1. Online Appendix

1.1. Details for the Proof of Part (ii) of Theorem 3.6

For $p \geq 2$, $k_1, \ldots, k_{p-1} \in \mathbb{Z}$ and $x_1, \ldots, x_p \in \mathbb{R}$, consider the *Laplace cumulant* of order p

$$\begin{aligned}\gamma_{k_1,\ldots,k_{p-1}}^{x_1,\ldots,x_p} &:= \operatorname{cum}\big[I\{X_{k_1} \leq x_1\}, I\{X_{k_2} \leq x_2\}, \ldots, I\{X_0 \leq x_p\}\big] \\ &= \sum_{\{\nu_1,\ldots,\nu_R\}} (-1)^{R-1}(R-1)! \prod_{j=1}^{R} \mathbb{P}\Big(X_{k_i} \leq x_i \,:\, i \in \nu_j\Big), \quad k_p := 0,\end{aligned}$$

where the summation runs over all partitions $\{\nu_1, \ldots, \nu_R\}$ of $\{1, \ldots, p\}$. All results in this part of the Appendix are established under the following condition on Laplace cumulants:

(CS) Let $p \geq 2, \delta > 0$. There exists a non-increasing function $a_p : \mathbb{N} \to \mathbb{R}^+$ such that

$$\sup_{x_1,\ldots,x_p} |\gamma_{k_1,\ldots,k_{p-1}}^{x_1,\ldots,x_p}| \leq a_p\big(\max_j |k_j|\big) \quad \text{and} \quad \sum_{k \in \mathbb{N}} k^\delta a_p(k) < \infty.$$

This condition follows from Assumption (C) but is in fact somewhat weaker.

Note that under assumption (CS) the following quantity, which we call *Laplace spectrum of order p*, exists as soon as $p - 1 < \delta$

$$\mathfrak{f}_{x_1,\ldots,x_p}(\omega_1, \ldots, \omega_{p-1}) := \frac{1}{(2\pi)^{p-1}} \sum_{k_1,\ldots,k_{p-1}=-\infty}^{\infty} \gamma_{k_1,\ldots,k_{p-1}}^{x_1,\ldots,x_p} \mathrm{e}^{-\mathrm{i}(\omega_1 k_1 + \ldots + \omega_{p-1} k_{p-1})}.$$

The existence of $\mathfrak{f}_{x_1,\ldots,x_p}(\omega_1, \ldots, \omega_{p-1})$ follows, since under (CS)

$$\begin{aligned}&\Big|\sum_{k_1,\ldots,k_{p-1}=-\infty}^{\infty} \gamma_{k_1,\ldots,k_{p-1}}^{x_1,\ldots,x_p} \mathrm{e}^{-\mathrm{i}(\omega_1 k_1 + \ldots + \omega_{p-1} k_{p-1})}\Big| \leq \sum_{k_1,\ldots,k_{p-1}=-\infty}^{\infty} a_p\big(\max_j |k_j|\big) \\ \leq \quad & a_p(0) + \sum_{m=1}^{\infty} a_p(m)\big|\{k_1, ..., k_{p-1} : \max_j |k_j| = m\}\big| < \infty,\end{aligned}$$

since $\big|\{k_1, ..., k_{p-1} : \max_j |k_j| = m\}\big| = O(m^{p-2})$.

The main result in this section is Lemma 1.5, giving an asymptotic expansion of the expectation $\mathbb{E}[\hat{G}_{n,U}(\tau_1, \tau_2; \omega)]$ that holds uniformly in τ_1, τ_2, and ω. Essentially, it is a uniform version, for Laplace spectra, of Theorems 7.4.1 and 7.4.2 in Brillinger (1975). The proof is based on a series of uniform reinforcements of results from Brillinger (1975).

We first prove the following version of Lemma P4.1 in Brillinger (1975) in the special case where no tapering is applied, so that the constant can be chosen as 2.

Lemma 1.1. *Let $h_n(u) := I\{0 \leq u < n\}$ and $\Delta_n(\lambda) := \sum_{t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\lambda t}$. Then, for any $K \in \mathbb{N}, K \geq 2$, $u_1, \ldots, u_{K-1} \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$,*

$$\Big|\sum_{t=0}^{n-1} h_n(t + u_1) \cdots h_n(t + u_{K-1})\mathrm{e}^{-\mathrm{i}\lambda t} - \Delta_n(\lambda)\Big| \leq 2(|u_1| + \ldots + |u_{K-1}|). \tag{1.1}$$

Proof. The left-hand side in (1.1) is bounded by

$$\sum_{j=1}^{K-1}\sum_{t=0}^{n-1}|h_n(t+u_j)-h_n(t)| \le 2\sum_{j=1}^{K-1}|u_j|. \qquad \square$$

Next, we extend Lemma P4.2, still from Brillinger (1975). Define

$$\begin{aligned} E_n(\tau_1,\ldots,\tau_K,\lambda_1,\ldots,\lambda_K) &:= \mathrm{cum}(d_{n,U}^{\tau_1}(\lambda_1),\ldots,d_{n,U}^{\tau_K}(\lambda_K)) \\ &\quad - \Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)\sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n}\gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}}\exp\Big(-\mathrm{i}\sum_{j=1}^{K-1}\lambda_j u_j\Big). \end{aligned}$$

Lemma 1.2. *Under (CS) with $p=K$ and $\delta>K$,*

$$\begin{aligned} &\Big|E_n(\tau_1,\ldots,\tau_K,\lambda_1,\ldots,\lambda_K)\Big| \\ &\qquad \le 2\sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n}(|u_1|+\ldots+|u_{K-1}|)\Big|\gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}}\Big| \le 2(K-1)C_K, \end{aligned}$$

for all $\tau_1,\ldots,\tau_K\in[0,1]$ and $\lambda_1,\ldots,\lambda_K\in\mathbb{R}$, where C_K does not depend on λ_i, q_{τ_i}.

Proof. By multi-linearity of the cumulants, we have

$$\begin{aligned} &\mathrm{cum}(d_{n,U}^{\tau_1}(\lambda_1),\ldots,d_{n,U}^{\tau_K}(\lambda_K)) \\ &= \sum_{t_1=0}^{n-1}\cdots\sum_{t_K=0}^{n-1}h_n(t_1)\ldots h_n(t_K)\exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j t_j\Big)\gamma_{t_1-t_K,\ldots,t_{K-1}-t_K}^{q_{\tau_1},\ldots,q_{\tau_K}} \\ &= \sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n}\exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j u_j\Big)\gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}} \\ &\qquad \times\sum_{t=0}^{n-1}h_n(t+u_1)\cdots h_n(t+u_{K-1})h_n(t)\exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j t\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} E_n(\tau_1,\ldots,\tau_K,\lambda_1,\ldots,\lambda_K) &= \sum_{|u_1|<n}\cdots\sum_{|u_{K-1}|<n}\exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j u_j\Big)\gamma_{u_1,\ldots,u_{K-1}}^{q_{\tau_1},\ldots,q_{\tau_K}} \\ \times\Bigg(\sum_{t=0}^{n-1}h_n(t+u_1)\cdots h_n(t+u_{K-1})h_n(t)&\exp\Big(-\mathrm{i}\sum_{j=1}^{K}\lambda_j t\Big)-\Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)\Bigg). \end{aligned}$$

Applying the triangle inequality and Lemma 1.1, and taking condition (CS) into account, completes the proof. $\square$

Finally, we establish a uniform version of Theorem 4.3.2 in Brillinger (1975). Recalling the definition of $d^{\tau}_{n,U}$ given in (2.6), let

$$\varepsilon_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K) := \mathrm{cum}(d^{\tau_1}_{n,U}(\lambda_1),\dots,d^{\tau_K}_{n,U}(\lambda_K)) \\ - (2\pi)^{K-1}\Delta_n\Big(\sum_{j=1}^{k}\lambda_j\Big)\mathfrak{f}_{q_{\tau_1},\dots,q_{\tau_K}}(\lambda_1,\dots,\lambda_K).$$

Theorem 1.3. *If (CS) holds with $p=K$ and $\delta > K+1$, then*

$$\sup_n \sup_{\substack{\tau_1,\dots,\tau_K\in[0,1]\\ \lambda_1,\dots,\lambda_K\in\mathbb{R}}} \Big|\varepsilon_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K)\Big| < \infty.$$

Proof. By the definition of $\mathfrak{f}_{q_{\tau_1},\dots,q_{\tau_K}}$, we have

$$\begin{aligned} &\mathrm{cum}(d^{\tau_1}_{n,U}(\lambda_1),\dots,d^{\tau_K}_{n,U}(\lambda_K)) \\ &= \Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)(2\pi)^{K-1}\mathfrak{f}_{q_{\tau_1},\dots,q_{\tau_K}}(\lambda_1,\dots,\lambda_{K-1}) \\ &\quad - \Delta_n\Big(\sum_{j=1}^{K}\lambda_j\Big)\sum_{|u_1|\vee\dots\vee|u_{K-1}|\ge n}\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}\exp\Big(-\mathrm{i}\sum_{j=1}^{K-1}\lambda_j u_j\Big) \\ &\quad + E_n(\tau_1,\dots,\tau_K,\lambda_1,\dots,\lambda_K). \end{aligned}$$

Noting that $|\Delta_n(\lambda)|\le n$, we have by condition condition (CS),

$$\begin{aligned} &\sup_{\substack{\tau_1,\dots,\tau_K\in[0,1]\\ \lambda_1,\dots,\lambda_K\in\mathbb{R}}}\Big|\sum_{|u_1|\vee\dots\vee|u_{K-1}|\ge n}\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}\exp\Big(-\mathrm{i}\sum_{j=1}^{K-1}\lambda_j u_j\Big)\Big| \\ &\le \sup_{\substack{\tau_1,\dots,\tau_K\in[0,1]\\ \lambda_1,\dots,\lambda_K\in\mathbb{R}}}\sum_{m=n}^{\infty}\sum_{|u_1|\vee\dots\vee|u_{K-1}|=m}|\gamma^{q_{\tau_1},\dots,q_{\tau_K}}_{u_1,\dots,u_{K-1}}| \le \sum_{m=n}^{\infty}O(m^{K-2})a(m) = O(1/n). \end{aligned}$$

The claim follows by applying Lemma 1.2 to E_n. □

In analogy to Theorem 5.2.2 in Brillinger (1975), we also have the following result.

Lemma 1.4. *Under (CS) with $K=2,\delta>3$,*

$$\mathbb{E}I^{\tau_1,\tau_2}_{n,U}(\omega) = \begin{cases} \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + \frac{1}{2\pi n}\Big[\frac{\sin(n\omega/2)}{\sin(\omega/2)}\Big]^2\tau_1\tau_2 + \varepsilon^{\tau_1,\tau_2}_n(\omega) & \omega\ne 0 \mod 2\pi \\ \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + \frac{n}{2\pi}\tau_1\tau_2 + \varepsilon^{\tau_1,\tau_2}_n(\omega) & \omega = 0 \mod 2\pi \end{cases} \tag{1.2}$$

with $\sup_{\tau_1,\tau_2\in[0,1],\omega\in\mathbb{R}}|\varepsilon^{\tau_1,\tau_2}_n(\omega)| = O(1/n)$.

Remark: For the Fourier frequencies $\omega = \frac{2\pi j}{n}$, $j \in \mathbb{Z}$, the second term in the right-hand side of (1.2) vanishes, leading to the useful simple result

$$\mathbb{E}I_n^{\tau_1,\tau_2}(\omega) = \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + \frac{n}{2\pi}\tau_1\tau_2 I\{\omega = 0 \mod 2\pi\} + \varepsilon_n^{\tau_1,\tau_2}(\omega).$$

Proof. First note that, by definition,

$$\mathbb{E}I_n^{\tau_1,\tau_2}(\omega) = \frac{1}{2\pi n}\Big(\operatorname{cum}(d_{n,U}^{\tau_1}(\omega), d_{n,U}^{\tau_2}(-\omega)) + (\mathbb{E}d_{n,U}^{\tau_1}(\omega))(\mathbb{E}d_{n,U}^{\tau_2}(-\omega))\Big)$$

The result follows from applying Theorem 1.3 and noting that

$$\mathbb{E}d_{n,U}^{\tau}(\omega) = \tau\sum_{t=0}^{n-1} e^{-\mathrm{i}\omega t} = \tau\frac{e^{-\mathrm{i}\omega n}-1}{e^{-\mathrm{i}\omega}-1},$$

for $\omega \neq 0 \mod 2\pi$, while, for $\omega = 0 \mod 2\pi$, obviously, $\mathbb{E}d_{n,U}^{\tau}(\omega) = n\tau$. □

Lemma 1.5. *Assume that (CS), with $p = 2$ and $\delta > k+1$, and (W) hold. Then, with the notation of Theorem 3.5,*

$$\sup_{\tau_1,\tau_2\in[0,1],\omega\in\mathbb{R}} \Big|\mathbb{E}\hat{G}_n(\tau_1,\tau_2;\omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1,\tau_2;\omega)\Big| = O((nb_n)^{-1}) + o(b_n^k).$$

Proof. By definition of $\hat{G}_n$ and Lemma 1.4, following the proof of Theorem 5.6.1 in Brillinger (1975), we have, uniformly in τ_1, τ_2 and ω,

$$\begin{aligned}
&\mathbb{E}\hat{G}_n(\tau_1,\tau_2;\omega)\\
&= \frac{1}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big)\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(2\pi s/n) + \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big)\varepsilon_n^{\tau_1,\tau_2}(2\pi s/n)\\
&= \int_0^{2\pi} W_n(\omega-\alpha)\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\alpha)\mathrm{d}\alpha + O(b_n^{-1}n^{-1})\\
&= b_n^{-1}\int_{-\infty}^{\infty} W(b_n^{-1}[\omega-\alpha])\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\alpha)\mathrm{d}\alpha + O(b_n^{-1}n^{-1})\\
&= \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) + B_n^{(k)}(\tau_1,\tau_2;\omega) + o(b_n^k) + O(b_n^{-1}n^{-1}),
\end{aligned}$$

where the last equality follows from the fact that (CS) implies that the function $\omega \mapsto \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ is k times continuously differentiable with derivatives that are bounded uniformly in τ_1, τ_2. □

1.2. Proofs for Section 3

In this Appendix, we give the proofs for Propositions 3.1, 3.2 and 3.4.

1.2.1. Proof of Proposition 3.1

Recall that, by the definition of cumulants,

$$\begin{aligned}
&|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})| \\
&= \Big| \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)!\mathbb{P}\Big(\bigcap_{i\in\nu_1} \{X_{t_i} \in A_i\}\Big) \cdots \mathbb{P}\Big(\bigcap_{i\in\nu_R} \{X_{t_i} \in A_i\}\Big)\Big| \qquad (1.3)
\end{aligned}$$

where the summation is performed with respect to all partitions $\{\nu_1, \dots, \nu_R\}$ of the set $\{1, \dots, p\}$. It suffices to establish that

$$\Big|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})\Big| \le K_p \alpha\Big(\lfloor p^{-1} \max_{i,j} |t_i - t_j|\rfloor\Big).$$

In the case $t_1 = ... = t_p$ this is obviously true. If at least two indices are distinct, choose j with $\max_{i=1,\dots,p-1}(t_{i+1} - t_i) = t_{j+1} - t_j > 0$ and let $(Y_{t_{j+1}}, \dots, Y_{t_p})$ be a random vector that is independent of $(X_{t_1}, \dots, X_{t_j})$ and possesses the same joint distribution as $(X_{t_{j+1}}, \dots, X_{t_p})$. By an elementary property of the cumulants [cf. Theorem 2.3.1 (iii) in Brillinger (1975)], we have

$$\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_j} \in A_j\}, I\{Y_{t_{j+1}} \in A_{j+1}\}, \dots, I\{Y_{t_p} \in A_p\}) = 0.$$

Therefore, we can write the cumulant of interest as

$$\begin{aligned}
&\Big|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\}) \\
&\quad - \operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_j} \in A_j\}, I\{Y_{t_{j+1}} \in A_{j+1}\}, \dots, I\{Y_{t_p} \in A_p\})\Big| \\
&= \Big| \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)![P_{\nu_1} \cdots P_{\nu_R} - Q_{\nu_1} \cdots Q_{\nu_R}]\Big|,
\end{aligned}$$

where the sum runs over all partitions $\{\nu_1, \dots, \nu_R\}$ of $\{1, \dots, p\}$,

$$P_{\nu_r} := \mathbb{P}\Big(\bigcap_{i\in\nu_r} \{X_{t_i} \in A_i\}\Big) \quad \text{and} \quad Q_{\nu_r} := \mathbb{P}\Big(\bigcap_{\substack{i\in\nu_r \\ i\le j}} \{X_{t_i} \in A_i\}\Big)\mathbb{P}\Big(\bigcap_{\substack{i\in\nu_r \\ i> j}} \{X_{t_i} \in A_i\}\Big),$$

$r = 1, \dots, R$, with $\mathbb{P}(\bigcap_{i\in\emptyset}\{X_{t_i} \in A_i\}) := 1$ by convention. Since X_t is α-mixing, it follows that, for any partition $\nu_1, ..., \nu_R$ and any $r = 1, ..., R$, we have $|P_{\nu_r} - Q_{\nu_r}| \le \alpha(t_{j+1} - t_j)$. Thus, for every partition $\nu_1, ..., \nu_R$,

$$|P_{\nu_1} \cdots P_{\nu_R} - Q_{\nu_1} \cdots Q_{\nu_R}| \le \sum_{r=1}^{R} |P_{\nu_r} - Q_{\nu_r}| \le R\alpha(t_{j+1} - t_j).$$

All together, this yields

$$|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})| \le \alpha(t_{j+1} - t_j) \sum_{\{\nu_1,\dots,\nu_R\}} R!.$$

Noting that $p(t_{j+1}-t_j) \geq \max_{i_1,i_2} |t_{i_1}-t_{i_2}|$ and observing that α is a monotone function, we obtain

$$|\operatorname{cum}(I\{X_{t_1} \in A_1\}, \dots, I\{X_{t_p} \in A_p\})| \leq K\alpha(\max|t_i - t_j|).$$

Now, additionally assume that $\alpha(n) \leq C\kappa^n$. Then

$$\begin{aligned}\alpha(\lfloor p^{-1}\max|t_i-t_j|\rfloor) &\leq C\kappa^{\lfloor p^{-1}\max|t_i-t_j|\rfloor} \leq C\kappa^{-1}(\kappa^{1/p})^{p\lfloor p^{-1}\max|t_i-t_j|\rfloor+1} \\ &\leq C\kappa^{-1}(\kappa^{1/p})^{\max|t_i-t_j|}.\end{aligned}$$

Setting $\rho = \kappa^{1/p} \in (0,1)$ completes the proof. □

1.2.2. Proof of Proposition 3.2

We follow the ideas of the proof of Proposition 2 in Wu and Shao (2004). Let $p \geq 2$ and assume without loss of generality that $t_1 \leq t_2 \leq \dots \leq t_p$. For $t > 0$, define the coupled random variables $X_t' := g(\dots,\varepsilon_{-1}^*,\varepsilon_0^*,\varepsilon_1,\dots\varepsilon_t)$. Choose an arbitrary $j \in \{1,\dots,p-1\}$ that satisfies $t_{j+1}-t_j = \max_i(t_{i+1}-t_i)$. In the case $\max_i(t_{i+1}-t_i) = 0$, there is nothing to prove. So, assume that $\max_i(t_{i+1}-t_i) > 0$. Define $V_i := I\{X_{t_i-t_j} \in A_i\}$ and $V_i' := I\{X'_{t_i-t_j} \in A_i\}$. Strict stationarity implies

$$\begin{aligned}&\operatorname{cum}(I\{X_{t_1} \in A_1\},\dots,I\{X_{t_j} \in A_j\},I\{X_{t_{j+1}} \in A_j\},\dots,I\{X_{t_p} \in A_p\}) \\ &\quad= \operatorname{cum}(V_1,\dots,V_p) \\ &\quad= \operatorname{cum}(V_1,\dots,V_j,V_{j+1}-V_{j+1}',V_{j+2},\dots,V_p) \\ &\qquad+\sum_{m=1}^{p-j-1}\operatorname{cum}(V_1,\dots,V_j,V_{j+1}',\dots,V_{j+m}',V_{j+1+m}-V_{j+1+m}',\dots,V_p) \\ &\qquad+\operatorname{cum}(V_1,\dots,V_j,V_{j+1}',\dots,V_p'). \end{aligned}\tag{1.4}$$

By an elementary property of cumulants, the last term in (1.4) is zero since the groups of random variables $(V_t)_{t<0}$ and $(V_t')_{t\geq 0}$ are independent by definition of the V_t'. Additionaly, by the definition of cumulants, uniform boundedness of indicators, and Assumption (G), we obtain the bounds

$$\begin{aligned}&\left|\operatorname{cum}(V_1,\dots,V_j,V_{j+1}-V_{j+1}',V_{j+2},\dots,V_p)\right| \leq C\mathbb{E}\left|V_{j+1}-V_{j+1}'\right| \leq C\sigma^{t_{j+1}-t_j}, \\ &\left|\operatorname{cum}(V_1,\dots,V_j,V_{j+1}',\dots,V_{j+m}',V_{j+1+m}-V_{j+1+m}',\dots,V_p)\right| \leq C\sigma^{t_{j+m+1}-t_j}.\end{aligned}$$

Observe that $\max_{i\neq l}|t_i-t_l| \geq p\max_i(t_{i+1}-t_i)$. The bound

$$|\operatorname{cum}(I\{X_{t_1} \in A_1\},\dots,I\{X_{t_p} \in A_p\})| \leq C(\sigma^{1/p})^{\max|t_i-t_j|}$$

follows from the fact that the number of summands in the sum is at most p. Setting $\rho := \sigma^{1/p}$ completes the proof. □

1.2.3. Proof of Proposition 3.4

It suffices to prove that

$$\Big(n^{-1/2}d^{\tau}_{n,R}(\omega)\Big)_{\tau\in[0,1]} \rightsquigarrow \Big(\mathbb{D}(\tau;\omega)\Big)_{\tau\in[0,1]} \quad \text{in } \ell^{\infty}([0,1]). \tag{1.5}$$

Now, for (1.5) to hold, it is sufficient that $\big(n^{-1/2}d^{\tau}_{n,U}(\omega)\big)_{\tau\in[0,1]}$ satisfies the following two conditions:

(i1) convergence of the finite-dimensional distributions, that is,

$$\big(n^{-1/2}d^{\tau_j}_{n,U}(\omega_j)\big)_{j=1,\dots,k} \xrightarrow{d} \big(\mathbb{D}(\tau_j;\omega_j)\big)_{j=1,\dots,k}, \tag{1.6}$$

for any $\tau_j \in [0,1]$ and fixed $\omega_j \neq 0 \mod 2\pi$, $j = 1,\dots,k$ and $k \in \mathbb{N}$;

(i2) stochastic equicontinuity: for any $x > 0$ and any $\omega \neq 0 \mod 2\pi$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\|\tau_1-\tau_2|\le\delta}}|n^{-1/2}(d^{\tau_1}_{n,U}(\omega)-d^{\tau_2}_{n,U}(\omega))|>x\Big)=0. \tag{1.7}$$

Indeed, under (i1) and (i2), an application of Theorems 1.5.4 and 1.5.7 in van der Vaart and Wellner (1996) yields

$$\Big(n^{-1/2}d^{\tau}_{n,U}(\omega)\Big)_{\tau\in[0,1]} \rightsquigarrow \Big(\mathbb{D}(\tau;\omega)\Big)_{\tau\in[0,1]} \quad \text{in } \ell^{\infty}([0,1]), \tag{1.8}$$

which, in combination with

$$\sup_{\tau\in[0,1]}|n^{-1/2}(d^{\tau}_{n,R}(\omega)-d^{\tau}_{n,U}(\omega))| = o_P(1), \quad \text{for } \omega\neq 0 \mod 2\pi, \tag{1.9}$$

which we prove below, yields the desired result that (1.5) holds. To prove (1.9), observe that, by (7.25), it suffices to bound the term

$$\sup_{\tau\in[0,1]} n^{-1/2}|d^{\hat{F}_n^{-1}(\tau)}_{n,U}(\omega)-d^{\tau}_{n,U}(\omega))|.$$

Now, for any $x>0$ and $\delta_n = o(1)$ such that $n^{1/2}\delta_n\to\infty$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\tau\in[0,1]} n^{-1/2}|d^{\hat{F}_n^{-1}(\tau)}_{n,U}(\omega)-d^{\tau}_{n,U}(\omega))|>x\Big)\\
&\le \mathbb{P}\Big(\sup_{\tau\in[0,1]}\sup_{|u-\tau|\le\delta_n}|d^{u}_{n,U}(\omega)-d^{\tau}_{n,U}(\omega)|>xn^{1/2},\ \sup_{\tau\in[0,1]}|\hat{F}_n^{-1}(\tau)-\tau|\le\delta_n\Big)\\
&\quad+\mathbb{P}\Big(\sup_{\tau\in[0,1]}|\hat{F}_n^{-1}(\tau)-\tau|>\delta_n\Big)=o(1)+o(1),
\end{aligned}$$

where the first $o(1)$ follows from (1.7), and the second one is a consequence of Lemma 7.5.

It thus remains to establish (1.6) and (1.7). First consider (1.7). Letting $T := (\tau_1 \wedge \tau_2, \tau_1 \vee \tau_2]$, we use the following moment inequality, which holds for $\omega \neq 0 \mod 2\pi$ and $\kappa \in (0,1)$, if $|\tau_1 - \tau_2|$ is small enough:

$$\mathbb{E}\big|n^{-1/2}(d_n^{\tau_1}(\omega) - d_n^{\tau_2}(\omega))\big|^{2L} = n^{-L}\mathbb{E}\prod_{m=1}^{2L} d_n^T\big((-1)^{m-1}\omega\big)$$

$$= n^{-L}\sum_{\{\nu_1,\ldots,\nu_R\}}\prod_{r=1}^{R}\mathrm{cum}\left(d_n^T\big((-1)^{m-1}\omega\big)\,:\, m\in\nu_r\right) \tag{1.10}$$

$$\leq n^{-L}\sum_{\{\nu_1,\ldots,\nu_R\}}\prod_{r=1}^{R}\Big[\tilde{C}\Big(\big|\Delta_n\big(\omega\sum_{m\in\nu_r}(-1)^{m-1}\big)\big|+1\Big)|\tau_1-\tau_2|^{\kappa}\Big] \tag{1.11}$$

$$\leq Cn^{-L}\sum_{R=1}^{2L} n^{R\wedge(2L-R)}|\tau_1-\tau_2|^{\kappa R} = C\sum_{R=1}^{2L} n^{-|R-L|}|\tau_1-\tau_2|^{\kappa R}. \tag{1.12}$$

Equality in (1.10) (summation is with respect to all partitions $\{\nu_1,\ldots,\nu_R\}$ of the set $\{1,\ldots,2L\}$) follows from Theorem 2.3.2 in Brillinger (1975). Inequality (1.11) follows from Lemma 7.4, and holds for arbitrary $\kappa \in (0,1)$ as long as $|\tau_1 - \tau_2|$ is small enough.

As for (1.12), note the fact that

$$\Delta_n(\omega) = \begin{cases} n & \omega = 0 \mod 2\pi, \\ \sin\big(\omega(n+1/2)\big)/\sin(\omega/2) & \omega \neq 0 \mod 2\pi, \end{cases}$$

implies $|\Delta_n(\omega)| \leq |\sin(\omega/2)|^{-1}$ if $\omega \neq 0 \mod 2\pi$. Therefore, (1.12) follows if we show that

$$|\{j = 1,\ldots,R \,:\, |\nu_j| \geq 2\}| \leq R \wedge (2L - R) \tag{1.13}$$

for any partition $\{\nu_1,\ldots,\nu_R\}$ of the set $\{1,\ldots,2L\}$. If $R \leq L$, the bound obviously holds true. For any $R > L$, let us show that

$$|\{j = 1,\ldots,R \,:\, |\nu_j| = 1\}| \geq 2(R - L) \tag{1.14}$$

holds for all $\{\nu_1,\ldots,\nu_R\}$. Denote by S the number of "singles" [sets ν_j with $|\nu_j| = 1$] in the given partition $\{\nu_1,\ldots,\nu_R\}$: the number of sets containing two or more elements is thus $R - S$, which implies that there are more than $2(R-S)+S = 2R-S$ elements in total. Inequality (1.14) follows, because if S were strictly smaller than $2(R-L)$, this would imply that the total number $2R - S$ of elements were strictly larger than $2L$.

Inequality (1.14) implies that the number of elements in sets with two or more elements is bounded by $2L - 2(R-L) = 4L - 2R$, which in turn implies that there are no more than $2L - R$ such sets, since each of them contains at least two elements; inequality (1.13), hence also (1.12), follow.

We now use the moment inequality (1.12) and Lemma 7.1 for establishing (1.7). Define $\Psi(x) := x^{2L}$, and note that, for $\omega \neq 0 \mod 2\pi$, $\gamma \in (0,\kappa)$ and $\tau_1, \tau_2 \in [0,1]$ with

$|\tau_1 - \tau_2| > n^{-1/\gamma}$, we have

$$\|n^{-1/2}(d_n^{\tau_1}(\omega) - d_n^{\tau_2}(\omega))\|_\Psi = (\mathbb{E}\big|n^{-1/2}(d_n^{\tau_1}(\omega) - d_n^{\tau_2}(\omega))\big|^{2L})^{1/(2L)}$$
$$\le \Big(\bar{C}\sum_{R=1}^{2L} n^{-|R-L|}|\tau_1-\tau_2|^{\kappa R}\Big)^{1/(2L)} \le \tilde{C}\sum_{R=1}^{2L} n^{-|R-L|/(2L)}|\tau_1-\tau_2|^{\kappa R/(2L)}$$
$$\le \tilde{C}\sum_{R=1}^{2L}|\tau_1-\tau_2|^{(\kappa R+\gamma|R-L|)/(2L)} \le C|\tau_1-\tau_2|^{\gamma/2} =: Cd(\tau_1,\tau_2). \quad (1.15)$$

Letting $\bar{\eta}_n := 2n^{-1/2}$ and choosing γ and L such that $\gamma L > 1$, Lemma 7.1 entails, for any $\eta \ge \bar{\eta}_n$,

$$\mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\le\delta}} n^{-1/2}|d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega)| > 2x\Big)$$
$$\le \Big(\frac{8K}{x}\Big[\int_{\bar{\eta}_n/2}^{\eta}\varepsilon^{-1/(\gamma L)}\mathrm{d}\varepsilon + (\delta+2\bar{\eta}_n)\eta^{-2/(\gamma L)}\Big]\Big)^{2L}$$
$$+\,\mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\le\bar{\eta}_n}} n^{-1/2}|d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega)| > x/2\Big). \quad (1.16)$$

Furthermore,

$$\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\le\bar{\eta}_n}} n^{-1/2}|d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega)| \le \sup_{\substack{\tau_1,\tau_2\in[0,1]\\ d(\tau_1,\tau_2)\le\bar{\eta}_n}} n^{-1/2}\sum_{t=0}^{n-1} I\{X_t\in(a\wedge b, a\vee b]\}$$
$$\le \sup_{|x-y|\le 2^{2/\gamma}n^{-1/\gamma}} n^{1/2}|\hat{F}_n(x)-\hat{F}_n(y)-(x-y)| + \sup_{|x-y|\le 2^{2/\gamma}n^{-1/\gamma}} n^{1/2}|x-y|$$
$$= O_P\Big((n^{2-1/\gamma}+n)^{1/(2k)}\big[n^{-1/\gamma}\big(|\log n|/\gamma\big)^{d_k}+n^{-1}\big]^{1/2} + n^{1/2-1/\gamma}\Big)$$
$$= o_P(1). \quad (1.17)$$

Together, (1.16) and (1.17) imply

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ |\tau_1-\tau_2|\le\delta}}|n^{-1/2}(d_{n,U}^{\tau_1}(\omega) - d_{n,U}^{\tau_2}(\omega))| > x\Big)$$
$$\le \left[\frac{8\tilde{K}}{x}\frac{\gamma L}{\gamma L-1}\eta^{(\gamma L-1)/(\gamma L)}\right]^{2L} + o(1)$$

for every $x, \eta > 0$. Condition (1.7) follows, since the integral in the right-hand side can be made arbitrarily small by choosing η accordingly.

Turning to (1.6), we employ Lemma 7.4 in combination with Lemma P4.5 and Theorem 4.3.2 from Brillinger (1975). More precisely, we have to verify that, for any $\tau_1, \dots, \tau_k \in [0,1]$, $k \in \mathbb{N}$, and $\omega_1, \dots, \omega_k \neq 0 \mod 2\pi$, all cumulants of the vector

$$n^{-1/2}\big(d_{n,U}^{\tau_1}(\omega_1), d_{n,U}^{\tau_1}(-\omega_1), \dots, d_{n,U}^{\tau_k}(\omega_k), d_{n,U}^{\tau_k}(-\omega_k)\big)$$

converge to the corresponding cumulants of the vector

$$\big(\mathbb{D}(\tau_1;\omega_1), \mathbb{D}(\tau_1;-\omega_1), \dots, \mathbb{D}(\tau_k;\omega_k), \mathbb{D}(\tau_k;-\omega_k)\big).$$

It is easy to see that the cumulants of order one converge as desired:

$$|\mathbb{E}(n^{-1/2}d_{n,U}^{\tau}(\omega))| = n^{-1/2}|\Delta_n(\omega)|\tau \leq n^{-1/2}\tau|\sin(\omega/2)|^{-1} = o(1),$$

for any $\tau \in [0,1]$ and fixed $\omega \neq 0 \mod 2\pi$. Furthermore, for the cumulants of order two, applying Theorem 4.3.1 in Brillinger (1975) to the bivariate process $(I\{X_t \leq q_{\mu_1}\}, I\{X_t \leq q_{\mu_2}\})$, we obtain

$$\mathrm{cum}(n^{-1/2}d_{n,U}^{\mu_1}(\lambda_1), n^{-1/2}d_{n,U}^{\mu_2}(\lambda_2)) = 2\pi n^{-1}\Delta_n(\lambda_1+\lambda_2)\mathfrak{f}_{q_{\mu_1},q_{\mu_2}}(\lambda_1) + o(1)$$

for any $(\lambda_1,\mu_1), (\lambda_2,\mu_2) \in \bigcup_{i=1}^k \{(\omega_i,\tau_i), (-\omega_i,\tau_i)\}$. which yields the correct second moment structure. Finally, the cumulants of order J, with $J \in \mathbb{N}$ and $J \geq 3$, all tend to zero as, in view of Lemma 7.4, with $\varepsilon := \min\{\mu_1, \dots, \mu_J\}$,

$$\begin{aligned}
&\mathrm{cum}(n^{-1/2}d_{n,U}^{\mu_1}(\lambda_1), \dots, n^{-1/2}d_{n,U}^{\mu_J}(\lambda_J)) \\
&\leq Cn^{-J/2}(|\Delta_n(\sum_{j=1}^{J}\lambda_j)| + 1)\varepsilon(|\log\varepsilon| + 1)^d = O(n^{-(J-2)/2}) = o(1)
\end{aligned}$$

for $(\lambda_1,\mu_1), \dots, (\lambda_J,\mu_J) \in \bigcup_{i=1}^k \{(\omega_i,\tau_i), (-\omega_i,\tau_i)\}$. This implies that the limit $\mathbb{D}(\tau;\omega)$ is Gaussian, and completes the proof of (1.6). Proposition 3.4 follows. □

1.3. Proofs of the results from Section 7.4

We begin this section by stating an auxiliary technical result that is used in the proofs of Lemmas 7.5, 7.6 and 7.7. Its proof relies on Lemma 7.4.

Lemma 1.6. *Assume that $(X_t)_{t\in\mathbb{Z}}$ is a strictly stationary process satisfying (C) and such that $X_0 \sim U[0,1]$. Denote by $\hat{F}_n$ the empirical distribution function of $X_0, ..., X_{n-1}$. Then, for any $k \in \mathbb{N}$, there exists a constant d_k depending on k only such that*

$$\begin{aligned}
&\sup_{x,y\in[0,1],|x-y|\leq\delta_n} \sqrt{n}|\hat{F}_n(x) - \hat{F}_n(y) - (x-y)| \\
&\qquad = O_P\Big((n^2\delta_n + n)^{1/2k}(\delta_n|\log\delta_n|^{d_k} + n^{-1})^{1/2}\Big)
\end{aligned}$$

as $\delta_n \to 0$.

Proof of Lemma 1.6. Observe the decomposition

$$\begin{aligned}|\hat F_n(x)-\hat F_n(y)-(x-y)| \;\le\; & \Big|\hat F_n(x)-\hat F_n\Big(\frac{\lfloor nx\rfloor}{n}\Big)-\Big(x-\frac{\lfloor nx\rfloor}{n}\Big)\Big| \\ & +\Big|\hat F_n(y)-\hat F_n\Big(\frac{\lfloor ny\rfloor}{n}\Big)-\Big(y-\frac{\lfloor ny\rfloor}{n}\Big)\Big| \\ & +\Big|\hat F_n\Big(\frac{\lfloor nx\rfloor}{n}\Big)-\hat F_n\Big(\frac{\lfloor ny\rfloor}{n}\Big)-\Big(\frac{\lfloor nx\rfloor}{n}-\frac{\lfloor ny\rfloor}{n}\Big)\Big|.\end{aligned}$$

Since $|\lfloor ny\rfloor/n-y|\le 1/n$, and by monotonicity of $\hat F_n$,

$$\begin{aligned}\Big|\hat F_n(y)-\hat F_n\Big(\frac{\lfloor ny\rfloor}{n}\Big)-\Big(y-\frac{\lfloor ny\rfloor}{n}\Big)\Big| \;\le\; & \hat F_n\Big(\frac{1+\lfloor ny\rfloor}{n}\Big)-\hat F_n\Big(\frac{\lfloor ny\rfloor}{n}\Big)+\frac{1}{n} \\ \le\; & \Big|\hat F_n\Big(\frac{1+\lfloor ny\rfloor}{n}\Big)-\hat F_n\Big(\frac{\lfloor ny\rfloor}{n}\Big)-\frac{1}{n}\Big|+\frac{2}{n}.\end{aligned}$$

A similar bound holds with x substituting y, so that, letting $M_n:=\{j/n|j=0,...,n\}$,

$$\begin{aligned}&\sup_{x,y\in[0,1],|x-y|\le\delta_n}|\hat F_n(x)-\hat F_n(y)-(x-y)| \\ &\qquad\qquad \le 3\max_{x,y\in M_n,|x-y|\le\delta_n+2n^{-1}}|\hat F_n(x)-\hat F_n(y)-(x-y)|+4/n.\end{aligned}$$

The cardinality of the set $\{x,y\in M_n:|x-y|\le\delta_n+2n^{-1}\}$ is of the order $O(n^2(\delta_n+n^{-1}))$. Recalling that $\max_{j=1,...,N}|Z_j|=O_P(N^{1/m})$ as $N\to\infty$ for any sequence $(Z_j)_{j\in\mathbb{Z}}$ of random variables with uniformly bounded moments of order m, the claim follows if we can show that, for any $k\in\mathbb{N}$,

$$\sup_{x,y\in[0,1],|x-y|\le\delta}\mathbb{E}\Big(\frac{n^{1/2}|\hat F_n(x)-\hat F_n(y)-(x-y)|}{((\delta(1+|\log\delta|)^d)\vee n^{-1})^{1/2}}\Big)^{2k}\le C_k.$$

Now, this latter inequality is a consequence of the fact that, for all $y>x$,

$$\begin{aligned}&\mathbb{E}(\hat F_n(x)-\hat F_n(y)-(x-y))^{2k} \\ &\quad = n^{-2k}\sum_{\nu_1,...,\nu_R,|\nu_j|\ge2}\prod_{r=1}^{R}\mathrm{cum}(d_n^{(x,y]}(0),...,d_n^{(x,y]}(0))\end{aligned}\tag{1.18}$$

where $d:=\max(d_1,...,d_k)$ [recall the notation $d_n^{(x,y]}(\omega)$ from Lemma 7.4] and the sum runs over all partitions of $\{1,...,2k\}$; in view of Lemma 7.4, this latter quantity in turn is bounded by

$$\tilde C_k n^{-2k}\sum_{j=1}^{k}n^j|x-y|^j(1+|\log(y-x)|)^{jd},$$

where the constant $\tilde C_k$ only depends on k, ρ, and K. This completes the proof of Lemma 1.6. □

1.3.1. Proof of Lemma 7.1

As in the proof of Theorem 2.2.4 in van der Vaart and Wellner (1996), we construct nested sets $T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_k \subset T$ such that every T_j is a maximal set of points with $d(s,t) > \eta 2^{-j}$, for all $s,t \in T_j$. Here, maximal means that no point can be added without destroying the validity of the inequality; stop adding subsets when k is such that $\Delta_k := \eta/2^k < \bar\eta \le \eta/2^{k-1}$.

For $s,t \in T$ with $d(s,t) \le \delta$, denote by $s',t' \in T_k$ the points closest to s and t, respectively. Then, since by construction $d(s,t) \ge \Delta_k \ge \bar\eta/2$ for any $s \neq t$, $s,t \in T_k$,

$$\begin{aligned}
\sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_t| &= \sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_{s'} - (\mathbb{G}_t - \mathbb{G}_{t'}) - (\mathbb{G}_{t'} - \mathbb{G}_{s'})| \\
&\le \sup_{\substack{d(s',t')\le\delta+2\Delta_k \\ s',t'\in T_k}} |\mathbb{G}_{t'} - \mathbb{G}_{s'}| + 2 \sup_{t'\in T_k} \sup_{t:d(t,t')\le\Delta_k} |\mathbb{G}_t - \mathbb{G}_{t'}| \\
&\le \sup_{\substack{d(s',t')\le\delta+2\bar\eta \\ s',t'\in T_k}} |\mathbb{G}_{t'} - \mathbb{G}_{s'}| + 2 \sup_{t'\in T_k} \sup_{t:d(t,t')\le\bar\eta} |\mathbb{G}_t - \mathbb{G}_{t'}|.
\end{aligned}$$

Adapting the proof of Theorem 2.2.4 in van der Vaart and Wellner (1996), let us show that

$$\Big\| \sup_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |\mathbb{G}_t - \mathbb{G}_s| \Big\|_\Psi \le 4K \Big[\int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon,d)\big) \mathrm{d}\epsilon + (\delta + 2\bar\eta)\Psi^{-1}\big(D^2(\eta,d)\big)\Big]. \quad (1.19)$$

By the definition of packing numbers, we have $|T_j| \le D(\eta 2^{-j}, d)$. Let every point $t_j \in T_j$ be linked to a unique $t_{j-1} \in T_{j-1}$ such that $d(t_j, t_{j-1}) \le \eta 2^{-j}$. This yields, for every t_k a chain $t_k, t_{k-1}, \ldots, t_0$ connecting t_k to a point $t_0 \in T_0$. For two arbitrary points $s_k, t_k \in T_k$, the difference of increments along their respective chains is bounded by

$$\begin{aligned}
|(\mathbb{G}_{s_k} - \mathbb{G}_{s_0}) - (\mathbb{G}_{t_k} - \mathbb{G}_{t_0})| &= |\sum_{j=0}^{k-1} (\mathbb{G}_{s_{j+1}} - \mathbb{G}_{s_j}) - \sum_{j=0}^{k-1} (\mathbb{G}_{t_{j+1}} - \mathbb{G}_{t_j})| \\
&\le 2\sum_{j=0}^{k-1} \max_{(u,v)\in L_j} |\mathbb{G}_u - \mathbb{G}_v|,
\end{aligned}$$

where L_j denotes the set of all links (u,v) from points $u \in T_{j+1}$ to points $v \in T_j$. Because the links were constructed by connecting any point in T_{j+1} to a unique point in T_j, we have $|L_j| = |T_{j+1}|$. By assumption,

$$\|\mathbb{G}_u - \mathbb{G}_v\|_\Psi \le C d(u,v) \le C\eta 2^{-j} \quad \text{for all} (u,v) \in L_j.$$

Therefore, it follows from Lemma 2.2.2 in van der Vaart and Wellner (1996) that

$$\Big\| \max_{s,t\in T_k} |(\mathbb{G}_s - \mathbb{G}_{s_0}) - (\mathbb{G}_t - \mathbb{G}_{t_0})| \Big\|_\Psi \le 2 \sum_{j=0}^{k-1} \tilde{K} \Psi^{-1}\big(D(\eta 2^{-(j+1)}, d)\big) C\eta 2^{-j}$$
$$\le K \sum_{j=0}^{k-1} \Psi^{-1}\big(D(\eta 2^{-j-1}, d)\big) 4\eta (2^{-j} - 2^{-j-1}) \le 4K \int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon, d)\big) \mathrm{d}\epsilon \quad (1.20)$$

for some constant K only depending on Ψ and C.

In (1.20), $s_0 = s_0(s)$ and $t_0 = t_0(t)$ are the endpoints of the chains starting at s and t, respectively. We therefore have

$$\Big\| \max_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |\mathbb{G}_t - \mathbb{G}_s| \Big\|_\Psi \le 4K \int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon, d)\big) \mathrm{d}\epsilon + \Big\| \max_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |\mathbb{G}_{s_0(s)} - \mathbb{G}_{t_0(t)}| \Big\|_\Psi . \quad (1.21)$$

To complete the proof, we use the same arguments as in van der Vaart and Wellner (1996). For every pair of endpoints $s_0(s), t_0(t)$ of chains starting at $s, t \in T_k$ with distance $d(s,t) \le \delta$, choose exactly one pair $s_k^0, t_k^0 \in T_k$, with $d(s_k^0, t_k^0) < \delta + 2\bar\eta$, whose chains end at s_0, t_0. Because $|T_0| = D(\eta 2^{-0}, d)$, there are at most $D^2(\eta, d)$ such (s_k^0, t_k^0) pairs. Therefore, we have the following bound for the second term in the right-hand side in (1.21):

$$\begin{aligned}
\Big\| \max_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |\mathbb{G}_{s_0(s)} - \mathbb{G}_{t_0(t)}| \Big\|_\Psi &\le \Big\| \max_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |(\mathbb{G}_{s_0(s)} - \mathbb{G}_{s_k^0}) - (\mathbb{G}_{t_0(t)} - \mathbb{G}_{t_k^0})| \Big\|_\Psi \\
&\quad + \Big\| \max_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |(\mathbb{G}_{s_k^0} - \mathbb{G}_{t_k^0})| \Big\|_\Psi \\
&= S_1^n + S_2^n, \quad \text{say.}
\end{aligned}$$

Noting that S_1^n is bounded by the right-hand side in (1.20), while S_2^n can be bounded by employing Lemma 2.2.2 in van der Vaart and Wellner (1996) again, we obtain the desired inequality from

$$\Big\| \max_{\substack{d(s,t)\le\delta+2\bar\eta \\ s,t\in T_k}} |(\mathbb{G}_{s_k^0} - \mathbb{G}_{t_k^0})| \Big\|_\Psi \le (\delta + 2\bar\eta) \Psi^{-1}\big(D^2(\eta, d)\big) K.$$

This completes the proof of Lemma 7.1. □

1.3.2. Proof of Lemma 7.2

The proof consists of two steps. In the first step, we derive the representation

$$\mathbb{E}|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)|^{2L} = \sum_{\substack{\{\nu_1,\dots,\nu_R\} \\ |\nu_j|\ge 2,\ j=1,\dots,R}} \prod_{r=1}^{R} \mathcal{D}_{a,b}(\nu_r) \quad (1.22)$$

where the summation runs over all partitions $\{\nu_1,\dots,\nu_R\}$ of $\{1,\dots,2L\}$ such that each set ν_j contains at least two elements, and

$$\mathcal{D}_{a,b}(\xi) := \sum_{\ell_{\xi_1},\dots,\ell_{\xi_q}\in\{1,2\}} n^{-3q/2} b_n^{q/2}\Big(\prod_{m\in\xi}\sigma_{\ell_m}\Big) \times \sum_{s_{\xi_1},\dots,s_{\xi_q}=1}^{n-1}\Big(\prod_{m\in\xi} W_n(\omega-2\pi s_m/n)\Big)\,\mathrm{cum}(D_{\ell_m,(-1)^{m-1}s_m}\,:\,m\in\xi),$$

for any set $\xi := \{\xi_1,\dots,\xi_q\}\subset\{1,\dots,2L\}$, where $q := |\xi|$ and

$$D_{\ell,s} := d_n^{M_1(\ell)}(2\pi s/n)d_n^{M_2(\ell)}(-2\pi s/n), \quad \ell=1,2, \quad s=1,\dots,n-1,$$

with the sets $M_1(1)$, $M_2(2)$, $M_2(1)$, $M_1(2)$ and the signs $\sigma_\ell\in\{-1,1\}$ defined in (1.24) below.

In step two of the proof, we employ assumption (7.26) to prove

$$\sup_{\substack{\xi\subset\{1,\dots,2L\}\\|\xi|=q}}\ \sup_{\|a-b\|_1\le\varepsilon} |\mathcal{D}_{a,b}(\xi)| \le C(nb_n)^{1-q/2}g(\varepsilon), \quad 2\le q\le 2L. \tag{1.23}$$

To conclude the proof of the lemma, it is sufficient to observe that, for any partition in (1.22),

$$\Big|\prod_{r=1}^{R}\mathcal{D}_{a,b}(\nu_r)\Big| \le Cg^R(\varepsilon)(nb_n)^{R-L}$$

[note that $\sum_{r=1}^R|\nu_r| = 2L$].

Step 1. For the proof of (1.22), let $a=(a_1,a_2)$ and $b=(b_1,b_2)$. Then

$$\mathbb{E}|\hat H_n(a;\omega)-\hat H_n(b;\omega)|^{2L} = n^{-3L}b_n^L\sum_{s_1,\dots,s_{2L}=1}^{n-1}\Big(\prod_{m=1}^{2L}W_n(\omega-2\pi s_m/n)\Big) \times \sum_{j_1,\dots,j_{2L}=0}^{n-1}\sum_{k_1,\dots,k_{2L}=0}^{n-1}\mathbb{E}\Big[\prod_{m=1}^{2L}A_{j_mk_m}(a,b)\Big]\exp\Big(-\frac{2\pi}{n}\mathrm{i}\sum_{m=1}^{2L}(-1)^{m-1}s_m(j_m-k_m)\Big),$$

where

$$\begin{aligned} A_{jk}(a,b) &:= B_{jk}(a,b)-\mathbb{E}B_{jk}(a,b)\\ B_{jk}(a,b) &:= I\{X_j\le a_1\}I\{X_k\le a_2\} - I\{X_j\le b_1\}I\{X_k\le b_2\}\\ &= \sigma_1 I\{X_j\in M_1(1)\}I\{X_k\in M_2(1)\} + \sigma_2 I\{X_j\in M_1(2)\}I\{X_k\in M_2(2)\}\end{aligned}$$

with

$$\begin{aligned} \sigma_1 &:= 2I\{a_1>b_1\}-1, & \sigma_2 &:= 2I\{a_2>b_2\}-1,\\ M_1(1) &:= (a_1\wedge b_1, a_1\vee b_1], & M_2(2) &:= (a_2\wedge b_2, a_2\vee b_2],\\ M_2(1) &:= \begin{cases}[0,a_2] & b_2\ge a_2\\ [0,b_2] & a_2>b_2,\end{cases} & M_1(2) &:= \begin{cases}[0,b_1] & b_2\ge a_2\\ [0,a_1] & a_2>b_2.\end{cases}\end{aligned} \tag{1.24}$$

Note that, for each $\ell = 1, 2$, $\mathbb{P}(X_0 \in M_\ell(\ell)) = \lambda(M_\ell(\ell)) \leq \|a - b\|_1 \leq \varepsilon$. The product theorem (Theorem 2.3.2 of Brillinger (1975)) entails

$$\mathbb{E}\Big[\prod_{\ell=1}^{2L} A_{j_\ell k_\ell}(a, b)\Big] = \sum_{\substack{\{\nu_1,\ldots,\nu_R\} \\ |\nu_j| \geq 2,\ j=1,\ldots,R}} \prod_{r=1}^{R} \mathrm{cum}(B_{j_i k_i}(a, b)\ :\ i \in \nu_r)$$

where the sum runs over all partitions $\{\nu_1, \ldots, \nu_R\}$ of $\{1, \ldots, 2L\}$. Note that $\mathbb{E}A_{jk}(a, b) = 0$; consequently, a summand is vanishing for any partition which has some ν_j with $|\nu_j| = 1$. Therefore, it suffices to consider summation over the partitions for which $|\nu_j| \geq 2$ for all $j = 1, \ldots, R$.

Furthermore,

$$\begin{aligned}
&\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} B_{jk}(a, b) \exp(-\mathrm{i}(2\pi/n)[s(j-k)]) \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \Big(\sigma_1 I\{X_j \in M_1(1)\} I\{X_k \in M_2(1)\} + \sigma_2 I\{X_j \in M_1(2)\} I\{X_k \in M_2(2)\}\Big) \\
&\qquad\qquad \times \exp(-\mathrm{i}(2\pi/n)[s(j-k)]) \\
&= \sigma_1 D_{1,s} + \sigma_2 D_{2,s},
\end{aligned}$$

which yields

$$\begin{aligned}
\mathbb{E}|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)|^{2L} &= n^{-3L} b_n^L \sum_{s_1,\ldots,s_{2L}=1}^{n-1} \Big(\prod_{m=1}^{2L} W_n(\omega - 2\pi s_m/n)\Big) \\
&\times \sum_{\substack{\{\nu_1,\ldots,\nu_R\} \\ |\nu_j| \geq 2,\ j=1,\ldots,R}} \prod_{r=1}^{R} \mathrm{cum}(\sigma_1 D_{1,(-1)^{m-1}s_m} + \sigma_2 D_{2,(-1)^{m-1}s_m}\ :\ m \in \nu_r) \\
&= \sum_{\substack{\{\nu_1,\ldots,\nu_R\} \\ |\nu_j| \geq 2,\ j=1,\ldots,R}} \prod_{r=1}^{R} \mathcal{D}_{a,b}(\nu_r),
\end{aligned}$$

and concludes the proof of (1.22).

Step 2. Still by the product theorem, letting $q = |\xi|$,

$$\begin{aligned}
\mathcal{D}_{a,b}(\xi) = &\sum_{\ell_{\xi_1},\ldots,\ell_{\xi_q} \in \{1,2\}} n^{-3q/2} b_n^{q/2} \sum_{s_{\xi_1},\ldots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m \in \xi} W_n(\omega - 2\pi s_m/n)\Big) \\
&\times \Big(\prod_{m \in \xi} \sigma_{\ell_m}\Big) \sum_{\{\mu_1,\ldots,\mu_N\}} \prod_{j=1}^{N} \mathrm{cum}(d_n^{M_k(\ell_m)}(2\pi(-1)^{k+m} s_m/n) : (m, k) \in \mu_j)
\end{aligned}$$

where the summation runs over all *indecomposable partitions* $\{\mu_1, \ldots, \mu_N\}$ (see Brillinger (1975), p. 20) of the scheme

$$\begin{array}{cc} (\xi_1, 1) & (\xi_1, 2) \\ \vdots & \vdots \\ (\xi_q, 1) & (\xi_q, 2). \end{array} \tag{1.25}$$

Note that for each $m \in \xi \subset \{1, \ldots, 2L\}$, there exists a $j \in \{1, 2\}$ such that $\mathbb{P}(X_0 \in M_j(\ell_m)) = \lambda(M_j(\ell_m)) \leq \|a - b\|_1 \leq \varepsilon$.

Now, by assumption (7.26),

$$\begin{aligned} |\mathcal{D}_{a,b}(\xi)| \quad \leq \quad & K n^{-3q/2} b_n^{q/2} 2^q \sum_{\{\mu_1, \ldots, \mu_N\}} \sum_{s_{\xi_1}, \ldots, s_{\xi_q}=1}^{n-1} \Big(\prod_{m \in \xi} \Big| W_n(\omega - 2\pi s_m / n) \Big| \Big) \\ & \times \Big(\Big| \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k) \in \mu_1} (-1)^{m+k} s_m \Big) \Big| + 1 \Big) \\ & \times \cdots \times \Big(\Big| \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k) \in \mu_N} (-1)^{m+k} s_m \Big) \Big| + 1 \Big) g(\varepsilon). \end{aligned}$$

An indecomposable partition $\{\mu_1, \ldots, \mu_N\}$ of the scheme (1.25) consists of at most $N \leq q+1$ sets, because any partition with $N \geq q+2$ is necessarily decomposable. To see this, note that there is only one partition with $N = 2q$ and that this partition is decomposable. Any partition with $N = 2q - i < 2q$ sets can be obtained by i steps of agglomeration (i.e., iteratively merging sets from the partition, where each step reduces the number of sets by one unit). Obviously, it requires at least $q-1$ steps to obtain an indecomposable partition. Therefore, any partition that is the result of a sequence of $q-2$ steps (or less) is decomposable. Any partition with at least $2q - (q-2) = q+2$ sets thus is decomposable.

We now follow an argument from Brillinger (cf. the proof of his Theorem 7.4.4) to complete the proof. As sketched there, we have, with the common convention that $\prod_{i \in \emptyset} a_i := 1$,

$$\begin{aligned} & \prod_{j=1}^{N} \Big(\Big| \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k) \in \mu_j} (-1)^{m+k} s_m \Big) \Big| + 1 \Big) \\ & \qquad = \sum_{I \subset \{1, \ldots, N\}} \prod_{j \in I} \Delta_n \Big(\frac{2\pi}{n} \sum_{(m,k) \in \mu_j} (-1)^{m+k} s_m \Big), \end{aligned}$$

by using the fact that

$$0 \leq \Delta_n \Big(\frac{2\pi}{n} k \Big) = \begin{cases} n & k \in n\mathbb{Z} \\ 0 & k \notin n\mathbb{Z}. \end{cases}$$

As explained by Brillinger, the functions Δ_n introduce linear constraints on summation with respect to s_m, $m \in \xi$. First, note that the case $|I| = q+1 = N$ is irrelevant. Indeed, we then have that

$$\sum_{s_{\xi_1},\ldots,s_{\xi_q}=1}^{n-1}\Big(\prod_{m\in\xi}\Big|W_n(\omega-2\pi s_m/n)\Big|\Big)\prod_{j\in I}\Delta_n\Big(\frac{2\pi}{n}\sum_{(m,k)\in\mu_j}(-1)^{k+m}s_m\Big)=0,$$

because $|I|>q$ implies that there exists an index $j\in I$ with $|\mu_j|=|\{(\bar m,\bar k)\}|=1$, which in turn implies $\sum_{(m,k)\in\mu_j}(-1)^{m+k}s_m=(-1)^{\bar m+\bar k}s_{\bar m}\notin n\mathbb{Z}$ for all $s_{\bar m}=1,\ldots,n-1$.

Next, consider the case $|I|\le q$. We have

$$\sum_{s_{\xi_1},\ldots,s_{\xi_q}=1}^{n-1}\Big(\prod_{m\in\xi}\Big|W_n(\omega-2\pi s_m/n)\Big|\Big)\prod_{j\in I}\Delta_n\Big(\frac{2\pi}{n}\sum_{(m,k)\in\mu_j}(-1)^{k+m}s_m\Big)$$
$$=\sum_{(s_{\xi_1},\ldots,s_{\xi_q})\in S_n(\mu,I)}\Big(\prod_{m\in\xi}\Big|W_n(\omega-2\pi s_m/n)\Big|\Big)n^{|I|},$$

where

$$S_n(\mu,I):=\Big\{(s_{\xi_1},\ldots,s_{\xi_q})\in\{1,\ldots,n-1\}^q\ \Big|\ \sum_{(m,k)\in\mu_j}(-1)^{k+m}s_m\in n\mathbb{Z},\ \forall\mu_j\in\mu,\ j\in I\Big\}.$$

Elementary linear algebra implies that there are $|I|$ linear constraints if $|I|<N$ and $|I|-1$ linear constraints if $|I|=N$. More precisely, for every element μ_j of the partition $\{\mu_1,...,\mu_N\}$, define a vector

$$w_m^{(j)}:=(-1)^{m+1}I\{(m,1)\in\mu_j\}+(-1)^{m+2}I\{(m,2)\in\mu_j\}\in\{-1,0,1\}^L$$

for $m=1,...,L$. Observe that the linear constraint that is introduced by the equality $\sum_{(m,k)\in\mu_j}(-1)^{k+m}s_m\in n\mathbb{Z}$ can be written as $(s_1,...,s_m)'w^{(j)}\in n\mathbb{Z}$. In particular, the linear constraints corresponding to $\mu_{j_1},...,\mu_{j_\ell}$ are linearly dependent if and only if $\sum_{k=1}^{\ell}w^{(j_k)}=0$, which follows from the special structure of the vectors $w^{(j)}$ [note, in particular, that at each position $k=1,...,2L$, at most two vectors $w^{(1)},...,w^{(N)}$ can have non-zero entries, and that in this case the entry in one vector is 1 and the entry in the other vector is -1]. However, for non-decomposable partitions $\sum_{k=1}^{\ell}w^{(j_k)}=0$ if and only if $\{j_1,...,j_\ell\}=\{1,....,N\}$.

To complete the proof of (1.23), it is therefore sufficient to show that

$$\sum_{(s_{\xi_1},\ldots,s_{\xi_q})\in S_n(\mu,I)}\left(\prod_{m\in\xi}\Big|W_n(\omega-2\pi s_m/n)\Big|\right)=O\Big((b_n^{-1})^{|I|-\lfloor|I|/N\rfloor}n^{q-(|I|-\lfloor|I|/N\rfloor)}\Big),\tag{1.26}$$

because this implies that $\mathcal{D}_{a,b}(\xi)$ is of the order

$$n^{-3q/2}b_n^{q/2}\max_{N\le q}\max_{|I|\le N}(b_n^{-1})^{|I|-\lfloor|I|/N\rfloor}n^{q-(|I|-\lfloor|I|/N\rfloor)}n^{|I|}g(\varepsilon)\asymp(nb_n)^{1-q/2}g(\varepsilon).$$

As for the proof of (1.26), it suffices to point out that $|I| - \lfloor |I|/N \rfloor$ of the s-indices can be expressed via the independent linear constraints and will take only a number of values which is less or equal to q. Then (1.26) follows from the fact that

$$n\frac{1}{n}\sum_{s=1}^{n-1}\Big|W_n(\omega - 2\pi s/n)\Big| \leq n\left(\int_{\mathbb{R}} b_n^{-1}\Big|W\big(b_n^{-1}(\omega-\beta)\big)\Big|\mathrm{d}\beta + o(1)\right) = O(n),$$

and $|W_n(\omega)| \leq \|W\|_\infty b_n^{-1} = O(b_n^{-1})$. The proof is thus complete. □

1.3.3. Proof of Lemma 7.3

Observe that

$$\begin{aligned}
&\mathrm{cum}(I\{X_0 \leq q_{a_1}\}, I\{X_k \leq q_{a_2}\}) - \mathrm{cum}(I\{X_0 \leq q_{b_1}\}, I\{X_k \leq q_{b_2}\}) \\
&\quad = \mathrm{cum}(I\{F(X_0) \leq a_1\}, I\{F(X_k) \leq a_2\}) \\
&\qquad - \mathrm{cum}(I\{F(X_0) \leq b_1\}, I\{F(X_k) \leq b_2\}) \\
&\quad = \sigma_1 \,\mathrm{cum}(I\{F(X_0) \in M_1(1)\}, I\{F(X_k) \in M_2(1)\}) \\
&\qquad + \sigma_2 \,\mathrm{cum}(I\{F(X_0) \in M_1(2)\}, I\{F(X_k) \in M_2(2)\})
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &:= 2I\{a_1 > b_1\} - 1, & \sigma_2 &:= 2I\{a_2 > b_2\} - 1, \\
M_1(1) &:= (a_1 \wedge b_1, a_1 \vee b_1], & M_2(2) &:= (a_2 \wedge b_2, a_2 \vee b_2], \\
M_2(1) &:= \begin{cases} [0, a_2] & b_2 \geq a_2 \\ [0, b_2] & a_2 > b_2, \end{cases} & M_1(2) &:= \begin{cases} [0, b_1] & b_2 \geq a_2 \\ [0, a_1] & a_2 > b_2. \end{cases}
\end{aligned}$$

In particular, observe that $\lambda(M_j(j)) \leq \|a - b\|_1$ for $j = 1, 2$. We thus have

$$\begin{aligned}
&\Big|\frac{\mathrm{d}^j}{\mathrm{d}\omega^j}\mathfrak{f}_{q_{a_1},q_{a_2}}(\omega) - \frac{\mathrm{d}^j}{\mathrm{d}\omega^j}\mathfrak{f}_{q_{b_1},q_{b_2}}(\omega)\Big| \\
&\quad \leq \sum_{k\in\mathbb{Z}} |k|^j |\,\mathrm{cum}(I\{F(X_0) \in M_1(1)\}, I\{F(X_k) \in M_2(1)\})| \\
&\qquad + \sum_{k\in\mathbb{Z}} |k|^j |\,\mathrm{cum}(I\{F(X_0) \in M_1(2)\}, I\{F(X_k) \in M_2(2)\}| \\
&\quad \leq 4\sum_{k=0}^{\infty} k^j \Big((K\rho^k) \wedge \|a - b\|_1\Big),
\end{aligned}$$

and the assertion follows by simple algebraic manipulations similar to those in the proof of Proposition 3.1. □

1.3.4. Proof of Lemma 7.4

By the definition of cumulants and strict stationarity, we have

$$
\begin{aligned}
&\operatorname{cum}(d_n^{A_1}(\omega_1),\dots,d_n^{A_p}(\omega_p))\\
&\quad= \sum_{t_1=0}^{n-1}\cdots\sum_{t_p=0}^{n-1}\operatorname{cum}(I_{\{X_{t_1}\in A_1\}},\dots,I_{\{X_{t_p}\in A_p\}})\exp\Big(-\mathrm{i}\sum_{j=1}^{p}t_j\omega_j\Big)\\
&\quad= \sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^{p}\omega_j\Big)\sum_{t_2,\dots,t_p=0}^{n-1}\exp\Big(-\mathrm{i}\sum_{j=2}^{p}\omega_j(t_j-t_1)\Big)\\
&\qquad\quad\times\operatorname{cum}(I_{\{X_0\in A_1\}},I_{\{X_{t_2-t_1}\in A_2\}}\dots,I_{\{X_{t_p-t_1}\in A_p\}})\\
&\quad= \sum_{u_2,\dots,u_p=-n}^{n}\operatorname{cum}(I_{\{X_0\in A_1\}},I_{\{X_{u_2}\in A_2\}}\dots,I_{\{X_{u_p}\in A_p\}})\exp\Big(-\mathrm{i}\sum_{j=2}^{p}\omega_j u_j\Big)\\
&\qquad\quad\times\sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^{p}\omega_j\Big)I_{\{0\le t_1+u_2<n\}}\cdots I_{\{0\le t_1+u_p<n\}}. \qquad (1.27)
\end{aligned}
$$

Lemma 1.1 implies that

$$
\Big|\Delta_n(\sum_{j=1}^{p}\omega_j)-\sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^{p}\omega_j\Big)I_{\{0\le t_1+u_2<n\}}\cdots I_{\{0\le t_1+u_p<n\}}\Big|\le 2\sum_{j=2}^{p}|u_j|. \quad (1.28)
$$

Let us show that, for any $p+1$ intervals $A_0,\dots,A_p\subset\mathbb{R}$ and any p-tuple $\kappa:=(\kappa_1,...,\kappa_p)\in\mathbb{R}_+^p$, $p\ge 2$

$$
\begin{aligned}
&\sum_{k_1,\dots,k_p=-\infty}^{\infty}\Big(1+\sum_{j=1}^{p}|k_j|^{\kappa_j}\Big)\big|\operatorname{cum}\big(I_{\{X_{k_1}\in A_1\}},\dots,I_{\{X_{k_p}\in A_p\}},I_{\{X_0\in A_0\}}\big)\big|\\
&\quad\le C\varepsilon(|\log\varepsilon|+1)^d. \qquad (1.29)
\end{aligned}
$$

To this end, define $k_0=0$ and consider the set

$$
T_m:=\big\{(k_1,...,k_p)\in\mathbb{Z}^p|\max_{i,j=0,\dots,p}|k_i-k_j|=m\big\}
$$

and note that $|T_m|\le c_p m^{p-1}$ for some constant c_p. With this notation, it follows from condition (C) and the bound

$$
|\operatorname{cum}(I\{X_{t_1}\in A_1\},...,I\{X_{t_p}\in A_p\})|\le C\min_{i=1,\dots,p}P(X_0\in A_i),
$$

which follows from the definition of cumulants and some simple algebra that

$$
\begin{aligned}
&\sum_{k_1,\dots,k_p=-\infty}^{\infty}\Big(1+\sum_{j=1}^{p}|k_j|^{\kappa_j}\Big)\big|\operatorname{cum}\big(I_{\{X_{k_1}\in A_1\}},\dots,I_{\{X_{k_p}\in A_p\}},I_{\{X_0\in A_0\}}\big)\big|\\
&\quad= \sum_{m=0}^{\infty}\sum_{(k_1,\dots,k_p)\in T_m}\Big(1+\sum_{j=1}^{p}|k_j|^{\kappa_j}\Big)\big|\operatorname{cum}\big(I_{\{X_{k_1}\in A_1\}},\dots,I_{\{X_{k_p}\in A_p\}},I_{\{X_0\in A_0\}}\big)\big|\\
&\quad\le \sum_{m=0}^{\infty}\sum_{(k_1,\dots,k_p)\in T_m}\Big(1+pm^{\max_j\kappa_j}\Big)\Big(\rho^m\wedge\varepsilon\Big)K_p
\end{aligned}
$$

$$\leq C_p \sum_{m=0}^{\infty} \Big(\rho^m \wedge \varepsilon\Big)|T_m| m^{\max_j \kappa_j}.$$

For $\varepsilon \geq \rho$, (1.29) follows trivially. For $\varepsilon < \rho$, set $m_\varepsilon := \log \varepsilon / \log \rho$ and note that $\rho^m \leq \varepsilon$ if and only if $m \geq m_\varepsilon$. Thus,

$$\begin{aligned} \sum_{m=0}^{\infty} \Big(\rho^m \wedge \varepsilon\Big) m^u &\leq \sum_{m \leq m_\varepsilon} m^u \varepsilon + \sum_{m > m_\varepsilon} m^u \rho^m \\ &\leq C\Big(\varepsilon m_\varepsilon^{u+1} + \rho^{m_\varepsilon} \sum_{m=0}^{\infty} (m + m_\varepsilon)^u \rho^m\Big). \end{aligned}$$

Observing that $\rho^{m_\varepsilon} = \varepsilon$ completes the proof of the desired inequality (1.29). The lemma then follows from (1.27), (1.28), (1.29) and the triangular inequality. □

1.3.5. Proof of Lemma 7.5

By the functional delta method applied to the map $F \mapsto F^{-1}$ [see Theorem 3.9.4 and Lemma 3.9.23(ii) in van der Vaart and Wellner (1996)], it suffices to show that $\sqrt{n}(\hat{F}_n(\tau) - \tau)$ converges to a tight Gaussian limit with continuous sample paths. This can be done by proving convergence of finite-dimensional distributions together with stochastic equicontinuity [see the discussion in the proof of Theorem 3.6(iii)]. The stochastic equicontinuity follows by an application of Lemma 7.1, Lemma 1.6 and (1.18). For the convergence of the finite-dimensional distributions, apply the cumulant central limit theorem [Lemma P4.5 in Brillinger (1975)] in combination with Lemma 7.4. □

1.3.6. Proof of Lemma 7.6

Let $T := [0, 1]$, $T_n := \{j/n : j = 0, ..., n\}$, and note that, for n large enough,

$$\sup_{\omega \in \mathcal{F}_n} \sup_{\tau \in T} |d_n^\tau(\omega)| \leq \max_{\omega \in \mathcal{F}_n} \max_{\tau \in T_n} |d_n^\tau(\omega)| + \max_{\omega \in \mathcal{F}_n} \max_{\tau \in T_n} \sup_{|\eta - \tau| \leq 1/n} |d_n^\tau(\omega) - d_n^\eta(\omega)|. \tag{1.30}$$

Expressing moments in terms of cumulants, straightforward arguments and Lemma 7.4 yield

$$\max_{\omega \in \mathcal{F}_n} \max_{\tau \in T_n} \mathbb{E}|d_n^\tau(\omega)|^{2k} \leq C_k n^k.$$

Thus $n^{-1/2} d_n^\tau(\omega)$ has uniformly bounded moments of order $2k$. Recall that an arbitrary sequence $(Z_j)_{j \in \mathbb{Z}}$ of random variables with uniformly bounded moments of order m is such that $\max_{j=1,\ldots,N} |Z_j| = O_P(N^{1/m})$. Thus,

$$\max_{\omega \in \mathcal{F}_n} \max_{\tau \in T_n} n^{-1/2} |d_n^\tau(\omega)| = O_P((n^2)^{1/2k}) = O_P(n^{1/k})$$

since the maximum is taken over $O(n^2)$ values. For the second term in the right-hand side of (1.30), note that

$$\max_{\omega \in \mathcal{F}_n} \Big| d_n^\tau(\omega) - d_n^\eta(\omega) \Big| \leq \sum_{t=0}^{n-1} I\{X_t \leq \tau \vee \eta\} - I\{X_t \leq \tau \wedge \eta\}.$$

Thus, by Lemma 1.6, we have

$$
\begin{aligned}
&\max_{\omega\in\mathcal{F}_n}\max_{\tau\in T_n}\sup_{|\eta-\tau|\le 1/n}|d_n^{\tau}(\omega)-d_n^{\eta}(\omega)|\\
&\le n\max_{\tau\in T_n}\sup_{|\eta-\tau|\le 1/n}|\hat{F}_n(\tau\vee\eta)-\hat{F}_n(\tau\wedge\eta)-\tau\vee\eta+\tau\wedge\eta|+C\\
&=O_P(n^{1/2+1/2k}(\log n)^{d_k}).
\end{aligned}
$$

for some constant d_k. This completes the proof. $\square$

1.3.7. Proof of Lemma 7.7

Without loss of generality, we can assume that $n^{-1}=o(\delta_n)$ [otherwise, enlarge the supremum by considering $\tilde{\delta}_n:=\max(n^{-1},\delta_n)$]. Letting $u=(u_1,u_2)$ and $v=(v_1,v_2)$,

$$
\hat{H}_n(u;\omega)-\hat{H}_n(v;\omega)=b_n^{1/2}n^{-1/2}\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)(K_{s,n}(u,v)-\mathbb{E}K_{s,n}(u,v))
$$

where [with $d_{n,U}$ defined in (2.6)]

$$
\begin{aligned}
K_{s,n}(u,v) &:= n^{-1}\big(d_{n,U}^{u_1}(2\pi s/n)d_{n,U}^{u_2}(-2\pi s/n)-d_{n,U}^{v_1}(2\pi s/n)d_{n,U}^{v_2}(-2\pi s/n)\big)\\
&= d_{n,U}^{u_1}(2\pi s/n)n^{-1}\big[d_{n,U}^{u_2}(-2\pi s/n)-d_{n,U}^{v_2}(-2\pi s/n)\big]\\
&\quad + d_{n,U}^{v_2}(-2\pi s/n)n^{-1}\big[d_{n,U}^{u_1}(2\pi s/n)-d_{n,U}^{v_1}(2\pi s/n)\big].
\end{aligned}
$$

Note that, by Lemma 7.6, we have, for any $k\in\mathbb{N}$,

$$
\sup_{y\in[0,1]}\sup_{\omega\in\mathcal{F}_n}|d_{n,U}^{y}(\omega)|=O_P\Big(n^{1/2+1/k}\Big). \tag{1.31}
$$

Furthermore, by Lemma 1.6, for any $\ell\in\mathbb{N}$,

$$
\begin{aligned}
&\sup_{\omega\in\mathbb{R}}\sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n}n^{-1}|d_{n,U}^{x}(\omega)-d_{n,U}^{y}(\omega)|\\
&\le \sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n}n^{-1}\sum_{t=0}^{n-1}|I\{Y_t\le x\}-I\{Y_t\le y\}|\\
&\le \sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n}|\hat{F}_n(x\vee y)-\hat{F}_n(x\wedge y)-F(x\vee y)+F(x\wedge y)|+C\delta_n\\
&=O_P\big(\rho_n(\delta_n,\ell)+\delta_n\big),
\end{aligned}
$$

with $\rho_n(\delta_n,\ell):=n^{-1/2}(n^2\delta_n+n)^{1/2\ell}(\delta_n|\log\delta_n|^{d_\ell}+n^{-1})^{1/2}$, where d_ℓ is a constant depending only on ℓ.

Combining these arguments and observing that $\sup_{\omega\in\mathbb{R}}\sum_{s=1}^{n-1}\Big|W_n(\omega-2\pi s/n)\Big|=O(n)$ yields

$$
\sup_{\omega\in\mathbb{R}}\sup_{\substack{u,v\in[0,1]^2\\ \|u-v\|_1\le\delta_n}}\Big|\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)K_{s,n}(u,v)\Big|=O_P\big(n^{3/2+1/k}(\rho(\delta_n,\ell)+\delta_n)\big). \tag{1.32}
$$

Next, define the intervals

$$M_1(1) := (u_1 \wedge v_1, u_1 \vee v_1], \qquad M_2(2) := (u_2 \wedge v_2, u_2 \vee v_2],$$

$$M_2(1) := \begin{cases} [0, u_2] & v_2 \geq u_2 \\ [0, v_2] & u_2 > v_2, \end{cases} \qquad M_1(2) := \begin{cases} [0, v_1] & v_2 \geq u_2 \\ [0, u_1] & v_2 > v_2. \end{cases}$$

With this notation, observe that

$$\begin{aligned} &\sup_{\|u-v\|_1 \leq \delta_n} \sup_{s=1,\ldots,n-1} |\mathbb{E}K_{s,n}(u,v)| \qquad (1.33) \\ &\quad \leq n^{-1} \sup_{\|u-v\|_1 \leq \delta_n} \sup_{s=1,\ldots,n-1} \left| \operatorname{cum}(d_{n,U}^{M_1(1)}(2\pi s/n), d_{n,U}^{M_2(1)}(-2\pi s/n)) \right| \\ &\quad + n^{-1} \sup_{\|u-v\|_1 \leq \delta_n} \sup_{s=1,\ldots,n-1} \left| \operatorname{cum}(d_{n,U}^{M_1(2)}(2\pi s/n), d_{n,U}^{M_2(2)}(-2\pi s/n)) \right| \end{aligned}$$

where we have used the fact that $\mathbb{E}d_{n,U}^{M}(2\pi s/n) = 0$. Lemma 7.4 and the fact that $\lambda(M_j(j)) \leq \delta_n$ (with λ denoting the Lebesgue measure over $\mathbb{R}$) for $j = 1, 2$ yield

$$\sup_{\|u-v\|_1 \leq \delta_n} \sup_{s=1,\ldots,n-1} |\operatorname{cum}(d_n^{M_1(j)}(2\pi s/n), d_n^{M_2(j)}(-2\pi s/n))| \leq C(n+1)\delta_n(1 + |\log \delta_n|)^d,$$

It follows that the right-hand side in (1.33) is $O(\delta_n |\log \delta_n|^d)$.

Therefore, since $\sup_{\omega \in \mathbb{R}} \sum_{s=1}^{n-1} \left| W_n(\omega - 2\pi s/n) \right| = O(n)$, we obtain

$$\sup_{\omega \in \mathbb{R}} \sup_{\|u-v\|_1 \leq \delta_n} \left| b_n^{1/2} n^{-1/2} \sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) \mathbb{E}K_{s,n}(u,v) \right| = O\big((nb_n)^{1/2} \delta_n |\log n|^d\big).$$

Observe that, in view of the assumption that $n^{-1} = o(\delta_n)$, we have $\delta_n = O(n^{1/2}\rho_n(\delta_n, \ell))$, which, in combination with (1.32), yields

$$\begin{aligned} &\sup_{\omega \in \mathbb{R}} \sup_{\|u-v\|_1 \leq \delta_n} |\hat{H}_n(u;\omega) - \hat{H}_n(v;\omega)| \\ &= O_P\Big((nb_n)^{1/2}[n^{1/2+1/k}(\rho_n(\delta_n, \ell) + \delta_n) + \delta_n |\log \delta_n|^d]\Big) \\ &= O_P\Big((nb_n)^{1/2} n^{1/2+1/k} \rho_n(\delta_n, \ell)\Big) \\ &= O_P\Big((nb_n)^{1/2} n^{1/k+1/\ell}(n^{-1} \vee \delta_n (\log n)^{d_\ell})^{1/2}\Big). \end{aligned}$$

This latter quantity is $o_P(1)$: indeed, for arbitrary k and ℓ,

$$O((nb_n)^{1/2} n^{1/k+1/\ell} \delta_n^{1/2} (\log n)^{d_\ell/2}) = O((nb_n)^{1/2-1/2\gamma} n^{1/k+1/\ell} (\log n)^{d_\ell/2});$$

in view of the assumptions on b_n, which imply $(nb_n)^{1/2-1/2\gamma} = o(n^{-\kappa})$ for some $\kappa > 0$, this latter quantity is $o(1)$ for k, ℓ sufficiently large. The term $(nb_n)^{1/2} n^{1/k+1/\ell} n^{-1/2}$ can be handled in a similar fashion. This concludes the proof. $\square$

References

Ahdesmäki, M., Lähdesmäki, H., Pearson, R., Huttunen, H. and Yli-Harja, O. (2005). Robust detection of periodic time series measured from biological systems. *BMC Bioinformatics* **6** 117.

Berghaus, B., Bücher, A. and Volgushev, S. (2014). Weak convergence of the empirical copula process with respect to weighted metrics (arXiv:1411.5888). *ArXiv e-prints.*

Blomqvist, N. (1950). On a measure of dependence between two random variables. *The Annals of Mathematical Statistics* **21** 593–600.

Brillinger, D. R. (1975). *Time Series: Data Analysis and Theory.* Holt, Rinehart and Winston, Inc., New York.

Brockwell, P. J. and Davis, R. A. (1987). *Time Series: Theory and Methods. Springer Series in Statistics.* Springer, New York.

Carcea, M. and Serfling, R. (2014). A Gini autocovariance function for time series modeling. *Preprint, UT Dallas.*

Cifarelli, D. M., Conti, P. L. and Regazzini, E. (1996). On the asymptotic distribution of a general measure of monotone dependence. *The Annals of Statistics* **24** 1386–1399.

Dahlhaus, R. (1988). Empirical spectral processes and their applications to time series analysis. *Stochastic Processes and their Applications* **30** 69–83.

Dahlhaus, R. and Polonik, W. (2009). Empirical spectral processes for locally stationary time series. *Bernoulli* **15** 1–39.

Davis, R. A. and Mikosch, T. (2009). The extremogram: A correlogram for extreme events. *Bernoulli* **15** 977–1009.

Davis, R. A., Mikosch, T. and Zhao, Y. (2013). Measures of serial extremal dependence and their estimation. *Stochastic Processes and their Applications* **123** 2575–2602.

Dette, H., Hallin, M., Kley, T. and Volgushev, S. (2013). Of copulas, quantiles, ranks and spectra: An L_1-approach to spectral analysis. *Bernoulli* **forthcoming**.

Ferguson, T. S., Genest, C. and Hallin, M. (2000). Kendall's tau for serial dependence. *The Canadian Journal of Statistics* **28** 587–604.

Fermanian, J. D., Radulović, D. and Wegkamp, M. H. (2004). Weak convergence of empirical copula processes. *Bernoulli* **10** 847-860.

Gasser, T., Muller, H.-G. and Mammitzsch, V. (1985). Kernels for Nonparametric Curve Estimation. *Journal of the Royal Statistical Society. Series B (Methodological)* **47** 238–252.

Genest, C., Carabarín-Aguirre, A. and Harvey, F. (2013). Copula parameter estimation using Blomqvist's beta. *Journal de la Société Française de Statistique* **154** 5–24.

Genest, C. and Rémillard, B. (2004). Test of independence and randomness based on the empirical copula process. *Test* **13** 335–369.

Giraitis, L. and Koul, H. L. (2013). On asymptotic distributions of weighted sums of periodograms. *Bernoulli* **forthcoming**.

Hagemann, A. (2013). Robust Spectral Analysis (arXiv:1111.1965v2). *ArXiv e-prints*.

Hájek, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *The Annals of Mathematical Statistics* **39** 325–346.

Hallin, M. (2012). Ranks. In *Encyclopedia of Environmetrics* 2nd ed. (W. Piegorsch and A. El Shaarawi, eds.) 2135–2152. Wiley.

Hallin, M. and Puri, M. L. (1992). Rank tests for time-series analysis: a survey. In *New Directions in Time Series Analysis* (E. P. D. Brillinger and M. Rosenblatt, eds.) 111–154. Springer-Verlag, New York.

Hallin, M. and Puri, M. L. (1994). Aligned rank tests for linear models with autocorrelated error terms. *Journal of Multivariate Analysis* **50** 175 - 237.

Han, H., Linton, O. B., Oka, T. and Whang, Y.-J. (2014). The cross-quantilogram: measuring quantile dependence and testing directional predictability between time series. *Available at:* `papers.ssrn.com/sol3/papers.cfm?abstract_id=2338468`.

Hill, J. B. and McCloskey, A. (2013). Heavy tail robust frequency domain estimation. `http://www.econ.brown.edu/fac/adam_mccloskey/Research_files/FDTTQML.pdf`.

Hong, Y. (1999). Hypothesis testing in time series via the empirical characteristic function: a generalized spectral density approach. *Journal of the American Statistical Association* **94** 1201–1220.

Hong, Y. (2000). Generalized spectral tests for serial dependence. *Journal of the Royal Statistical Society Series B (Statistical Methodology)* **62** 557–574.

Katkovnik, V. (1998). Robust M-Periodogram. *IEEE Transactions on Signal Processing* **46** 3104–3109.

Kendall, M. G. (1938). A new measure of rank correlation. *Biometrika* **30** 81–93.

Kleiner, R., Martin, R. D. and Thomson, D. J. (1979). Robust estimation of power spectra. *J. Royal Statist. Soc. B* **41** 313–351.

Kley, T. (2014a). Quantile-based spectral analysis in an object-oriented framework and a reference implementation in R: The **quantspec** package (arXiv:1408.6755). *ArXiv e-prints*.

Kley, T. (2014b). quantspec: Quantile-based spectral analysis functions R package version 1.0-1.99.

Klüppelberg, C. and Mikosch, T. (1994). Some limit theory for the self-normalised periodogram of stable processes. *Scandinavian Journal of Statistics* **21** 485–491.

Koenker, R. and Bassett, G. (1978). Regression quantiles. *Econometrica* **46** 33–50.

Koenker, R. and Xiao, Z. (2006). Quantile autoregression. *Journal of the American Statistical Association* **101** 980–990.

Lee, J. and Rao, S. S. (2012). The quantile spectral density and comparison based tests for nonlinear time series (arXiv:1112.2759v2). *ArXiv e-prints*.

Li, T. H. (2008). Laplace periodogram for time series analysis. *Journal of the American Statistical Association* **103** 757–768.

Li, T.-H. (2012). Quantile periodograms. *Journal of the American Statistical Association* **107** 765–776.

Li, T.-H. (2013). *Time Series with Mixed Spectra: Theory and Methods*. CRC Press, Boca Raton.

Linton, O. and Whang, Y. J. (2007). The quantilogram: with an application to evaluating directional predictability. *Journal of Econometrics* **141** 250–282.

Liu, W. and Wu, W. B. (2010). Asymptotics of spectral density estimates. *Econometric Theory* **26** 1218–1245.

Maronna, R. A., Martin, D. R. and Yohai, V. J. (2006). *Robust Statistics: Theory and Methods. Wiley Series in Probability and Statistics*. Wiley.

Mikosch, T. (1998). Periodogram estimates from heavy-tailed data. In *A Practical Guide to Heavy Tails: Statistical Techniques for Analysing Heavy-Tailed Distributions* (R. A. Adler, R. Feldman and M. S. Taqqu, eds.) 241–258. Birkhäuser, Boston.

Nelsen, R. B. (1998). Concordance and Gini's measure of association. *Journal of Nonparametric Statistics* **9** 227–238.

Priestley, M. B. (1981). *Spectral Analysis and Time Series: Multivariate series, prediction and control. Probability and Mathematical Statistics*. Academic Press.

Schechtman, E. and Yitzhaki, S. (1987). A measure Of association based on Gini's mean difference. *Communications in Statistics - Theory and Methods* **16** 207–231.

Schmid, F., Schmidt, R., Blumentritt, T., Gaisser, S. and Ruppert, M. (2010). Copula-based measures of multivariate association. In *Copula Theory and Its Applications*, (P. Jaworski, F. Durante, W. K. Härdle, T. Rychlik, P. Bickel, P. Diggle, S. Fienberg, U. Gather, I. Olkin and S. Zeger, eds.). *Lecture Notes in Statistics* **198** 209-236. Springer, Berlin, Heidelberg.

Shao, X. (2011). Testing for white noise under unknown dependence and its applications to diagnostic checking for time series models. *Econometric Theory* **27** 312.

Shao, X. and Wu, W. B. (2007). Asymptotic spectral theory for nonlinear time series. *The Annals of Statistics* **35** 1773–1801.

Skaug, H. J. and Tjøstheim, D. (1993). A nonparametric test of serial independence based on the empirical distribution function. *Biometrika* **80** 591–602.

Skowronek, S., Volgushev, S., Kley, T., Dette, H. and Hallin, M. (2014). Quantile spectral analysis for locally stationary time series (arXiv:1404.4605). *ArXiv e-prints*.

Tjøstheim, D. (1996). Measures of dependence and tests of independence. *Statistics: A Journal of Theoretical and Applied Statistics* **28** 249–284.

van der Vaart, A. and Wellner, J. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer, New York.

Wald, A. and Wolfowitz, J. (1943). An exact test for randomness in the nonparametric case based on serial correlation. *The Annals of Mathematical Statistics* **14** 378–388.

Wu, W. B. and Shao, X. (2004). Limit theorems for iterated random functions. *Journal of Applied Probability* **41** 425–436.