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Proof of Theorem 4.1 — Supplement to "An empirical likelihood approach for symmetric α-stable processes"

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We give the proof for Theorem 4.1 in this supplement since the proof for the vector case is more technical than the scalar case.

Keywords: heavy tail, confidence region, Whittle likelihood, symmetric α-stable process, empirical likelihood ratio, self-normalized periodogram, normalized power transfer function.

Appendix A: Proof for Theorem 4.1

For our purpose, it is sufficient to derive the asymptotics of $\boldsymbol{P}_n(\boldsymbol{\theta}_0)$ and $\boldsymbol{S}_n(\boldsymbol{\theta}_0)$, since the evaluation of other terms is the same as what we derived in the scalar case. The random vectors in the underscript will not be written in boldface, and it will not be confused with random variables.

Lemma A.1. *Suppose $\{X(t)\}_{t=0}^{\infty}$ satisfies (4.1) with (4.2). Then*

$$\boldsymbol{I}_{n,X}(\omega) = \Psi(\omega)\boldsymbol{I}_{n,Z}(\omega)\Psi(\omega)^* + \boldsymbol{R}_n(\omega).$$

If $\phi(\omega)$ is a $d \times d$ matrix-valued continuous function on $[-\pi, \pi]$, then

$$x_n \int_{-\pi}^{\pi} \mathrm{tr}[\,\boldsymbol{R}_n(\omega)\phi(\omega)\,]d\omega \xrightarrow{\mathcal{P}} 0.$$

Proof. We follow the proof of the univariate case in Mikosch et al. (1995).

$$\begin{aligned}
d_{n,X}(\omega) &= n^{-1/\alpha}\sum_{t=1}^{n}\boldsymbol{X}(t)\exp(i\omega t) = n^{-1/\alpha}\sum_{t=1}^{n}\exp(i\omega t)\Big(\sum_{j=0}^{\infty}\Psi(j)\boldsymbol{Z}(t-j)\Big) \\
&= \Psi(\omega)d_{n,Z}(\omega) + n^{-1/\alpha}\sum_{j=0}^{\infty}\Psi(j)\exp(ij\omega)\boldsymbol{Y}_{n,j}(\omega), \\
&= \boldsymbol{J}_{n,Z}(\omega) + n^{-1/\alpha}\boldsymbol{Y}_n(\omega) \quad \text{(say)},
\end{aligned}$$

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where

$$\boldsymbol{Y}_{n,j}(\omega) = \sum_{t=1-j}^{n-j} \boldsymbol{Z}(t)\exp(i\omega t) - \sum_{t=1}^{n} \boldsymbol{Z}(t)\exp(i\omega t).$$

Then we have

$$\boldsymbol{R}_n(\omega) = n^{-1/\alpha}\boldsymbol{Y}_n(\omega)\boldsymbol{J}_{n,Z}(\omega)^* + n^{-1/\alpha}\boldsymbol{J}_{n,Z}(\omega)\boldsymbol{Y}_n(\omega)^* + n^{-2/\alpha}\boldsymbol{Y}_n(\omega)\boldsymbol{Y}_n(\omega)^*.$$

$\sum_{j=0}^{\infty}\Psi(j)\exp(ij\omega) \leq \sum_{j=0}^{\infty}\|\Psi(j)\| < \infty$, so that $\|\Psi(\omega)\|$ is bounded. Since every element of $\boldsymbol{Z}(t)$ is in the domain of attraction of a stable law with a parameter α, $\boldsymbol{J}_{n,Z}(\omega)$ is also stochastically bounded. As results in the proof of lemma 6.2 in Mikosch et al. (1995), we know that for each $l \in \{1, 2, \ldots, d\}$,

$$\sum_{j=0}^{\infty}\Psi(j)_{kl}\exp(ij\omega)\boldsymbol{Y}_{n,j}(\omega)_l = O_p(1)$$

and

$$\int_{-\pi}^{\pi} n^{-2/\alpha}|\sum_{j=0}^{\infty}\Psi(j)_{kl}\exp(ij\omega)\boldsymbol{Y}_{n,j}(\omega)_l|^2 d\omega = o_p(x_n^{-2}).$$

Combining these two results, it is easy to see that $\boldsymbol{Y}_n(\omega) = O_p(1)$, and by the boundedness of $\phi(\omega)$,

$$\begin{aligned}
x_n\Big|\int_{-\pi}^{\pi}\mathrm{tr}[\boldsymbol{R}_n(\omega)\phi(\omega)]\,d\omega\Big| &\leq x_n\int_{-\pi}^{\pi}|\mathrm{tr}[\,\boldsymbol{R}_n(\omega)\phi(\omega)\,]|\,d\omega \\
&\leq x_n\int_{-\pi}^{\pi}\|\boldsymbol{R}_n(\omega)\|_E\|\phi(\omega)\|_E\,d\omega \\
&\leq c_1x_n\int_{-\pi}^{\pi}\|n^{-1/\alpha}\boldsymbol{Y}_n(\omega)\boldsymbol{J}_{n,Z}(\omega)^*\|_E + \|n^{-1/\alpha}\boldsymbol{J}_{n,Z}(\omega)\boldsymbol{Y}_n(\omega)^*\|_E \\
&\quad +\|n^{-2/\alpha}\boldsymbol{Y}_{n,Z}(\omega)\boldsymbol{Y}_{n,Z}(\omega)^*)\|_E\,d\omega \\
&\leq c_2x_n\Big\{\Big(\int_{-\pi}^{\pi}\|\boldsymbol{J}_{n,Z}(\omega)\|_E^2\,d\omega\Big)^{1/2}\Big(\int_{-\pi}^{\pi}n^{-2/\alpha}\|\boldsymbol{Y}_n(\omega)\|_E^2\,d\omega\Big)^{1/2} \\
&\quad +\int_{-\pi}^{\pi}n^{-2/\alpha}\|\boldsymbol{Y}_n(\omega)\|_E^2\,d\omega\Big\}. \\
&\xrightarrow{\mathcal{P}} 0.
\end{aligned}$$

□

Before looking into the asymptotics of $\boldsymbol{P}_n(\boldsymbol{\theta}_0)$, we have to show the existence of the limit matrix of the autocovariance matrix in distribution. If the case is the same as what we suppose in section 4, i.e., the components of the vector $\boldsymbol{Z}$ are mutually independent, then we have the lemma due to Davis et al. (1986) by applying continuous mapping theorem.

Suppose $y_n = (n\log n)^{1/\alpha}$. It is obvious that $Z(1)_k$'s satisfy followings:

$$P(|Z(1)_i| > x) = x^{-\alpha}L(x), \quad i = 1, 2, \ldots, d \tag{A.1}$$

with $\alpha > 0$ and $L(x)$ a slowly varying function at ∞ and

$$\frac{P(Z(1)_i > x)}{P(|Z(1)_i| > x)} \to p, \quad \frac{P(Z(1)_i < -x)}{P(|Z(1)_i| > x)} \to q \tag{A.2}$$

as $x \to \infty$, $0 \le p \le 1$ and $q = 1 - p$.

Lemma A.2. *Let $\{\boldsymbol{Z}(t)\}$ be a sequence of i.i.d. random vectors satisfying (6.19) and (6.20) with $0 < \alpha < 2$ and $E|Z(1)_i|^\alpha = \infty$ for all $i = 1, 2, \dots, d$. Then*

$$\left(n^{-2/\alpha} \sum_{t=1}^{n} \boldsymbol{Z}(t)\boldsymbol{Z}(t)', y_n^{-1} \sum_{t=1}^{n} \boldsymbol{Z}(t)\boldsymbol{Z}(t+1)', \dots, y_n^{-1} \sum_{t=1}^{n} \boldsymbol{Z}(t)\boldsymbol{Z}(t+h)' \right) \xrightarrow{\mathcal{L}} (S(0), S(1), \dots, S(h)),$$

where $S(0), S(1), \dots, S(h)$ are independent stable random matrices; the components of $S(0)$ are all positive with index $\alpha/2$, and $S(1), \dots, S(h)$ are identically distributed with index α.

The self-normalized parameter, denoted by $\|Z\|_n$, is defined as follows:

$$\|Z\|_n \equiv \sqrt{\sum_{t=1}^{n} \sum_{i=1}^{d} Z(t)_i^2}.$$

It is well known that $Z(t)_i^2$ is in the domain of attraction of a stable limit with $\alpha/2$, and the linear transformation of stable distribution with nonrandom scale is also stable with the same characteristic exponent. Thus the sum $\sum_{i=1}^{d} Z(t)_i^2$ is also in the domain of attraction of a stable limit with $\alpha/2$. The normalized form of vectors is written as

$$\tilde{Z}(t)_i = \frac{Z(t)_i}{\|Z\|_n}, \quad i = 1, \dots, d.$$

Lemma A.3. *Let $(\boldsymbol{X}(t))_{t\in\mathbb{Z}}$ be a linear process defined as (4.1) with coefficient matrices $(\Psi(j))_{j\in\mathbb{Z}}$ satisfying (4.2) with $\alpha \in (0, 2)$. Also, let $\phi_k(\omega), k = 1, \dots, d$, be $d \times d$ matrix-valued 2π-periodic continuous function with $\phi_k(\omega) = \phi_k(\omega)^*$ such that the Fourier coefficients of $\Psi(\cdot)\phi_k(\cdot)\Psi(\cdot)^*$ satisfy*

$$\sum_{t=1}^{\infty} \left\| \int_{-\pi}^{\pi} \Psi(\omega)^* \phi_k(\omega) \Psi(\omega) \exp(it\omega) d\omega \right\|_E^{\mu} < \infty$$

for some $\mu \in (0, \alpha)$ and all $k = 1, \cdots, d$. Then

$$(n^{-2/\alpha}\|Z\|_n^2, x_n \int_{-\pi}^{\pi} \mathrm{tr} \left[\{\boldsymbol{I}_{n,X}(\omega) - \Psi(\omega)\hat{\Gamma}_{n,Z}(0)\Psi(\omega)^*\}\phi_k(\omega) \right] d\omega) \xrightarrow{\mathcal{L}} (\, S_{\alpha/2}, \sum_{i,j=1}^{d} \sum_{h=1}^{\infty} S(h)_{ij} \int_{-\pi}^{\pi} (A(\omega) + \overline{A(\omega)})_{ij}\, d\omega\,),$$

where

$$A(\omega) = \Psi(\omega)^* \phi_k(\omega) \Psi(\omega) \exp(ih\omega),$$

and $S(h)_{ij}$ is the (i, j)-component of the limit stable random matrix $S(h)$, where

$$x_n \hat{\Gamma}_{n,Z}(h) \Rightarrow S(h) \quad \textit{for } h = 1, 2, \dots.$$

Proof. From Lemma A.2, we can see that

$$\left(n^{-2/\alpha}\hat{\Gamma}_{n,Z}(0), y_n^{-1}\hat{\Gamma}_{n,Z}(k), k=1\dots,h\right) \Rightarrow (S(0),S(1),\dots,S(h)).$$

Note that $\operatorname{tr}\hat{\Gamma}_{n,Z}(0)=\|Z\|_n^2$, according to the continuous mapping theorem, the statement holds true if we show

$$x_n\int_{-\pi}^{\pi}\operatorname{tr}\Big[\{\boldsymbol{I}_{n,X}(\omega)-\Psi(\omega)\hat{\Gamma}_{n,Z}(0)\Psi(\omega)^*\}\phi_k(\omega)\Big]\,d\omega$$

$$\xrightarrow{\mathcal{L}}\sum_{i,j=1}^{d}\sum_{h=1}^{\infty}S(h)_{ij}\int_{-\pi}^{\pi}(A(\omega)+\overline{A(\omega)})_{ij}\,d\omega.$$

As a result of Lemma 6.1 in Klüppelberg and Mikosch (1996), under Assumption 4.3,

$$x_n\left(\sum_{i,j=1}^{d}\left[\sum_{h=1}^{n-1}\hat{\Gamma}_{n,Z}(h)\right]_{ij}[\exp(-ih\omega)\Psi(-\omega)^*\phi_k(-\omega)\Psi(-\omega)]_{ij}\right.$$
$$\left.+\sum_{i,j=1}^{d}\left[\sum_{h=1}^{n-1}\hat{\Gamma}_{n,Z}(h)\right]_{ij}[\exp(ih\omega)\Psi(\omega)^*\phi_k(\omega)\Psi(\omega)]_{ij}\right)$$
$$\xrightarrow{\mathcal{L}}\sum_{i,j=1}^{d}\sum_{h=1}^{\infty}S(h)_{ij}\int_{-\pi}^{\pi}(A(\omega)+\overline{A(\omega)})_{ij}\,d\omega$$

holds. From Lemma A.1,

$$\begin{aligned}
&x_n\int_{-\pi}^{\pi}\operatorname{tr}\Big[\{\boldsymbol{I}_{n,X}(\omega)-\Psi(\omega)\hat{\Gamma}_{n,Z}(0)\Psi(\omega)^*\}\phi_k(\omega)\Big]\,d\omega\\
=\;&x_n\int_{-\pi}^{\pi}\operatorname{tr}\Big[\{\Psi(\omega)\boldsymbol{I}_{n,Z}(\omega)\Psi(\omega)^*-\Psi(\omega)\hat{\Gamma}_{n,Z}(0)\Psi(\omega)^*+\boldsymbol{R}(\omega)\}\phi_k(\omega)\Big]\,d\omega\\
=\;&x_n\int_{-\pi}^{\pi}\operatorname{tr}\Big[\{\Psi(\omega)(\boldsymbol{I}_{n,Z}(\omega)-\hat{\Gamma}_{n,Z}(0))\Psi(\omega)^*\}\phi_k(\omega)\Big]\,d\omega+x_n\int_{-\pi}^{\pi}\operatorname{tr}[\boldsymbol{R}(\omega)\phi_k(\omega)]\,d\omega\\
=\;&x_n\int_{-\pi}^{\pi}\operatorname{tr}\left[\left(\sum_{h=1}^{n-1}\hat{\Gamma}_{n,Z}(h)\exp(-ih\omega)\right)\Psi(\omega)^*\phi_k(\omega)\Psi(\omega)\right]d\omega\\
&+x_n\int_{-\pi}^{\pi}\operatorname{tr}\left[\left(\sum_{h=1}^{n-1}\hat{\Gamma}_{n,Z}(h)'\exp(ih\omega)\right)\Psi(\omega)^*\phi_k(\omega)\Psi(\omega)\right]d\omega+o_p(1)\\
=\;&x_n\left(\int_{-\pi}^{\pi}\sum_{i,j=1}^{d}\left[\sum_{h=1}^{n-1}\hat{\Gamma}_{n,Z}(h)\right]_{ij}[\exp(-ih\omega)\Psi(-\omega)^*\phi_k(-\omega)\Psi(-\omega)]_{ij}\,d\omega\right.\\
&\left.+\int_{-\pi}^{\pi}\sum_{i,j=1}^{d}\left[\sum_{h=1}^{n-1}\hat{\Gamma}_{n,Z}(h)'\right]_{ij}[\exp(ih\omega)\Psi(\omega)^*\phi_k(\omega)\Psi(\omega)]_{ij}\,d\omega\right)+o_p(1)\\
\xrightarrow{\mathcal{L}}\;&\sum_{i,j=1}^{d}\sum_{h=1}^{\infty}S(h)_{ij}\int_{-\pi}^{\pi}(A(\omega)+\overline{A(\omega)})_{ij}\,d\omega.
\end{aligned}$$

$\square$

***Remark* A.1.** The assumption on the components of $\boldsymbol{Z}(t)$ is for simplicity and for simulation. The condition of regular variation on the vector case is crucial for the convergence of $\boldsymbol{Z}(t)$ with some other technical conditions. For detail, we recommend to refer to Bartkiewicz et al. (2011).

Similar to the univariate case, we have

$$\sum_{t=1}^{n}\sum_{i=1}^{d}\tilde{Z}(t)_i^2 = 1 \quad \text{almost surely,}$$

which shows the second moment of $\tilde{\boldsymbol{Z}}(t)$ is finite. By the properties that the components of vectors are mutually independent and they are symmetry around 0, we assume generally

$$E\left[\tilde{Z}(t)_i\tilde{Z}(s)_i\right] = \Sigma_{\tilde{Z}} = \begin{cases}\frac{\sigma_{ii}}{n}, & \text{if } t = s,\\ 0 & \text{if } t \neq s.\end{cases} \tag{A.3}$$

Lemma A.4. *Assume the covariance matrix of self-normalized process $\{\tilde{Z}(t);\ t \in \mathbb{Z}\}$ is given by $\Sigma_{\tilde{Z}}$. If $\alpha \in [1, 2)$, then*

$$(n^{-2/\alpha}\|Z\|_n^2)^{-2}\boldsymbol{S}_n(\boldsymbol{\theta}_0) \xrightarrow{\mathcal{P}} \boldsymbol{W},$$

where the (a, b)-component of $\boldsymbol{W}$ satisfies

$$\begin{aligned}W_{ab} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigg(&\operatorname{tr}\left[\tilde{\boldsymbol{g}}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_a}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\tilde{\boldsymbol{g}}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_b}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\right]\\ &+\operatorname{tr}\left[\tilde{\boldsymbol{g}}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_a}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\right]\operatorname{tr}\left[\tilde{\boldsymbol{g}}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_b}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\right]\Bigg)d\omega,\end{aligned}$$

where $\tilde{\boldsymbol{g}}(\omega)$ is defined as

$$\tilde{\boldsymbol{g}}(\omega) = \Psi(\omega)\Sigma_{\tilde{Z}}\Psi(\omega)^*.$$

Proof. Write

$$\tilde{\boldsymbol{I}}_{n,X}(\omega) \equiv (n^{-2/\alpha}\|Z\|_n^2)^{-1}\boldsymbol{I}_{n,X}(\omega) \tag{A.4}$$

Recalling the decomposition in Lemma A.1 again and using self-normalized form, we can see that

$$\tilde{\boldsymbol{I}}_{n,X}(\omega) = \Psi(\omega)\boldsymbol{I}_{n,\tilde{Z}}(\omega)\Psi(\omega)^* + (n^{-2/\alpha}\|Z\|_n^2)^{-1}\boldsymbol{R}_n(\omega), \tag{A.5}$$

where the last term $(n^{-2/\alpha}\|Z\|_n^2)^{-1}\boldsymbol{R}_n(\omega)$ is asymptotically negligible. Taking the expectation of the product of periodogram of $\tilde{\boldsymbol{Z}}$, we obtain

$$\begin{aligned}&E(I_{n,\tilde{Z}}(\omega_1)_{pq}I_{n,\tilde{Z}}(\omega_2)_{rs})\\ &= E\left(\sum_{m,l,k,j}\tilde{Z}_p(m)\tilde{Z}_q(l)\tilde{Z}_r(k)\tilde{Z}_s(j)\exp\{\,i((j-k)\omega_1 - (l-m)\omega_2)t\}\right)\\ &= \begin{cases}\sigma_{pq}\sigma_{rs} + \sigma_{pr}\sigma_{qs} + o_p(1) & \text{if } \omega_1 = \omega_2,\\ \sigma_{pq}\sigma_{rs} + \sigma_{ps}\sigma_{qr} + o_p(1) & \text{if } \omega_1 = -\omega_2.\end{cases}\end{aligned}$$

Therefore, if we write $g(\omega)_{ab} = (\sum_{j=0}^{n} \Psi(j)\exp(-ij\omega))_{ab}$, then

$$\begin{aligned}\lim_{n\to\infty} E(\tilde{I}_{n,X}(\omega)_{pq}\tilde{I}_{n,X}(\omega)_{rs}) &= \sum_{k,l,m,n} g(\omega)_{pk}\overline{g(\omega)_{ql}}\ \overline{g(\omega)_{rm}}g(\omega)_{sn}(\sigma_{pq}\sigma_{rs}+\sigma_{pr}\sigma_{qs})\\ &= \tilde{\boldsymbol{g}}(\omega)_{pq}\tilde{\boldsymbol{g}}(\omega)_{rs}+\tilde{\boldsymbol{g}}(\omega)_{pr}\tilde{\boldsymbol{g}}(\omega)_{qs}.\end{aligned}$$

If $\alpha \in [1,2)$, we have

$$\begin{aligned}&E[(n^{-2/\alpha}\|Z\|_n^2)^{-2}\boldsymbol{S}_n(\boldsymbol{\theta}_0)_{ab}]\\ &\to \frac{1}{2\pi}\Big\{\int_{-\pi}^{\pi}\sum_{\beta_1,\beta_2,\beta_3,\beta_4=1}^{d}\tilde{\boldsymbol{g}}(\omega)_{\beta_1\beta_2}\tilde{\boldsymbol{g}}(\omega)_{\beta_3\beta_4}\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{\beta_2\beta_1}}{\partial\theta_a}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{\beta_4\beta_3}}{\partial\theta_b}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}d\omega\\ &\quad+\int_{-\pi}^{\pi}\sum_{\beta_1,\beta_2,\beta_3,\beta_4=1}^{d}\tilde{\boldsymbol{g}}(\omega)_{\beta_1\beta_3}\tilde{\boldsymbol{g}}(\omega)_{\beta_2\beta_4}\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{\beta_2\beta_1}}{\partial\theta_a}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{\beta_4\beta_3}}{\partial\theta_b}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}d\omega\Big\}. \qquad (A.6)\\ &= W_{ab}\end{aligned}$$

The convergence of $(n^{-2/\alpha}\|Z\|_N^2)^{-2}\boldsymbol{S}_n(\boldsymbol{\theta_0})$ in probability is guaranteed by the result that

$$\sum_{k\neq l}\mathrm{Cov}(I_{n,\tilde{Z}}(\lambda_k)_{pq}^2, I_{n,\tilde{Z}}(\lambda_l)_{rs}^2) = O(n).$$

□

Corollary A.1. *If all the elements of $\boldsymbol{Z}(t)$ are i.i.d symmetric α stable random variables, then*

$$(n^{-2/\alpha}\|Z\|_n^2)^{-2}\boldsymbol{S}_n(\boldsymbol{\theta}_0)\xrightarrow{\mathcal{P}}\boldsymbol{W},$$

where the (i,j)-component of $\boldsymbol{W}$ is

$$\begin{aligned}W_{ab} = \frac{1}{2\pi d^2}\int_{-\pi}^{\pi}\Bigg(&\mathrm{tr}\left[\boldsymbol{g}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_a}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\boldsymbol{g}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_b}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\right]\\ &+\mathrm{tr}\left[\boldsymbol{g}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_a}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\right]\mathrm{tr}\left[\boldsymbol{g}(\omega)\frac{\partial \boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}}{\partial\theta_b}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}}\right]\Bigg)d\omega. \qquad (A.7)\end{aligned}$$

Proof of Theorem 4.1. Apply Lemma A.3 to $x_n\boldsymbol{P}_n(\boldsymbol{\theta}_0)$, we can see that

$$\begin{aligned}x_n\boldsymbol{P}_n(\boldsymbol{\theta}_0) &= \frac{x_n}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\boldsymbol{\theta}}\mathrm{tr}\left[\boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}\boldsymbol{I}_{n,X}(\omega)\right]d\omega\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\\ &= \frac{x_n}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\boldsymbol{\theta}}\mathrm{tr}\left[\boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}\{\boldsymbol{I}_{n,X}(\omega)-\Psi(\omega)(\hat{\Gamma}_{n,Z}(0))\Psi(\omega)^*\}\right]d\omega\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\\ &= \frac{x_n}{2\pi}\begin{pmatrix}\int_{-\pi}^{\pi}\mathrm{tr}\left[\frac{\partial}{\partial\theta_1}\boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}\{\boldsymbol{I}_{n,X}(\omega)-\Psi(\omega)(\hat{\Gamma}_{n,Z}(0))\Psi(\omega)^*\}\right]d\omega\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\\ \int_{-\pi}^{\pi}\mathrm{tr}\left[\frac{\partial}{\partial\theta_2}\boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}\{\boldsymbol{I}_{n,X}(\omega)-\Psi(\omega)(\hat{\Gamma}_{n,Z}(0))\Psi(\omega)^*\}\right]d\omega\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\\ \vdots\\ \int_{-\pi}^{\pi}\mathrm{tr}\left[\frac{\partial}{\partial\theta_q}\boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}\{\boldsymbol{I}_{n,X}(\omega)-\Psi(\omega)(\hat{\Gamma}_{n,Z}(0))\Psi(\omega)^*\}\right]d\omega\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\end{pmatrix}\end{aligned}$$

$$\xrightarrow{\mathcal{L}} \quad \frac{1}{2\pi}\sum_{i,j=1}^{d}\sum_{h=1}^{\infty} S(h)_{ij} \begin{pmatrix} \int_{-\pi}^{\pi}(B_1(\omega)+\overline{B_1(\omega)})_{ij}d\omega \\ \int_{-\pi}^{\pi}(B_2(\omega)+\overline{B_2(\omega)})_{ij}d\omega \\ \vdots \\ \int_{-\pi}^{\pi}(B_q(\omega)+\overline{B_q(\omega)})_{ij}d\omega. \end{pmatrix}$$

where

$$B_k(\omega) = \Psi(\omega)^* \frac{\partial}{\partial\theta_k}\boldsymbol{f}(\omega;\boldsymbol{\theta})^{-1}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta_0}} \Psi(\omega) \quad k=1,\dots,q.$$

Remember that

$$n^{-2/\alpha}\|Z\|_n^2 \xrightarrow{\mathcal{L}} S_{\alpha/2},$$

$$\frac{x_n\boldsymbol{P}_n(\boldsymbol{\theta}_0)}{n^{-2/\alpha}\|Z\|_n^2} \xrightarrow{\mathcal{L}} \frac{1}{2\pi}\sum_{i,j=1}^{d}\sum_{h=1}^{\infty}\frac{S(h)_{ij}}{S_{\alpha/2}} \begin{pmatrix} \int_{-\pi}^{\pi}(B_1(\omega)+\overline{B_1(\omega)})_{ij}d\omega \\ \int_{-\pi}^{\pi}(B_2(\omega)+\overline{B_2(\omega)})_{ij}d\omega \\ \dots \\ \int_{-\pi}^{\pi}(B_q(\omega)+\overline{B_q(\omega)})_{ij}d\omega \end{pmatrix}.$$

Thus the limit of $-2(x_n^2/n)\log R(\theta_0)$ is

$$-2\frac{x_n^2}{n}\log R(\boldsymbol{\theta}_0) \xrightarrow{\mathcal{L}} \boldsymbol{V}'\boldsymbol{W}^{-1}\boldsymbol{V},$$

where $\boldsymbol{V}$ and $\boldsymbol{W}$ are defined in Theorem 4.1. □