell, Ins.

[3] H. D. Curtis. 2010. Orbital Mechanics for engineering students. 2nded, Elsevier Aerospace Engineering Series p 163, pp. 168-170.

[4] Mikkola. 1987. A cubic approximation for Kepler's equation. Celestial Mech 40. pp. 329-334.

[5] Markley. 1995. Kepler equation solver. Celestial Mech Dyn Astr 63. pp. 101-111.

[6] S. A. Feinstein, C. A. McLaughlin. 2006. Dynamic discretization method for solving Kepler's equation. Celestial Mech Dyn Astr 96. pp. 49-62.

[7] D. Mortari, A. Clocchiatti. 2007. Solving Kepler's equation using Bezier curves. Celestial Mech Dyn Astr 99. pp. 45-57.

[8] J. T. Betts. 2010. Practical Methods for Optimal Control and Estimation Using Nonlinear Programming. Second Edition, Siam, Philadelphia. pp. 3.

[9] Edward W .NG. 1979. A general algorithm for the solution of Kepler's equation for elliptic orbits. Celestial Mech *20*. pp. 243-249.

[10] Smith. 1979. A simple efficient staring value for the iterative solution of Kepler's equation. Celestial Mech 19. pp. 163-166.

[11] J.M.A. Danby, T. M. Burkardt .T.M. 1983. The solution of Kepler's equation, I. Celestial Mech 31. pp. 95-107.

[12] J.M.A. Danby. 1987. The solution of Kepler's equation, III. Celestial Mech *40*. pp. 303-312.

[13] R.A. Serafin. 1986. Bounds on the solution to Kepler's equation. Celestial Mech 38. pp. 111-121.

[14] B. A. Conway. 1986. An improved algorithm due to laguerre for the solution of Kepler's equation. Celestial Mech 39. pp. 199-211.

[15] A. W. Odell, R. H. Gooding. 1986. Procedures for solving Kepler's equation. Celestial Mech 38. pp. 307-344.

[16] L. G. Taff, T. A. Brenan. 1989. On solving Kepler's equation. Celestial Mech Dyn Astr 46. pp. 163-176.

[17] Nijenhuis. 1991. Solving Kepler's equation with high efficiency and accuracy. Celestial Mech Dyn Astr 51. pp. 319-330.

[18] Chobotov. 1996. Orbital Mechanics. 2nded, AIAA education series.

[19] D. A. Vallado. 2001. Fundamentals of Astrodynamics and Application. Space Technology Series, McGraw-Hill. pp. 211, pp. 230-244.

[20] E. D. Charles, J. B. Tatum. 1998. The convergence of Newton-Raphson iteration with Kepler's equation. Celestial Mech Dyn Astr 69. pp. 357-372.

[21] A. Tewari. 2006. Atmospheric and space flight dynamics. Birkhauser. pp. 105.