Coefficient Problem for Certain Classes of Analytic Functions using Hankel Determinant

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Abstract: **In this paper, we introduce some classes of analytic-univalent functions and for any real μ, determine the sharp upper bounds of the functional $|a_2a_4 - \mu a_3^2|$ for the functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to such classes in the unit disc $E = \{z : |z| < 1\}$.**

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I. INTRODUCTION

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

in the unit disk $\Delta = \{z : |z| < 1\}$.

Let S be the class of functions $f(z) \in A$ and univalent in Δ. In the present paper, we consider the following subclasses of A.

Definition 1.1.
Let

$$S^*_{S(\lambda)} = \left\{ f \in A, Re\left[\frac{2[\lambda z^2 f''(z) + zf'(z)]}{\lambda z[f'(z) + f'(-z)] + (1-\lambda)[f(z) - f(-z)]}\right] > 0, 0 \le \lambda \le 1 \right\},$$

the class of starlike functions with respect to symmetric points.

$$K_{S(\lambda)} = \left\{ f \in A, Re\left[\frac{2[\lambda z^2 f''(z) + zf'(z)]'}{[\lambda z[f'(z) + f'(-z)] + (1-\lambda)[f(z) - f(-z)]]'}\right] > 0, 0 \le \lambda \le 1 \right\},$$

the class of convex functions with respect to symmetric points.

$$C_{S(\lambda)} = \left\{ f \in A, Re\left[\frac{2[\lambda z^2 f''(z) + zf'(z)]}{\lambda z[g'(z) + g'(-z)] + (1-\lambda)[g(z) - g(-z)]}\right] > 0, g \in S^*_{S(\lambda)}, z \in \Delta \right\},$$

the class of close-to-convex functions with respect to symmetric points.

The classes K_S and C_S were introduced by Das and Singh [2].

$$C_{S1(\lambda)} = \left\{ f \in A, Re\left[\frac{2[\lambda z^2 f''(z) + zf'(z)]}{\lambda z[h'(z) + h'(-z)] + (1-\lambda)[h(z) - h(-z)]}\right] > 0, h \in K_{S(\lambda)}, z \in \Delta \right\},$$

$$C'_{S1(\lambda)} = \left\{ f \in A, Re\left[\frac{2[\lambda z^2 f''(z) + zf'(z)]'}{[\lambda z[h'(z) + h'(-z)] + (1-\lambda)[h(z) - h(-z)]]'}\right] > 0, h \in K_{S(\lambda)}, z \in \Delta \right\}.$$

Definition 1.2.
Let

$$S_{S(\gamma,\lambda)} = \left\{ f \in A : Re\left[\frac{2[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma - \lambda) z^2 f''(z) + zf'(z)]}{\gamma\lambda z^2[f''(z) - f''(-z)] + (\gamma - \lambda) z[f'(z) + f'(-z)] + (1-\gamma+\lambda)[f(z) - f(-z)]}\right] > 0, \; 0 \le \lambda \le \gamma \le 1 \right\},$$

the class of starlike functions with respect to symmetric points (1.1)

$$K_{S(\gamma,\lambda)} = \left\{ f \in A : Re\left[\frac{2[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma - \lambda) z^2 f''(z) + zf'(z)]'}{[\gamma\lambda z^2[f''(z) - f''(-z)] + (\gamma - \lambda) z[f'(z) + f'(-z)] + (1-\gamma+\lambda)[f(z) - f(-z)]]'}\right] > 0, \; 0 \le \lambda \le \gamma \le 1 \right\},$$

the class of convex functions with respect to symmetric points.

$$C_{S(\gamma,\lambda)} = \left\{ f \in A : Re\left[\frac{2[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma - \lambda) z^2 f''(z) + zf'(z)]}{\gamma\lambda z^2[g''(z) - g''(-z)] + (\gamma - \lambda) z[g'(z) + g'(-z)] + (1-\gamma+\lambda)[g(z) - g(-z)]}\right] > 0, \; g \in S^*_S, z \in \Delta \right\},$$

the class of close-to-convex functions with respect to symmetric points.

$$C_{S2(\gamma,\lambda)} = \left\{ f \in A : Re\left[\frac{2[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma - \lambda) z^2 f''(z) + zf'(z)]}{\gamma\lambda z^2[h''(z) - h''(-z)] + (\gamma - \lambda) z[h'(z) + h'(-z)] + (1-\gamma+\lambda)[h(z) - h(-z)]}\right] > 0, \; h \in K_S, z \in \Delta \right\},$$

$$C'_{S2(\gamma,\lambda)} = \left\{ f \in A : Re\left[\frac{2[\gamma\lambda z^3 f'''(z) + (2\gamma\lambda + \gamma - \lambda) z^2 f''(z) + zf'(z)]'}{[\gamma\lambda z^2[h''(z) - h''(-z)] + (\gamma - \lambda) z[h'(z) + h'(-z)] + (1-\gamma+\lambda)[h(z) - h(-z)]]'}\right] > 0, \; h \in K_S, z \in \Delta \right\},$$

In 1976, Noonan and Thomas [12] stated the q^{th} Hankel determinant for $q \ge 1$ and $n \ge 1$ as

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$$Hq(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & & \cdots \\ a_{n+q+1} & \cdots & & a_{n+2q-2} \end{vmatrix}$$

This determinant has also been considered by several authors, for example, Noor [13] determined the rate of growth at $Hq(n)$ as $n \to \infty$ for functions given by Equation (1.1) with bounded boundary. Ehrenborg [3] studied the Hankel determinant of exponential polynomials and in [6], the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman.Also Hankel determinant was studied by various authors including Hayman [5] and Pommerenke [14] and recently by Choc and Janteng [1], Mehrok and Singh [9] and Janteng et al. [10, 11].

Easily, one can observe that the Fekete and Szeg¨o functional is $H_2(1)$. Fekete and Szeg¨o [4] then further generalized the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$$

In this paper, we seek upper bound for the functional $|a_2a_4 - \mu a_3^2|$ where μ is real, for the functions belonging to the above defined classes.

II. Preliminary Results

Let P be the family of all functions p analytic in Δ for which $Re(P(z)) > 0$ and

$$P(z) = 1 + p_1z + p_2z^2 + \cdots \text{ for } z \in \Delta.(2.1)$$

Lemma 2.1. *[14]*

If $p \in P$, then $|p_k| \le 2$ $(k = 1,2,3,...)$

Lemma 2.2. *[7, 8]*

If $p \in P$, then $2p_2 = p_1^2 + (4 - p_1^2)x$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z, satisfying $|x| \le 1$, $|z| \le 1$ *and* $p_1 \in [0,2]$.

III. Main Result

Theorem 3.1.

If $f(z) \in C_{S(\lambda)}$ *then*

$$\left(\frac{[(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\mu]^2}{[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]} - (1+\lambda)(1+3\lambda)\mu \right.$$

$$|a_2a_4 - \mu a_3^2| \le \begin{cases} (1+2\lambda)^2 - (1+\lambda)(1+3\lambda)\mu & i\,0 \le \mu \le \tfrac{1}{2}f \\ (1+\lambda)(1+3\lambda)\mu & if,if\ \mu \ge 1. \\ \frac{[(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2]^2}{[2(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2]} + \mu(1+\lambda)(1+3\lambda) & \end{cases}$$

Proof:

Since $f \in C_S$, by definition we have

$$\frac{2[\lambda z^2 f''(z) + zf'(z)]}{\lambda z[g'(z) + g'(-z)] + (1-\lambda)[g(z) - g(-z)]} = p(z \quad (3.1)$$

where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_{S^*(\lambda)} \quad (3.2)$$

Using (1.1), (2.1) and (3.1), (3.2) gives

$$1 + 2(1+\lambda)a_2z + (1+2\lambda)3a_3z^2 + (1+3\lambda)4a_4z^3$$
$$= (1 + p_1z + p_2z^2 + \cdots)[1 + (1+2\lambda)b_3z^2 + \cdots] \quad (3.3)$$

On equating co-efficients in (3.9), we get

$$2(1+\lambda)a_2 = p_1,$$
$$3a_3(1+2\lambda) = (1+2\lambda)b_3 + p_2,$$
$$4a_4(1+3\lambda) = (1+2\lambda)p_1b_3 + p_3 \quad (3.4)$$

From (3.2), we can easily verify that

$$b_3 = \frac{p_2}{2(1+2\lambda)}$$

So (3.4) yields

$$a_2 = \frac{p_1}{2(1+\lambda)}, \quad a_3 = \frac{p_2}{2(1+2\lambda)}, \quad a_4 = \frac{p_3}{4(1+3\lambda)} + \frac{p_2p_1}{8(1+3\lambda)} \quad (3.5)$$

From (3.5),

$$|a_2a_4 - \mu a_3^2| = \left| \frac{p_1p_3}{8(1+\lambda)(1+3\lambda)} + \frac{p_2p_1^2}{16(1+\lambda)(1+3\lambda)} - \mu \frac{p_2^2}{4(1+2\lambda)^2} \right|$$

.

Let $X = 32(1+\lambda)(1+3\lambda)(1+2\lambda)^2$.

Using Lemma 2.2, it gives

$$|a_2a_4 - \mu a_3^2| = \frac{1}{X} \left| \begin{array}{l} 2[(1+2\lambda)^2 - (1+\lambda)(1+3\lambda)\mu]p_1^4 \\ +[3(1+2\lambda)^2 - 4(1+\lambda)(1+3\lambda)\mu](4-p_1^2)p_1^2x \\ +2(1+2\lambda)^2p_1(4-p_1^2)[1-|x|^2]z \\ -(4-p_1^2)x^2[[(1+2\lambda)^2 - 2(1+\lambda)(1+3\lambda)\mu]p_1^2 \\ +8(1+\lambda)(1+3\lambda)\mu] \end{array} \right|$$

Suppose now that $p_1 = p$, $p \in [0,2]$ and using triangle inequality, we get

$$|a_2a_4 - \mu a_3^2| \le \frac{1}{X} \left[\begin{array}{l} 2|(1+2\lambda)^2 - (1+\lambda)(1+3\lambda)\mu|p^4 \\ +2(1+2\lambda)^2p(4-p^2) \qquad \text{if } \mu \le 0, \\ +|3(1+2\lambda)^2 - 4(1+\lambda)(1+3\lambda)\mu|p^2(4-p^2)\rho \\ +(4-p^2)([|(1+2\lambda)^2 - 2(1+\lambda)(1+3\lambda)\mu|p^2 \\ +8|(1+\lambda)(1+3\lambda)\mu| - 2(1+2\lambda)^2p)\rho^2 \end{array} \right]$$

$$= F(\rho) \text{ with } \rho = |x| \le 1.$$

This gives rise to

$$F'(\rho)=\begin{cases}\frac{1}{X}\begin{bmatrix}[3(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\mu](4-p^2)p^2\\+2(4-p^2)+[[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\-8(1+\lambda)(1+3\lambda)\mu-2(1+2\lambda)^2p]\rho\end{bmatrix} & \text{if } \mu<0,\\[2ex] \frac{1}{X}\begin{bmatrix}[3(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\mu](4-p^2)p^2\\+2(4-p^2)+[[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+8(1+\lambda)(1+3\lambda)\mu-2(1+2\lambda)^2p]\rho\end{bmatrix} & \text{if } 0\le\mu\le\frac{1}{2},\\[2ex] \frac{1}{X}\begin{bmatrix}[3(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\mu](4-p^2)p^2\\+2(4-p^2)+[-[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+8(1+\lambda)(1+3\lambda)\mu-2(1+2\lambda)^2p]\rho\end{bmatrix} & \frac{1}{2}\le\mu\le\frac{3}{4}\end{cases}$$

if,

$$\frac{1}{X}\begin{bmatrix}-[3(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\mu](4-p^2)p^2\\+2(4-p^2)+[-[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+8(1+\lambda)(1+3\lambda)\mu-2(1+2\lambda)^2p]\rho\end{bmatrix} \text{if } \mu\ge\frac{3}{4}.$$

and for all the cases above, $F^{0}(\rho)>0$ for $\rho>0$; implying that

Max $F(\rho)=F(1)$

Now let

$$G(p)=F(1)=\frac{1}{X}\begin{bmatrix}2[(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\mu]p^4\\+2(1+2\lambda)^2p(4-p^2)+[3(1+2\lambda)^2\\-4(1+\lambda)(1+3\lambda)\mu]p^2(4-p^2)\\+(4-p^2)[[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+8[(1+\lambda)(1+3\lambda)\mu]-2(1+2\lambda)^2p]\end{bmatrix} \quad (3.7)$$

Now we discuss the following cases:

Case I: For $\mu\le 0$,

$$G(p)=\frac{1}{X}\begin{bmatrix}-2[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^4\\+16[(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\mu]p^2\\-32(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

and

$$G'(p)=\frac{1}{4(1+\lambda)(1+3\lambda)(1+2\lambda)^2}p\begin{bmatrix}-[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+4(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

Easy calculation reveals that G attains its maximum value at $p=2\sqrt{\frac{(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\mu}{(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu}}$.

The upper bound for equation (3.6) corresponds to $\rho=1$ and $p=2\sqrt{\frac{(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\mu}{(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu}}$, in which case

$$|a_2a_4-\mu a_3^2|\le\frac{[(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\mu]^2}{[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]}-(1+\lambda)(1+3\lambda)\mu$$

Case II: For $0\le\mu\le\frac{1}{2}$,

$$G(p)=\frac{1}{16(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\begin{bmatrix}[-(1+2\lambda)^2+2(1+\lambda)(1+3\lambda)\mu]p^4\\+[8(1+2\lambda)^2-16(1+\lambda)(1+3\lambda)\mu]p^2\\+16(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

and

$$G'(p)=\frac{1}{4(1+\lambda)(1+3\lambda)(1+2\lambda)^2}p\begin{bmatrix}-[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+4(1+2\lambda)^2-8(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

where G attains its maximum vlaue at $p=2$, Hence we obtain

$$|a_2a_4-\mu a_3^2|\le(1+2\lambda)^2-(1+\lambda)(1+3\lambda$$

Case III: For $\frac{1}{2}\le\mu\le 1$ we consider two subcases. **Subcase (i):** When $\frac{1}{2}\le\mu\le\frac{3}{4}$

$$G(p)=\frac{1}{4(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\begin{bmatrix}[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^2\\+4(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

here G attains its maximum value at $p=0$

Hence $|a_2a_4-\mu a_3^2|\le(1+\lambda)(1+3\lambda)\mu$

Subcase (ii): When $\frac{3}{2}\le\mu\le 1$

$$G(p)=\frac{1}{16(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\begin{bmatrix}[3(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\mu]p^4\\+8[-(1+2\lambda)^2+(1+\lambda)(1+3\lambda)\mu]p^2\\+16(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

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In this case $G(p)$ is a decreasing function so if attains its maximum value at $p=0$.

Case IV: Finally, for $\mu\ge 1$,

$$G(p)=\frac{1}{16(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\begin{bmatrix}[(1+2\lambda)^2-2(1+\lambda)(1+3\lambda)\mu]p^4\\+8[\mu(1+\lambda)(1+3\lambda)-(1+2\lambda)^2]p^2\\+16(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

and

$$G'(p)=\frac{1}{4(1+\lambda)(1+3\lambda)(1+2\lambda)^2}p\begin{bmatrix}-[2(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2]p^2\\+4[(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2]\end{bmatrix}$$

Here G attains its maximum value at $p=2\sqrt{\frac{\mu(1+\lambda)(1+3\lambda)-(1+2\lambda)^2}{2(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2}}$. Hence

$$|a_2a_4-\mu a_3^2|\le\frac{[(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2]^2}{[2(1+\lambda)(1+3\lambda)\mu-(1+2\lambda)^2]}+\mu(1+\lambda)(1+3\lambda)$$

.

For $\mu=1$, Theorem 3.1 gives the following result.

Corollary 3.1.

If $f(z)\in C_{S(\lambda)}$ then $|a_2a_4-a_3^2|\le(1+2\lambda)^2$.

Theorem 3.2.

If $f(z)\in S^*_{S(\lambda)}$, then we obtain the same result as in Theorem 3.1 on the same lines, we have

Theorem 3.3.

If $f(z)\in K^*_{S(\lambda)}$, then

$$|a_2a_4-\mu a_3^2| \le \begin{cases} \frac{(9(1+2\lambda)^2-8(1+\lambda)(1+3\lambda)\mu)^2}{72(9(1+2\lambda)^2-16(1+\lambda)(1+3\lambda)\mu)} - \frac{1}{9}\mu(1+\lambda)(1+3\lambda) & \text{if } \mu \le 0, \\ \frac{1}{8}(1+2\lambda)^2 - \frac{1}{9}(1+\lambda)(1+3\lambda)\mu & 0 \le \mu \le \frac{9}{16} \\ \frac{1}{9}(1+\lambda)(1+3\lambda)\mu & \frac{9}{16} \le \mu \le \frac{9}{8}, \end{cases}$$

For $\mu = 1$ Theorem 3.3 gives

Corollary 3.2.

If $f(z) \in K_{S(\lambda)}$, then $|a_2a_4 - a_3^2| \le \frac{1}{9}.$

Theorem 3.4.

If $f(z) \in C_{S1(\lambda)}$ then

$$|a_2a_4-\mu a_3^2| \le \begin{cases} \frac{[135(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]^2}{324[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]} - \frac{41}{81}(1+\lambda)(1+3\lambda)\mu & \text{if } \mu \le 0 \\ \frac{[135(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu]^2}{324[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]} + \frac{49}{81}(1+\lambda)(1+3\lambda)\mu & \text{if } 0 \le \mu \le \frac{135}{196}, \\ \frac{49}{81}(1+\lambda)(1+3\lambda)\mu & \text{if } \frac{135}{196} \le \mu \le \frac{135}{98}, \\ \frac{[98(1+\lambda)(1+3\lambda)\mu-135(1+2\lambda)^2]^2}{324[98(1+\lambda)(1+3\lambda)\mu-81(1+2\lambda)^2]} + \frac{49}{81}(1+\lambda)(1+3\lambda)\mu & \text{if } \mu \ge \frac{135}{98}. \end{cases}$$

Proof:

Since $f \in C_{S1(\lambda)}$ by definition we have

$$\frac{2[\lambda z^2 f''(z) + zf'(z)]}{\lambda z[h'(z)+h'(-z)] + (1-\lambda)[h(z)-h(-z)]} = p(z) \tag{3.8}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in K_{S(\lambda)} \tag{3.9}$$

Using (1.1), (2.1) and (3.8), (3.9) gives

$$1 + 2(1+\lambda)a_2z + (1+2\lambda)3a_3z^2 + (1+3\lambda)4a_4z^3$$
$$= (1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \cdots)(1 + (1+\lambda)b_3z^2 + \cdots] \tag{3.10}$$

On equating co-efficients in (3.10) we get

$$2(1+\lambda)a_2 = p_1,$$
$$3a_3(1+2\lambda) = (1+2\lambda)b_3 + p_2,$$
$$4a_4(1+3\lambda) = (1+2\lambda)p_1b_3 + p_3 \tag{3.11}$$

From (3.9) we can easily verify that $b_3 = \frac{p_2}{6(1+2\lambda)}$

So (3.11) yields

$$a_2 = \frac{p_1}{2(1+\lambda)}, \quad a_3 = \frac{7p_2}{18(1+2\lambda)}, \quad a_4 = \frac{p_3}{4(1+3\lambda)} + \frac{p_2p_1}{24(1+3\lambda)} \tag{3.12}$$

From (3.12)

$$|a_2a_4-\mu a_3^2| = \left|\frac{p_1p_3}{8(1+\lambda)(1+3\lambda)} + \frac{p_1^2p_2}{48(1+\lambda)(1+3\lambda)} - \mu\frac{49p_2^2}{324(1+2\lambda)^2}\right|$$

Let $X_1 = 2592(1+\lambda)(1+3\lambda)(1+2\lambda)^2$.

Using Lemma 2.2 it gives

$$|a_2a_4-\mu a_3^2| \le \frac{1}{X_1}\left|\begin{array}{l} [108(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p_1^4 \\ +[189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu][4-p_1^2)p_1^2x \\ +162(1+2\lambda)^2(4-p_1^2)p_1(1-|x|^2)z-(4-p_1^2)x^2 \\ [p_1^2[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu) \\ +392(1+\lambda)(1+3\lambda)\mu \end{array}\right|$$

Suppose now that $p_1 = p$, $p \in [0,2]$ and using triangle inequality we get

$$|a_2a_4-\mu a_3^2| \le \frac{1}{X_1}\left[\begin{array}{l} |108(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu|p^4 \\ +162(1+2\lambda)^2p(4-p^2) \\ +|189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu|p^2(4-p^2)\rho \\ +(4-p^2)(|81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu|p^2 \\ +392|(1+\lambda)(1+3\lambda)\mu|-162(1+2\lambda)^2p)\rho^2 \end{array}\right]$$

$= F(\rho)$ with $\rho = |x| \le 1$.

This gives rise to

$$F'(\rho) = \begin{cases} \frac{1}{X_1}\left\{\begin{array}{l} [189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu][4-p^2]p^2 \\ +2(4-p^2)[[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p^2 \\ -392(1+\lambda)(1+3\lambda)\mu-162(1+2\lambda)^2p]\rho \end{array}\right\} & \text{if } \mu \le 0, \\ \frac{1}{X_1}\left\{\begin{array}{l} [189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu][4-p^2]p^2 \\ +2(4-p^2)[[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p^2 \\ +392(1+\lambda)(1+3\lambda)\mu-162(1+2\lambda)^2p]\rho \end{array}\right\} & \text{if } 0 \le \mu \le \frac{81}{98}, \\ \frac{1}{X_1}\left\{\begin{array}{l} [189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu][4-p^2]p^2 \\ +2(4-p^2)[-[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p^2 \\ +392(1+\lambda)(1+3\lambda)\mu-162(1+2\lambda)^2p]\rho \end{array}\right\} & \text{if } \frac{81}{98} \le \mu \le \frac{189}{196}, \\ \frac{1}{X_1}\left\{\begin{array}{l} [-(189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu](4-p^2)p^2 \\ +2(4-p^2)[-[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p^2 \\ +392(1+\lambda)(1+3\lambda)\mu-162(1+2\lambda)^2p]\rho \end{array}\right\} & \text{if } \mu \ge \frac{189}{196} \end{cases}$$

$$G(p)=\frac{1}{1296(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\times\begin{bmatrix}[-81(1+2\lambda)^2+98(1+\lambda)(1+3\lambda)\mu]p^4\\+4[135(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu]p^2\\+784(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

We consider two subcases,

Subcase (i): When $0\leq\mu\leq\frac{135}{196}$,

$$G'(p)=\frac{1}{324(1+\lambda)(1+3\lambda)(1+2\lambda)^2}p\begin{bmatrix}[-81(1+2\lambda)^2+98(1+\lambda)(1+3\lambda)\mu]p^2\\+270(1+2\lambda)^2-392(1+\lambda)(1+3\lambda)\mu]\end{bmatrix}$$

Here G attains its maximum value at $p=\sqrt{\frac{270(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu}{81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu}}$, in which case

$$|a_2a_4-\mu a_3^2|\leq\frac{[135(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu]^2}{324[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]}+\frac{49}{81}(1+\lambda)(1+3\lambda)\mu$$

Subcase (ii): When $\frac{135}{196}\leq\mu\leq\frac{81}{98}$,

$$G(p)=\frac{1}{1296(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\times\begin{bmatrix}[-81(1+2\lambda)^2+98(1+\lambda)(1+3\lambda)\mu]p^4\\-4[196(1+\lambda)(1+3\lambda)\mu-135(1+2\lambda)^2]p^2\\+784(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

In this case $G(p)$ attains its maximum value at $p = 0$. So

$$|a_2a_4-\mu a_3^2|\leq\frac{49}{81}(1+\lambda)(1+3\lambda)\mu$$

Case III: For $\frac{81}{98}\leq\mu\leq\frac{108}{98}$, we consider two subcases.

Subcase (i): When $\frac{81}{98}\leq\mu\leq\frac{189}{196}$,

$$G(p)=\frac{1}{162(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\times\begin{bmatrix}[27(1+2\lambda)^2-49(1+\lambda)(1+3\lambda)\mu]p^2\\+98(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

here G attains its maximum value at $p = 0$. Hence

$$|a_2a_4-\mu a_3^2|\leq\frac{49}{81}(1+\lambda)(1+3\lambda)\mu$$

Subcase (ii): When $\frac{189}{196}\leq\mu\leq\frac{108}{98}$,

$$G(p)=\frac{1}{1296(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\times\begin{bmatrix}[189(1+2\lambda)^2-196(1+\lambda)(1+3\lambda)\mu]p^4\\+4[-135(1+2\lambda)^2+98(1+\lambda)(1+3\lambda)\mu]p^2\\+784(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

In this case $G(p)$ is a decreasing function so it attains its maximum value at $p = 0$.

Case IV: Finally for $\mu\geq\frac{108}{98}$,

$$G(p)=\frac{1}{1296(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\times\begin{bmatrix}[(81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p^4\\+4[98(1+\lambda)(1+3\lambda)\mu-135(1+2\lambda)^2]p^2\\+784(1+\lambda)(1+3\lambda)\mu\end{bmatrix}$$

we consider two subcases:

Subcase (i): When $\frac{108}{98}\leq\mu\leq\frac{135}{98}$, $G(p)$ is a decreasing function so it attains its maximum value at $p = 0$.

Subcase (ii): When $\mu\geq\frac{135}{98}$,

$$G'(p)=\frac{1}{324(1+\lambda)(1+3\lambda)(1+2\lambda)^2}p\times\begin{bmatrix}[81(1+2\lambda)^2-98(1+\lambda)(1+3\lambda)\mu]p^2\\+2[98(1+\lambda)(1+3\lambda)\mu-135(1+2\lambda)^2]\end{bmatrix}$$

here G attains its maximum value at $p=\sqrt{\frac{196(1+\lambda)(1+3\lambda)\mu-270(1+2\lambda)^2}{98(1+\lambda)(1+3\lambda)\mu-81(1+2\lambda)^2}}$.

Hence

$$|a_2a_4-\mu a_3^2|\leq\frac{[98(1+\lambda)(1+3\lambda)\mu-135(1+2\lambda)^2]^2}{324[98(1+\lambda)(1+3\lambda)\mu-81(1+2\lambda)^2]}+\frac{49}{81}(1+\lambda)(1+3\lambda)\mu$$

For $\mu = 1$, Theorem 3.4 gives the following result.

Corollary 3.3.

If $f(z)\in C_{S1(\lambda)}$ then $|a_2a_4-a_3^2|\leq\frac{49}{81}(1+2\lambda)^2$ *on the same lines, we can easily prove the following theorem:*

Theorem 3.5.

If $f(z)\in C'_{S1(\lambda)}$ *then we get the same result as in Theorem 3.3.*

Theorem 3.6.

If $f(z)\in C_{S(\gamma,\lambda)}$ then

$$|a_2a_4-\mu a_3^2|\leq\begin{cases}\frac{[A_1^2-B_1C_1\mu]^2}{[A_1^2-2B_1C_1\mu]}-B_1C_1\mu & i\ \mu\leq 0,\\ A_1^2-B_1C_1\mu & \text{if } 0\leq\mu\leq\frac{1}{2},\\ B_1C_1\mu & \text{if } \frac{1}{2}\leq\mu\leq 1,\\ \frac{[B_1C_1\mu-A_1^2]^2}{[2B_1C_1\mu-A_1^2]}+B_1C_1\mu & \text{if } \mu\geq 1.\end{cases}$$

where $A_1 = (1 + 6\gamma\lambda + 2\gamma - 2\lambda)$

$B_1 = (1 + 12\gamma\lambda + 3\gamma - 3\lambda)$

$C_1 = (1 + 2\gamma\lambda + \gamma - \lambda)$

Proof:

Since $f\in C_{S(\gamma,\lambda)}$ we have

$$\frac{2[\gamma\lambda z^3f'''(z)+(2\gamma\lambda+\gamma-\lambda)z^2f''(z)+zf'(z)}{\gamma\lambda z^2[g''(z)-g''(-z)]+(\gamma-\lambda)z[g'(z)+g'(-z)]+(1-\gamma+\lambda)[g(z)-g(-z)]}=p(z)\qquad(3.15)$$

where

∞

$$g(z) = z + \mathrm{X}_{k=2} b_k z^k \in S_{s*(\gamma,\lambda)} \tag{3.16}$$

Using (1.1), (2.1) and (3.15), (3.16) gives

$$1 + 2(2\gamma\lambda + \gamma - \lambda + 1)a_2 z + 3a_3[6\gamma\lambda + 2\gamma - 2\lambda + 1]z^2$$
$$+ 4a_4[1 + 6\gamma\lambda + 3(2\gamma\lambda + \gamma - \lambda)]z^3 + \cdots$$
$$= (1 + p_1 z + p_2 z^2 + \cdots)(1 + b_3(1 + 6\gamma\lambda + 2\gamma - 2\lambda)z^2 + \cdots) \tag{3.17}$$

On equating coefficients in (3.17) we get

$$2[2\gamma\lambda + \gamma - \lambda + 1)a_2 = p_1,$$
$$3a_3[6\gamma\lambda + 2\gamma - 2\lambda + 1] = b_3[1 + 6\gamma\lambda + 2\gamma - 2\lambda) + p_2,$$
$$4a_4[1 + 6\gamma\lambda + 3(2\gamma\lambda + \gamma - \lambda)] = p_1 b_3[1 + 6\gamma\lambda + 2\gamma - 2\lambda) + p_3 \tag{3.18}$$

From (3.16), we can easily verify that

$$b_3 = \frac{p_2}{2[1 + 2\gamma - 2\lambda + 6\gamma\lambda]} \tag{3.19}$$

So (3.18) yields

$$a_2 = \frac{p_1}{2[1 + 2\gamma\lambda + \gamma - \lambda]},$$
$$a_3 = \frac{p_2}{2[1 + 6\gamma\lambda + 2\gamma - 2\lambda]},$$
$$a_4 = \frac{p_1 p_2}{8[1 + 12\gamma\lambda + 3\gamma - 3\lambda]} + \frac{p_3}{4[1 + 12\gamma\lambda + 3\gamma - 3\lambda]}$$

$$\frac{1}{32A_1^2B_1C_1}\begin{bmatrix} -[3A_1^2 - 4B_1C_1\mu](4-p^2)p^2 + 2(4-p^2) \\ [-(A_1^2 - 2B_1C_1\mu)p^2 + 8B_1C_1\mu - 2A_1^2 p]\rho \end{bmatrix}$$

From (3.20) define

$$A_1 = (1 + 6\gamma\lambda + 2\gamma - 2\lambda)$$
$$B_1 = (1 + 12\gamma\lambda + 3\gamma - 3\lambda)$$
$$C_1 = (1 + 2\gamma\lambda + \gamma - \lambda)$$

$$|a_2a_4 - \mu a_3^2| \leq \left| \frac{p_1^2 p_2}{16B_1C_1} + \frac{p_1p_3}{8B_1C_1} - \mu\frac{p_2^2}{4A_1^2} \right|$$

Using Lemma 2.2 it gives

$$= \frac{1}{32A_1^2B_1C_1}\left| \begin{matrix} 2[A_1^2 - B_1C_1\mu]p_1^4 + [3A_1^2 - 4B_1C_1\mu](4-p_1^2)p_1^2 x \\ +2A_1^2 p_1(4-p_1^2)(1-|x|^2]z - (4-p_1^2)x^2 \\ [A_1^2 - 2B_1C_1\mu)p_1^2 + 8B_1C_1\mu] \end{matrix} \right|$$

Suppose now that $p_1 = p$, $p \in [0,2]$ and using triangle inequality, we get

$$F'(\rho) = \begin{cases} \frac{1}{32A_1^2B_1C_1}\begin{bmatrix} [3A_1^2 - 4B_1C_1\mu](4-p^2)p^2 + 2(4-p^2) \\ [(A_1^2 - 2B_1C_1\mu)p^2 - 8B_1C_1\mu - 2A_1^2 p]\rho \end{bmatrix} \\ \frac{1}{32A_1^2B_1C_1}\begin{bmatrix} [3A_1^2 - 4B_1C_1\mu](4-p^2)p^2 + 2(4-p^2) \\ [(A_1^2 - 2B_1C_1\mu)p^2 + 8B_1C_1\mu - 2A_1^2 p]\rho \end{bmatrix} \\ \frac{1}{32A_1^2B_1C_1}\begin{bmatrix} [3A_1^2 - 4B_1C_1\mu](4-p^2)p^2 + 2(4-p^2) \\ [-(A_1^2 - 2B_1C_1\mu)p^2 + 8B_1C_1\mu - 2A_1^2 p]\rho \end{bmatrix} \end{cases}$$

if $\mu \geq \frac{3}{4}$.

and for all the cases above $F^0(\rho) > 0$ for $\rho > 0$, implying that Max $F(\rho) = F(1)$ Now let (3.20)

$$G(P) = F(1) = \frac{1}{32A_1^2B_1C_1}\begin{bmatrix} 2|A_1^2 - B_1C_1\mu|p^4 + 2A_1^2 p(4-p^2) \\ +|3A_1^2 - 4B_1C_1\mu|p^2(4-p^2) \\ +(4-p^2)[|A_1^2 - 2B_1C_1\mu|p^2 + 8|B_1C_1\mu| - 2A_1^2 p] \end{bmatrix} \tag{3.22}$$

Now we discuss the following cases:

Case I: For $\mu \leq 0$,

$$G(P) = \frac{1}{32A_1^2B_1C_1}\left[-2[A_1^2 - 2B_1C_1\mu]p^4 + 16[A_1^2 - B_1C_1\mu]p^2 - 32B_1C_1\mu \right]$$

and

$$G'(P) = \frac{1}{4A_1^2B_1C_1}p\left[-[A_1^2 - 2B_1C_1\mu]p^2 + 4A_1^2 - 4B_1C_1\mu \right]$$

Easy calculation reveals that G attainsits

$$= F(\rho) \text{ with } \rho = |x| \leq 1 \tag{3.21}$$

This gives rise to

$$|a_2a_4 - \mu a_3^2| \leq \frac{1}{32A_1^2B_1C_1}\begin{bmatrix} 2|A_1^2 - B_1C_1\mu|p_1^4 + 2A_1^2 p(4-p_1^2) \\ +|3A_1^2 - 4B_1C_1\mu|p^2(4-p^2)\rho + (4-p^2) \\ [|A_1^2 - 2B_1C_1\mu|p^2 + 8|B_1C_1\mu| - 2A_1^2 p]\rho^2 \end{bmatrix}$$

if $\mu \leq \mu \leq \frac{1}{2} < 0$,

if 0,

if $\frac{1}{2} \leq \mu \leq \frac{3}{4}$

maximum value at $p = 2\sqrt{\frac{A_1^2 - B_1C_1\mu}{A_1^2 - 2B_1C_1\mu}}$.

The upper bound for equation (3.21) corresponds to $\rho = 1$ and $p = 2\sqrt{\frac{A_1^2 - B_1C_1\mu}{A_1^2 - 2B_1C_1\mu}}$, in which case

$$|a_2a_4 - \mu a_3^2| \leq \frac{[A_1^2 - B_1C_1\mu]^2}{A_1^2 - 2B_1C_1\mu} - B_1C_1\mu$$

Case II: For $0 \leq \mu \leq \frac{1}{2}$,

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$$G(P) = \frac{1}{16A_1^2B_1C_1}\left[\ (-A_1^2 + 2B_1C_1\mu)p^4 + [8A_1^2 - 16B_1C_1\mu]p^2 + 16B_1C_1\mu\right.$$

$$G'(P) = \frac{1}{4A_1^2B_1C_1}p\left[\ -(A_1^2 - 2B_1C_1)p^2 + 4A_1^2 - 8B_1C_1\mu\ \right]$$

where G attains maximum value at $p = 2$,

Hence $|a_2a_4 - \mu a_3^2| \leq A_1^2 - B_1C_1\mu$.

Case III: For $\frac{1}{2} \leq \mu \leq 1$, we consider two subcases.

Subcase (i):

$\frac{1}{2} \leq \mu \leq \frac{3}{4}$,

$$G(P) = \frac{1}{4A_1^2B_1C_1}\left[\ (A_1^2 - 2B_1C_1\mu)p^2 + 4B_1C_1\mu\right]\ ]$$

Hence G attains its maximum value at $p = 0$ hence $|a_2a_4 - \mu a_3^2| \leq B_1C_1\mu$.

Subcase (ii): $\frac{3}{4} \leq \mu \leq 1$,

$$G(P) = \frac{1}{16A_1^2B_1C_1}\left[\ (3A_1^2 - 4B_1C_1\mu)p^4 + 8(-A_1^2 + B_1C_1\mu)p^2 + \right.$$

In this case $G(p)$ is a decreasing function so it attains its maximum value at $p = 0$.

Case IV: Finally, for $\mu \geq 1$,

$$G(P) = \frac{1}{16A_1^2B_1C_1}\left[\ (A_1^2 - 2B_1C_1\mu)p^4 + 8(B_1C_1\mu - A_1^2)p^2 + 16B_1C_1\mu\ \right]$$

and

$$G'(P) = \frac{1}{4A_1^2B_1C_1}p\left[\ -(2B_1C_1\mu - A_1^2)p^2 + 4(B_1C_1\mu - A_1^2)\ \right]$$

here G attains maximum value at $p = 2\sqrt{\frac{B_1C_1\mu - A_1^2}{2B_1C_1\mu - A_1^2}}$.

Hence $|a_2a_4 - \mu a_3^2| \leq \frac{(B_1C_1\mu - A_1^2)^2}{(2B_1C_1\mu - A_1^2)} + \mu B_1C_1$.

For $\mu = 1$, Theorem 3.6 gives the following result:

Corollary 3.4.

If $f(z) \in C_{S(\gamma,\lambda)}$, then $|a_2a_4 - a_3^2| \leq A_1^2$.

Theorem 3.7.

If $f(z) \in S_{S^*(\gamma,\lambda)}$ then we obtain the same result as in Theorem 3.6. On the same lines we have

Theorem 3.8.

If $f(z) \in K_{S(\gamma,\lambda)}$ then

$$\begin{cases} \frac{(9A_1^2 - 8B_1C_1\mu)^2}{72(9A_1^2 - 16B_1C_1\mu)} - \frac{1}{9}B_1C_1\mu & \text{if } \mu \leq 0, \\ & \text{if } 0 \leq \mu \leq \frac{9}{16}, \text{ if } \frac{9}{16} \leq \mu \leq \frac{9}{8}, \\ & \text{if } \mu \geq \frac{9}{8}. \end{cases}$$

For $\mu = 1$ Theorem 3.8 gives:

Corollary 3.5.

If $f(z) \in K_{S(\gamma,\lambda)}$, then $|a_2a_4 - a_3^2| \leq \frac{A_1^2}{9}$.

Theorem 3.9. *If $f(z) \in C_{S2(\gamma,\lambda)}$, then*

$$|a_2a_4 - \mu a_3^2| \leq \begin{cases} \frac{(135A_1^2 - 19\mu B_1C_1)^2}{324(81A_1^2 - 98B_1C_1\mu)} + \frac{49}{81}B_1C_1\mu \\ \frac{49}{81}B_1C_1\mu \end{cases}$$

$$\begin{cases} \frac{(135A_1^2 - 98B_1C_1\mu)^2}{324(81A_1^2 - 98B_1C_1\mu)} - \frac{49}{81}B_1C_1\mu & \text{if } \mu \leq 0, \\ & \text{if } 0 \leq \mu \leq \frac{135}{196}, \text{ if } \frac{135}{196} \leq \mu \leq \frac{135}{198}, \\ \frac{(98B_1C_1\mu - 135A_1^2)^2}{324(98B_1C_1\mu - 81A_1^2)} + \frac{49}{81}B_1C_1\mu & \text{if } \mu \geq \frac{135}{98}. \end{cases}$$

For $\mu = 1$ Theorem 3.9 gives the following result:

Corollary 3.6.

If $f(z) \in C_{S2(\gamma,\lambda)}$, then $|a_2a_4 - a_3^2| \leq \frac{49}{81}A_1^2$ *on the same lines we can easily prove the following theorem:*

Theorem 3.10.

If $f(z) \in C'_{S2(\gamma,\lambda)}$, then we get the same result as in Theorem 3.8.

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