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NONLINEAR SYSTEM OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

This paper explores some nonlinear systems of singular partial differential equations written in the form $t^A D_t U = \Lambda(t, x)U + f(t, x, \zeta U, t^A D_x U)$. Under an assumption on Λ, unique solvability theorems are provided in the space of functions that are holomorphic in x on an open set, differentiable with respect to t on a real interval $]0, r]$ and extending to a continuous function at $t = 0$. The studied systems contain Fuchsian systems.

Introduction

Consider a system of differential equations $t^k D_t U = f(t, U)$, where k is an integer ≥ 2 and f is holomorphic in a neighbourhood of $\{0\} \times \mathbb{C}$. We know that such a problem generally does not have analytic solution, see, for example, [2, 3, 9, 13]. Any formal solution belongs to a Gevrey class of order > 1, we could refer to [6, 7, 8, 12] among others. The purpose of this paper is to investigate nonlinear systems of the type

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$$t^A D_t U = \Lambda(t, x)U + f(t, x, \zeta U, t^A D_x U), \tag{0.1}$$

where A is a real diagonal matrix, $\Lambda(t, x) \in \mathcal{M}_N(\mathbb{C})$ and f is a function which is continuous with respect to t in a real interval $[0, r_0]$ and holomorphic in the remaining variables. This regularity assumption also appears in [1, 4, 14, 15]. The linear parts of our equations are irregular at $t = 0$ in the sense of [5]. However, we are interested in solutions extending continuously at $t = 0$. Under a reasonable assumption on Λ, we show that (0.1) has a unique solution $(t, x) \mapsto U(t, x)$ holomorphic in x on an open set, differentiable with respect to t on a real interval $]0, r]$ and continuous on $[0, r]$. To achieve our statement, we first invert the operator $t^A D_t - \Lambda(0, 0)$, which then leads us to a fixed point problem. We prepare some estimations that allow to apply the contraction mapping principle. Our main results are Theorem 1.1 and Theorem 1.4.

Partially holomorphic system

We will make use of the following notations:

$$t \in \mathbb{R},\ x = (x_1, \ldots, x_n) \in \mathbf{C}^n, \quad D_t = \frac{\partial}{\partial t}, \quad D_j = \frac{\partial}{\partial x_j},$$

$$\mathbb{N} = \{0, 1, 2, \ldots\},\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n,\ |\alpha| = \sum_{j=1}^{n} \alpha_j,\ D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

1. Statement of Results

Given an interval $I \subset \mathbb{R}$, an open set $\Omega \subset \mathbb{C}^n$, a Banach space E and an integer $0 \leq k \leq \infty$, we denote by $\mathcal{C}^{k,\omega}(I \times \Omega; E)$ the algebra of functions $u : I \times \Omega \to E$ such that for $0 \leq l \leq k$, the partial derivative $D_t^l u : I \times \Omega \to E$ exists, is continuous and for any $t \in I$, the mapping $x \in \Omega \mapsto D_t^l u(t, x) \in E$ is holomorphic. It is easily checked, using Cauchy's integral formula, that this space is stable by differentiation with respect to x and that we have

$$D_t^l D^\alpha u = D^\alpha D_t^l u \text{ for any } \alpha \in \mathbb{N}^n,\ 0 \le l \le k.$$

When $E = \mathbb{C}$, the previous space will be simply denoted by $\mathcal{C}^{k,\omega}(I \times \Omega)$.

Let us consider a system of partial differential equations of the form

$$t^A D_t U(t, x) = \Lambda(t, x) U(t, x) + f(t, x, (\zeta U)(t, x), (t^A D_x U)(t, x)), \quad (1.1)$$

in which $U = (u_1, \dots, u_N)$ is the unknown, $A = \mathrm{diag}(a_1, \dots, a_N)$ is a diagonal matrix with real coefficients $(a_i)_{i \in [1, N]}$ that are all ≥ 1, Λ is an upper triangular matrix of order N whose coefficients are functions of $(t, x) \in \mathbb{R} \times \mathbb{C}^n$, $\zeta U \equiv (\zeta_1 u_1, \dots, \zeta_N u_N)$ where each ζ_i is a function of $(t, x) \in \mathbb{R} \times \mathbb{C}^n$ satisfying $\zeta_i(0, 0) = 0$, $t^A D_x U$ denotes the nN-tuple $((t^{a_i} D_j u_i)_{i \in [1, N]})_{j \in [1, n]}$, f is a function of the variables $t \in \mathbb{R}$, $x \in \mathbb{C}^n$, $y = (y_i)_{i \in [1, N]} \in \mathbb{C}^N$, $z = ((z_{ij})_{i \in [1, N]})_{j \in [1, n]} \in \mathbb{C}^{nN}$.

We assume there are $r_0 > 0$ and an open neighbourhood Ω_0 (resp., $\mathcal{O}_0$) of the origin in $\mathbb{C}_x^n$ (resp., $\mathbb{C}_y^N \times \mathbb{C}_z^{nN}$) such that the coefficients of Λ, like the functions ζ_i, belong to $\mathcal{C}^{0,\omega}([0, r_0] \times \Omega_0)$ and

$$f \in \mathcal{C}^{0,\omega}([0, r_0] \times (\Omega_0 \times \mathcal{O}_0); \mathbb{C}^N).$$

Let $\mathcal{Z}$ be the zero set of the polynom $P(\lambda) \equiv \det(\lambda I - \Lambda(0, 0))$.

Theorem 1.1. *Suppose $\mathcal{Z}$ is included in the half-plane $\Re e \lambda < 0$. Then, there exist $r \in\ ]0, r_0]$ and an open neighbourhood $\Omega \subset \Omega_0$ of the origin in $\mathbb{C}_x^n$ such that system* (1.1) *has a unique solution $U \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ and necessarily $t^A D_t U \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$.*

Suppose all $a_i \ge 0$, then Theorem 1.1 can be extended as follows.

Let $\mathcal{I}$ denote the set of $i \in [\![1, N]\!]$ such that $a_i \geq 1$ and assume $[\![1, N]\!] \setminus \mathcal{I}$ is not empty. Suppose

$$\zeta_i(0, 0) = 0 \text{ for } i \in \mathcal{I} \quad \text{and} \quad \zeta_i \equiv 1 \text{ for } i \notin \mathcal{I}. \tag{1.2}$$

For each $i \in [\![1, N]\!] \setminus \mathcal{I}$, let w_i be a holomorphic function on Ω_0. We consider system (1.1) under (1.2) with the initial conditions

$$u_i(0, x) = w_i(x) \quad \text{for } i \in [\![1, N]\!] \setminus \mathcal{I}. \tag{1.3}$$

Impose

$$D_x w_i(0) = 0 \text{ only if } a_i = 0. \tag{1.4}$$

The set $\mathcal{I}$ is uniquely written as

$$\mathcal{I} = \{i_1, \dots, i_p\} \quad \text{with} \quad 1 \leq i_1 < i_2 < \dots < i_p \leq N.$$

Then, we associate with the matrix $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq N}$, the square sub-matrix of order p

$$\widetilde{\Lambda} = \mathcal{M}_{\mathcal{I}, \mathcal{I}}(\Lambda) = (\lambda_{i_k i_l})_{1<k, l<p}.$$

Let us name $\widetilde{\mathcal{Z}}$ the zero set of the polynom $\widetilde{P}(\lambda) \equiv \det(\lambda I - \widetilde{\Lambda}(0, 0))$. We then have the following result:

Theorem 1.2. *Suppose $\widetilde{\mathcal{Z}}$ is included in the half-plane $\Re e\, \lambda < 0$. Then, there exist $r \in]0, r_0]$ and an open neighbourhood $\Omega \subset \Omega_0$ of the origin in $\mathbb{C}_x^n$ such that the problem* (1.1)-(1.2)-(1.3) *has a unique solution $U \in \mathcal{C}^{1, \omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0, \omega}([0, r] \times \Omega; \mathbb{C}^N)$ and so $t^A D_t U \in \mathcal{C}^{0, \omega}([0, r] \times \Omega; \mathbb{C}^N)$.*

Here is an example about this theorem.

Example 1.3. For all $(a, b) \in \mathbb{C}^2$ and $(\alpha, \beta) \in \mathbb{R}^2$, there is $r > 0$ and an open neighbourhood Ω of the origin in $\mathbb{C}_x$ such that the problem

$$\begin{cases} t^2 D_t u_1 = -u_1 + au_2 + [1 + xu_1 + \sqrt{t}(\partial u_2 / \partial x)]^{\alpha}, \\ \sqrt{t} D_t u_2 = bu_2 + [1 + u_2 + t^2(\partial u_1 / \partial x)]^{\beta}, \\ u_2(0, x) = 0, \end{cases}$$

has a unique solution $(u_1, u_2) \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^2) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^2)$ as a result $(t^2 D_t u_1, \sqrt{t} D_t u_2) \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^2)$.

Now we turn our attention to a system of the form:

$$t^a D_t U(t, x) = \Lambda(t, x) U(t, x) + f(t, x, (\zeta U)(t, x), (t^a D_x U)(t, x)), \quad (1.5)$$

where a is a positive real number and Λ is a square matrix of order N whose coefficients belong to the space $\mathcal{C}^{0,\omega}([0, r_0] \times \Omega_0)$. We are then able to state the following result.

Theorem 1.4. (1) *If $0 \leq a < 1$, take $\zeta \equiv (1, \ldots, 1)$ and let $W : \Omega_0 \to \mathbb{C}^N$ be an holomorphic function (with $D_x W(0) = 0$ only when $a = 0$). Then, there exist $r \in]0, r_0]$ and an open neighbourhood $\Omega \subset \Omega_0$ of the origin in $\mathbb{C}_x^n$ such that system* (1.5) *with the initial data $U(0, x) = W(x)$, has a unique solution $U \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ and $t^a D_t U \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$.*

(2) *If $a \geq 1$ and if the zero set of the polynom $\lambda \mapsto \det(\lambda I - \Lambda(0, 0))$ is included in the half-plane $\Re e\lambda < 0$, then, there exist $r \in]0, r_0]$ and an open neighbourhood $\Omega \subset \Omega_0$ of the origin in $\mathbb{C}_x^n$ such that system* (1.5) *has a unique solution $U \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ and $t^a D_t U \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$.*

Remark 1.5. When $a = 1$, we obtain a more general system of equations than a Fuchsian system; indeed, we do not need to take $\zeta_i(t, x) = t$ but we simply have $\zeta_i(0, 0) = 0$. Recall ([10, 11]) that in the Fuchsian case we have already studied nonlinear equations in spaces of holomorphic functions.

2. Reformulation

In order to prove Theorem 1.1, we first transform the problem.

By writing $\Lambda = \Lambda(0) + \mathcal{E}$, where $\mathcal{E} \equiv \Lambda - \Lambda(0)$ satisfies all the same assumptions as ζ, we may suppose that Λ is an upper triangular constant matrix, namely,

$$\Lambda = (\lambda_{ij})_{1 \leq i, j \leq N} \in \mathcal{M}_N(\mathbb{C}), \quad \text{where} \quad \lambda_{ij} = 0 \text{ if } j < i. \tag{2.1}$$

Furthermore,

$$\Re e\, \lambda_{ii} < 0 \quad \text{for} \quad i = 1, \ldots, N. \tag{2.2}$$

Next, we will need the following result:

Let $a \geq 1, \lambda \in \mathbb{C}$ and let $\mathcal{P}$ be the elementary operator $\mathcal{P} \equiv t^a D_t - \lambda$.

Lemma 2.1. *Suppose* $\Re e\, \lambda < 0$. *Let* $r > 0$ *and let* Ω *be an open neighbourhood of the origin in* $\mathbb{C}_x^n$. *Then, for every* $v \in \mathcal{C}^{0, \omega}([0, r] \times \Omega)$, *the equation* $\mathcal{P}u = v$ *has a unique solution* $u \in \mathcal{C}^{1, \omega}(]0, r] \times \Omega) \cap \mathcal{C}^{0, \omega}([0, r] \times \Omega)$. *In addition,* $t^a D_t u \in \mathcal{C}^{0, \omega}([0, r] \times \Omega)$ *with* $t^a D_t u(0, x) = 0$. *This solution is defined by*

$$u(t, x) = (\mathcal{P}^{-1} v)(t, x) \equiv e^{\lambda} \varphi(t) \int_0^t v(\tau, x) \frac{e^{-\lambda \varphi(\tau)}}{\tau^a} d\tau, \textit{ where } \varphi(t) = \begin{cases} \dfrac{t^{1-a}}{1-a} & \textit{if } a > 1, \\ \ln t & \textit{if } a = 1. \end{cases} \tag{2.3}$$

Moreover, $D^\alpha u = \mathcal{P}^{1} D^\alpha v$ *for any* $\alpha \in \mathbb{N}^n$.

Proof. Let us start with the uniqueness. Suppose $u \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega)$ satisfies $\mathcal{P}u = 0$. Let $x \in \Omega$. Since $D\varphi(t) = t^{-a}$, the derivative of the mapping $t \mapsto u(t, x)e^{-\lambda\varphi(t)}$ is equal to zero on $]0, r]$, so there exists $c_x \in \mathbb{C}$ such that

$$u(t, x) \equiv c_x e^{\lambda\varphi(t)} \quad \text{for} \quad t \in]0, r].$$

This function has a finite limit at $t = 0$ only if $c_x = 0$ because $\lim_0 \varphi = -\infty$. It follows that $u \equiv 0$ on $[0, r] \times \Omega$. Concerning the existence, we shall prove that the formal solution (2.3) is of class $\mathcal{C}^{1,\omega}(]0, r] \times \Omega) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega)$. Let Δ be an open polydisk such that $\overline{\Delta} \subset \Omega$; denote

$$M = \max_{[0,r]\times\overline{\Delta}} |v|.$$

The formula (2.3) can be written equivalently

$$u(t, x) = te^{\lambda\varphi(t)} \int_0^1 v(\sigma t, x) \frac{e^{-\lambda\varphi(\sigma t)}}{(\sigma t)^a} d\sigma. \tag{2.4}$$

We first consider the case $a > 1$. Then,

$$\frac{|e^{-\lambda\varphi(t)}|}{t^a} = \frac{e^{-(\Re e\,\lambda)\varphi(t)}}{t^a} \underset{t\to 0}{\to} 0,$$

and we may set

$$c_a \equiv \max_{\tau\in[0,r]} \frac{|e^{-\lambda\varphi(\tau)}|}{\tau^a}.$$

The mapping $(\sigma, t, x) \in]0, 1] \times]0, r] \times \Delta \mapsto v(\sigma t, x) \dfrac{e^{-\lambda\varphi(\sigma t)}}{(\sigma t)^a}$ is continuous, holomorphic with respect to x and bounded by Mc_a. Thus, from (2.4), we have $u \in \mathcal{C}^{0,\omega}(]0, r] \times \Omega)$. Now we show that u has a unique continuous extension to a function of the space $\mathcal{C}^{0,\omega}([0, r] \times \Omega)$. Let $b \in \Omega$. Considering

$$e^{\lambda\varphi(t)} \int_0^t \frac{e^{-\lambda\varphi(\tau)}}{\tau^a} \, d\tau = -1 / \lambda,$$

we notice that

$$u(t, x) + v(0, b) / \lambda = e^{\lambda\varphi(t)} \int_0^t [v(\tau, x) - v(0, b)] \frac{e^{-\lambda\varphi(\tau)}}{\tau^a} \, d\tau.$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that: let $(t, x) \in [0, r] \times \Omega$ with $|t| \leq \delta$ and $|x_j - b_j| \leq \delta$ for $1 \leq j \leq n$, then we have

$$|v(t, x) - v(0, b)| \leq -(\Re e \, \lambda)\varepsilon,$$

therefore,

$$|u(t, x) + v(0, b) / \lambda| \leq e^{(\Re e \, \lambda)\varphi(t)} \int_0^t -(\Re e \, \lambda)\varepsilon \, \frac{e^{-(\Re e \, \lambda)\varphi(\tau)}}{\tau^a} \, d\tau = \varepsilon,$$

and the statement follows with

$$u(0, x) = -v(0, x) / \lambda, \quad \forall x \in \Omega. \tag{2.5}$$

We next consider the case $a = 1$. Then (2.4) is reduced to the well-known formula

$$u(t, x) = \int_0^1 \frac{v(\sigma t, x)}{\sigma^{\lambda+1}} \, d\sigma, \tag{2.6}$$

where the mapping $(\sigma, t, x) \in \,]0, 1] \times [0, r] \times \Delta \mapsto \dfrac{v(\sigma t, x)}{\sigma^{\lambda+1}} \in \mathbb{C}$ is continuous, holomorphic with respect to x and bounded by the integrable function $\sigma \mapsto M / \sigma^{\lambda+1}$. It directly results that $u \in \mathcal{C}^{0,\omega}([0, r] \times \Omega)$ with (2.5) anew since

$$u(0, x) = \int_0^1 \frac{v(0, x)}{\sigma^{\lambda+1}} \, d\sigma = -v(0, x) / \lambda, \quad \forall x \in \Omega.$$

In all cases, we naturally have $D^{\alpha}u = \mathcal{P}^{-1}D^{\alpha}v$ for any $\alpha \in \mathbb{N}^n$. If $t \in]0, r]$, then the partial derivative of (2.3) exists and is given by

$$D_t u(t, x) = t^{-a}[\lambda u(t, x) + v(t, x)],$$

which leads to $u \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega)$, $t^a D_t u \in \mathcal{C}^{0,\omega}([0, r] \times \Omega)$ with $t^a D_t u(0, x) = 0$ seen (2.5); further, $\mathcal{P}u = v$ on $[0, r] \times \Omega$. This completes the proof of our lemma. □

Now we can consider the system of differential equations:

$$t^A D_t U(t, x) = \Lambda U(t, x) + V(t, x), \tag{2.7}$$

where Λ is a matrix satisfying (2.1)-(2.2) and $V = (v_1, \dots, v_N)$ is assumed to be of class $\mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$. For every $i \in [\![1, N]\!]$, denote

$$\mathcal{P}_i \equiv t^{a_i} D_t - \lambda_{ii}.$$

Then (2.7) is written as

$$\mathcal{P}_i u_i = \sum_{j=i+1}^{N} \lambda_{ij} u_j + v_i, \quad 1 \le i \le N. \tag{2.8}$$

Given $i \in [\![1, N]\!]$, for every $p \in [\![1, N - i + 1]\!]$, we set

$$G_i^p = \{\gamma = (\gamma_1, \dots, \gamma_p) \in [\![i, N]\!]^p;\ i = \gamma_1 < \gamma_2 < \cdots < \gamma_p\}.$$

The cardinal of G_i^p is equal to the number of $(p-1)$-combinations from $[\![i+1, N]\!]$, that is to say $\binom{N-i}{p-1}$. From Lemma 2.1, one has $u_N = \mathcal{P}_N^{-1} v_N$ and, by finite induction,

$$u_i = \sum_{p=1}^{N-i+1} \sum_{\gamma \in G_i^p} c_\gamma Q_\gamma V, \text{ where } c_\gamma = \prod_{l=1}^{p-1} \lambda_{\gamma_l \gamma_{l+1}} \text{ and } Q_\gamma V = \mathcal{P}_{\gamma_1}^{-1} \circ \cdots \circ \mathcal{P}_{\gamma_p}^{-1} v_{\gamma_p}. \tag{2.9}$$

These considerations and Lemma 2.1 prove the following result.

Lemma 2.2. *Problem* (2.7) *has a unique solution* $U = (u_1, \dots, u_N) \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ *and* $t^{a_i} D_t u_i \in \mathcal{C}^{0,\omega}([0, r] \times \Omega)$ *with* $t^{a_i} D_t u_i(0, x) = 0$ *for any* $i \in [\![1, N]\!]$. *Furthermore, each* u_i *is a finite linear combination of up to* 2^{N-i} *terms of the form*

$$\mathcal{P}_i^{-1} \circ \mathcal{P}_{\gamma_2}^{-1} \circ \cdots \circ \mathcal{P}_{\gamma_p}^{-1} v_{\gamma_p}, \textit{ where } \gamma_2, \dots, \gamma_p \in [\![i, N]\!] \textit{ and } p \in [\![1, N - i + 1]\!]. \tag{2.10}$$

Denoting by $\mathcal{R}V$ the solution of problem (2.7), we define an endomorphism $\mathcal{R}$ of the vector space $\mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$. If we set $\mathscr{U}(t, x) = t^A D_t U(t, x) - \Lambda U(t, x)$, i.e., $U = \mathcal{R}\mathscr{U}$, problem (1.1) is converted into $\mathscr{U} = \mathcal{F}\mathscr{U}$, where $\mathcal{F}$ denotes the operator

$$(\mathcal{F}\mathscr{U})(t, x) = f(t, x, (\zeta\mathcal{R}\mathscr{U})(t, x), (t^A D_x \mathcal{R}\mathscr{U})(t, x)). \tag{2.11}$$

The next section aims to apply the contraction mapping principle to $\mathcal{F}$ in a Banach space that we are going to introduce.

3. Framework and Estimates for the Operator $\mathcal{F}$

Given a majorant function $\phi \in \mathbb{R}_+\{\xi\}$ with a radius of convergence $\geq R > 0$, let ρ be a parameter ≥ 1 and let $r > 0$ be such that $\rho r < R$.

Definition 3.1. We define

$$\Omega \equiv \Omega_{R,\rho,r} = \{x \in \mathbb{C}^n;\ \sum_{j=1}^{n} |x_j| < R - \rho r\},$$

and $\mathscr{B} \equiv \mathscr{B}_{\phi, R, \rho, r}$ by the set of functions $u \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; E)$ for which there exists $c \geq 0$ such that

$$\forall t \in [0, r], \quad u(t, x) \ll c\phi(\rho t + \xi), \quad \text{where} \quad \xi = \sum_{j=1}^{n} x_j. \tag{3.1}$$

This precisely means $\|D^{\alpha} u(t, 0)\|_E \leq cD^{|\alpha|}\phi(\rho t, 0)$ for all $\alpha \in \mathbb{N}^n$ and all $t \in [0, r]$.

Obviously, $\mathscr{B}$ is a vector subspace of $\mathcal{C}^{0,\omega}([0, r] \times \Omega; E)$ and the smallest $c \geq 0$ for which (3.1) is satisfied is a norm on $\mathscr{B}$ denoted by $\|\cdot\|_{\phi, R, \rho, r}$ or simply $\|\cdot\|$ if no confusion is possible.

Lemma 3.2. *The space $\mathscr{B}$ is a Banach space.*

Proof. Let (U_n) be a Cauchy sequence in $\mathcal{B}$ and let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, for all $n, n' \geq N$ and all $t \in [0, r]$,

$$(U_n - U_{n'})(t, x) \ll \varepsilon\phi(\rho t + \xi). \tag{3.2}$$

If K is a compact subset of Ω, then we have

$$\max_{[0,r]\times K} \|U_n - U_{n'}\|_E \leq \varepsilon C_K,$$

where $C_K = \max_{x \in K} \phi(\rho r + \sum_{j=1}^{n} |x_j|)$ is $< +\infty$ since the mapping $x \mapsto \phi(\rho r + \sum_{j=1}^{n} |x_j|)$ is continuous on Ω. This shows that (U_n) is a Cauchy sequence in $\mathcal{C}^{0,\omega}([0, r] \times \Omega; E)$ so it converges compactly to a function $U \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; E)$; a fortiori, for all $t \in [0, r]$ and $\alpha \in \mathbb{N}^n$, the sequence $(D^{\alpha} U_n(t, 0))_n$ converges to $D^{\alpha} U(t, 0)$. By letting n' tend to infinity into (3.2), we get $U_n - U \in \mathcal{B}$ and $\|U_n - U\| \leq \varepsilon$, therefore $U \in \mathcal{B}$ and (U_n) converges to U in $\mathcal{B}$. □

The following lemma will be useful to study the forthcoming operators.

Let $a \geq 1$ and $\lambda < 0$. If $t > 0$, we set

$$S_k(t) = e^{\lambda\varphi(t)} \int_0^t \tau^k \frac{e^{-\lambda\varphi(\tau)}}{\tau^a} d\tau, \quad \forall k \in \mathbb{N}. \tag{3.3}$$

Recall that $S_0 \equiv -1/\lambda > 0$. When $k \in N^*$, given Lemma 2.1, S_k extends continuously to 0 with $S_k(0) = 0$.

Lemma 3.3. *There exists* $c_0 = c_0(\lambda) > 0$ *such that, for all* $t \geq 0$

$$S_k(t) \leq c_0 t^k, \quad \forall k \in \mathbb{N}. \tag{3.4}$$

There exists $c_1 = c_1(a, \lambda, r_0) > 0$ *such that, for all* $t \in [0, r_0]$

$$t^a S_k(t) \leq c_1 \frac{t^{k+1}}{k+1}, \quad \forall k \in \mathbb{N}. \tag{3.5}$$

Proof. Since $\tau^k \leq t^k$, we have (3.4) with $c_0 = S_0$. To show (3.5), we notice that

$$\left(1 - \frac{a}{k+1}\right) \int_0^t \tau^k \frac{e^{-\lambda\varphi(\tau)}}{\tau^a} d\tau = \frac{t^{k+1}}{k+1} \frac{e^{-\lambda\varphi(t)}}{t^a} + \frac{\lambda}{k+1} \int_0^t \tau^{k+1} \frac{e^{-\lambda\varphi(\tau)}}{\tau^{2a}} d\tau, \tag{3.6}$$

as long as $k + 1 \geq a$. We then consider integers $k > a - 1$, i.e., $k \geq \lfloor a \rfloor$ which is the smallest integer larger than or equal to a; since λ is < 0, it ensues that

$$t^a S_k(t) \leq c \frac{t^{k+1}}{k+1}, \quad \text{where} \quad c \equiv \frac{\lfloor a \rfloor + 1}{\lfloor a \rfloor + 1 - a} > 0.$$

Let $k \leq a - 1$ be an integer. From (3.4), one has, for all $t \in [0, r_0]$

$$t^a S_k(t) \leq c_0 t^{a+k} \leq c_0 t^{k+1} r_0^{a-1} \leq c' \frac{t^{k+1}}{k+1}, \quad \text{where} \quad c' = c_0 a r_0^{a-1} > 0.$$

The result follows with $c_1 = \max(c, c')$. $\square$

Remark 3.4. When $a = 1$, then $c_1 = \max(2, 1/|\lambda|)$ does not depend on r_0 and (3.5) is valid for all $t \geq 0$. Otherwise, one can see that $\lim_{t \to +\infty} S_k(t) = +\infty$.

Here are the estimates involving operator $\mathcal{P}^{-1}$ of Lemma 2.1.

Lemma 3.5. *There exists* $c = c(a, \lambda, r_0) > 0$ *such that, for* $R > 0$, $\rho r < R$ *and* $u \in \mathcal{C}^{0,\omega}([0, r] \times \Omega)$, *the function* $\mathcal{P}^{-1}u \in \mathcal{C}^{0,\omega}([0, r] \times \Omega)$ *satisfies*

$$\forall t \in [0, r], \quad u(t, x) \ll \phi(\rho t + \xi) \ \Rightarrow \ \forall t \in [0, r], \quad \mathcal{P}^{-1}u(t, x) \ll c\phi(\rho t + \xi), \tag{3.7}$$

and

$$\forall t \in [0, r], \quad u(t, x) \ll D\phi(\rho t + \xi) \ \Rightarrow \ \forall t \in [0, r], \quad t^a \mathcal{P}^{-1}u(t, x) \ll c\rho^{-1}\phi(\rho t + \xi). \tag{3.8}$$

Proof of (3.7). For all $t \in [0, r]$ and $\alpha \in \mathbb{N}^n$, one has

$$|D^\alpha u(t, 0)| \leq D^{|\alpha|}\phi(\rho t) = \sum_{k=0}^{\infty} (\rho t)^k \frac{D^{k+|\alpha|}\phi(0)}{k!},$$

and from Lemma 2.1, $D^\alpha \mathcal{P}^{-1} u = \mathcal{P}^{-1} D^\alpha u$, hence

$$|D^\alpha \mathcal{P}^{-1}u(t, 0)| \leq \sum_{k=0}^{\infty} \rho^k S_k(t) \frac{D^{k+|\alpha|}\phi(0)}{k!},$$

where S_k is defined by (3.3) in which we substitute $\Re e\lambda$ to λ. The assertion is confirmed by (3.4).

Proof of (3.8). As above, we get in this case

$$|D^\alpha t^a \mathcal{P}^{-1}u(t, 0)| \leq \sum_{k=0}^{\infty} \rho^k t^a S_k(t) \frac{D^{k+1+|\alpha|}\phi(0)}{k!} \leq c_1\rho^{-1} \sum_{k=0}^{\infty} (\rho t)^{k+1} \frac{D^{k+1+|\alpha|}\phi(0)}{(k+1)!}$$

from (3.5), and the conclusion follows. □

We then consider the expressions $\zeta\mathcal{R}U$ and $t^A D_x\mathcal{R}U$. Let us denote by $\mathcal{R}_i$ the i-th component of $\mathcal{R}$. From now on, we will take $E = \mathbb{C}^N$ and $\|(u_1, \dots, u_n)\|_E = \max_{1\le i\le N}|u_i|$.

Lemma 3.6. *There exists* $c = c(A, \Lambda(0, 0), r_0) > 0$ *such that, for* $R > 0$, $\rho r < R$ *and* $U \in \mathscr{B}$, *we have, for every* $i \in [\![1, N]\!]$, $j \in [\![1, n]\!]$

$$\forall t \in [0, r], \quad \mathcal{R}_i U(t, x) \ll c\|U\|\phi(\rho t + \xi), \tag{3.9}$$

and

$$\forall t \in [0, r], \quad t^{a_i} D_j \mathcal{R}_i U(t, x) \ll c\rho^{-1}\|U\|\phi(\rho t + \xi). \tag{3.10}$$

Proof. From Lemma 2.2, this result must be established with terms like

$$\mathcal{P}_i^{-1} \circ \mathcal{P}_{\gamma_2}^{-1} \circ \dots \circ \mathcal{P}_{\gamma_p}^{-1} u_{\gamma_p} \quad \text{for (3.9)},$$

and terms like

$$t^{a_i}\mathcal{P}_i^{-1} \circ \mathcal{P}_{\gamma_2}^{-1} \circ \dots \circ \mathcal{P}_{\gamma_p}^{-1} D_j u_{\gamma_p} \quad \text{for (3.10)},$$

where $p \in [\![1, N]\!]$ and $\gamma_2, \dots, \gamma_p \in [\![1, N]\!]$. Let $U \in \mathscr{B}$, then $u_{\gamma_p}(t, x) \ll \|U\|\phi(\rho t + \xi)$ for all $t \in [0, r]$. Using (3.7) p-times, we obtain (3.9).

Otherwise, one has $D_j u_{\gamma_p}(t, x) \ll \|U\| D\phi(\rho t + \xi)$ for all $t \in [0, r]$. Applying (3.7) $(p-1)$-times and (3.8) once, we get (3.10). □

Let us specify hereafter the majorant function we shall employ.

Given $R > 0$, we consider the entire serie (2.1) of [15]

$$\phi(\xi) = K^{-1}\sum_{p=0}^{\infty}\frac{(\xi / R)^p}{(p+1)^2}, \tag{3.11}$$

where the constant $K > 0$ is such that $\phi^2 \ll \phi$. Recall that ϕ also satisfies the following properties. Let $\eta > 0$, there exists $c = c(\eta) > 0$, such that $\eta R / (\eta R - \bullet) \ll c\phi$ and necessarily

$$\frac{\eta R}{\eta R - (\rho t + \xi)} \ll c\phi(\rho t + \xi) \quad \text{for all } t \in [0, r]. \tag{3.12}$$

Lemma 3.7. *Let* $u \in \mathcal{C}^{0,\omega}([0, r] \times \Omega)$ *and* $0 \leq c < R'$ *be such that* $u(t, x) \ll c\phi(\rho t + \xi)$ *for all* $t \in [0, r]$, *then* u *is bounded by* c *on* $[0, r] \times \Omega$, *the function* $R' / (R' - u)$ *belongs to the space* $\mathcal{C}^{0,\omega}([0, r] \times \Omega)$ *and*

$$\frac{R'}{R' - u(t, x)} \ll \left(K + \frac{c}{R' - c}\right)\phi(\rho t + \xi) \quad \textit{for all } t \in [0, r]. \tag{3.13}$$

Concerning the operator $\mathcal{F}$, we are going to set up

Proposition 3.8. *There is* $a_0 > 0$ *such that, for all* $a \geq a_0$, *the following holds: there exist* $R \in \,]0, R_0]$, $\rho \geq 1$, $r \in \,]0, r_0]$ *with* $\rho r < R$ *such that the mapping* $\mathcal{F}$ *is a strict contraction in the closed ball* $B'(0, a)$ *of the Banach space* $\mathscr{B}$.

Let us observe now that we can write

$$\begin{aligned} f(t, x, y, z) - f(t, x, y', z') = &\sum_{i \in [\![1, N]\!]} g_i(t, x, y, z, y', z')(y_i - y'_i) \\ &+ \sum_{(i,j) \in [\![1, N]\!] \times [\![1, n]\!]} h_{i,j}(t, x, y, z, y', z')(z_{i,j} - z'_{i,j}), \end{aligned} \tag{3.14}$$

where

$$g_i,\, h_{i,j} \in \mathcal{C}^{0,\omega}([0, r_0] \times \Omega_0 \times \mathcal{O}_0 \times \mathcal{O}_0;\, \mathbb{C}^N).$$

For $R > 0$, we set

$$\Delta_R = \{x \in \mathbb{C}^n;\ \max_{j \in [\![1, n]\!]} |x_j| < R\},$$

and

$$O_R = \{(y, z) \in \mathbb{C}^N \times \mathbb{C}^{nN};\ \max_{i \in [\![1, N]\!]} |y_i| < R,\ \max_{(i,j) \in [\![1, N]\!] \times [\![1, n]\!]} |z_{i,j}| < R\}.$$

We fix, once and for all, $\eta > 1$, $R_0 > 0$ and $R' > 0$ such that $\overline{\Delta}_{\eta R_0} \subset \Omega_0$ and $\overline{O}_{R'} \subset \mathcal{O}_0$. Consequently, the functions $f \in \mathcal{C}^{0,\omega}([0, r_0] \times (\Delta_{\eta R_0} \times O_{R'}); \mathbb{C}^N)$ and $g_i, h_{i,j} \in \mathcal{C}^{0,\omega}([0, r_0] \times (\Delta_{\eta R_0} \times O_{R'} \times O_{R'}); \mathbb{C}^N)$ are bounded, say by a constant $M > 0$.

We put

$$\varepsilon(r, R) = \max_{\substack{(t,x) \in [0, r] \times \overline{\Delta}_{\eta R} \\ i \in [\![1, N]\!]}} |\zeta_i(t, x)| \quad \text{for} \quad (r, R) \in \,]0, r_0] \times \,]0, R_0].$$

This function has limit 0 as (r, R) tends to $(0, 0)$. From Cauchy's inequalities and Lemma 3.7, one has

$$\zeta_i(t, x) \ll \varepsilon(r, R) \frac{\eta R}{\eta R - \xi} \ll c(\eta) \varepsilon(r, R) \phi(\xi),$$

and, given $\phi(\xi) \ll \phi(\rho t + \xi)$ (since $\phi \gg 0$ and $\rho r < R$), it comes

$$\zeta_i(t, x) \ll c(\eta) \varepsilon(r, R) \phi(\rho t + \xi) \quad (\text{for all } t \in [0, r],\ i \in [\![1, N]\!]). \tag{3.15}$$

Similarly, we have

$$f(t, x, y, z) \ll c(\eta) M \phi(\rho t + \xi) \prod_i \frac{R'}{R' - y_i} \prod_{i,j} \frac{R'}{R' - z_{ij}}, \tag{3.16}$$

and

$$g_i,\ h_{i,j}(t, x, y, z, y', z') \ll c(\eta) M \phi(\rho t + \xi) \prod_i \frac{R'}{R' - y_i} \frac{R'}{R' - y'_i} \times \prod_{i,j} \frac{R'}{R' - z_{ij}} \frac{R'}{R' - z'_{ij}}. \tag{3.17}$$

Proof of Proposition 3.8. Let $a > 0$ and $U \in \mathscr{B}$ be such that $\|U\| \leq a$. In what follows, any constant ≥ 0 that does not depend on the parameters a, R, ρ, r will be denoted by c. From (3.9)-(3.15) and (3.10), we have

$$\begin{cases} \zeta_i \mathcal{R}_i U(t, x) & \ll c\varepsilon(r, R) a\phi(\rho t + \xi), \\ t^{a_i} D_j \mathcal{R}_i U(t, x) & \ll c\rho^{-1} a\phi(\rho t + \xi). \end{cases} \tag{3.18}$$

Then, under a condition like

$$c\varepsilon(r, R)a \leq R'/2 \quad \text{and} \quad c\rho^{-1}a \leq R'/2, \tag{3.19}$$

it follows from (3.16) and Lemma 3.7 that $\mathcal{F}U$ is well-defined on $[0, r] \times \Omega$, belongs to $\mathcal{C}^{0, \omega}([0, r_0] \times \Omega; \mathbb{C}^N)$ and

$$\forall t \in [0, r], \quad \mathcal{F}U(t, x) \ll c\phi(\rho t + \xi).$$

This proves the existence of a $a_0 > 0$ sufficiently large $(a_0 > c)$, such that

$$\mathcal{F}(B'(0, a)) \subset B'(0, a) \quad \text{for all } a \geq a_0. \tag{3.20}$$

Let $U' \in B'(0, a)$. As explained for f, if (3.19) is satisfied, we also have

$$g_i, h_{i,j}(t, x, \zeta\mathcal{R}U, t^A D_x \mathcal{R}U, \zeta\mathcal{R}U', t^A D_x \mathcal{R}U') \ll c\phi(\rho t + \xi).$$

Thence, from (3.14) and Lemma 3.6, we obtain

$$\|\mathcal{F}U - \mathcal{F}U'\| \leq c(\varepsilon(r, R) + \rho^{-1})\|U - U'\|.$$

Let $a \geq a_0$. We first take $\rho \geq 1$ such that $c\rho^{-1}a \leq R'/2$ and $c\rho^{-1} < 1/2$. Next, we choose $(r, R) \in \,]0, r_0] \times \,]0, R_0]$ with $\rho r < R$ (for instance $r = R/2\rho$), such that $c\varepsilon(r, R)a \leq R'/2$ and $c\varepsilon(r, R) < 1/2$. Thus we have (3.19), (3.20) and $c(\varepsilon(r, R) + \rho^{-1}) < 1$. We get the desired result therefrom. $\square$

4. Proof of Theorem 1.1

By Proposition 3.8, the mapping $\mathcal{F}$ has an unique fixed point $U \in B'(0, a_0) \subset \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ and $U_1 = \mathcal{R}U \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ is a solution of (1.1). Let us show the uniqueness of this solution. Let $U_2 \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ be a solution of (1.1), then $U' = t^A D_t U_2 - \Lambda U_2 \in \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ is a fixed point of $\mathcal{F}$. There is $R_1 > 0$ such that $\overline{\Delta}_{\eta R_1} \subset \Omega$, so, from Cauchy inequalities, $U' \in \mathscr{B}_{\phi, R_1, 1, r}$. We take $a = \max(a_0, \|U'\|_{\phi, R_1, 1, r})$. Using Proposition 3.8 again, there exist $S \in]0, R_1]$, $\rho \geq 1$, $s \in]0, r]$ with $\rho s < S$ such that $U = U'$ on $[0, s] \times \Omega_{S, \rho, s}$, i.e., on $[0, s] \times \Omega$ since Ω is a connected open set. We shall prove that the real number

$$t_0 \equiv \max\{t \in]0, r]; U_1 = U_2 \text{ on } [0, t] \times \Omega\}$$

is equal to r. For this purpose, we assume $0 < t_0 < r$ and we set $W(x) = U_1(t_0, x) = U_2(t_0, x)$. This function W is holomorphic on Ω and the functions U_i belong to $\mathcal{C}^{1,\omega}([t_0, r] \times \Omega; \mathbb{C}^N)$. Thus, writing

$$U_i(t, x) = W(x) + (t - t_0)\mathcal{U}_i(t, x),$$

we define uniquely $\mathcal{U}_i \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([t_0, r] \times \Omega; \mathbb{C}^N)$ and we find that these $\mathcal{U}_i$ are solutions of

$$t^A((t - t_0)D_t + I)\mathcal{U} = \Lambda(t - t_0)\mathcal{U} + f(t, x, \zeta W + \zeta(t - t_0)\mathcal{U}, t^A D_x W + t^A(t - t_0)D_x\mathcal{U}), \tag{4.1}$$

namely,

$$((t - t_0)D_t + I)\mathcal{U} = g(t, x, (t - t_0)\mathcal{U}, (t - t_0)D_x\mathcal{U}), \tag{4.2}$$

where $g \in \mathcal{C}^{0,\omega}([t_0, r] \times \Omega \times \mathcal{O}; \mathbb{C}^N)$, $\mathcal{O} \subset \mathcal{O}_0$ is an open neighbourhood of the origin in $\mathbb{C}_y^N \times \mathbb{C}_z^{nN}$ defined, from (3.19), at least for

$$\max_{i \in [\![1, N]\!]} |y_i| < R' / 2\varepsilon(r_0, R_0) \quad \text{and} \quad \max_{(i, j) \in [\![1, N]\!] \times [\![1, n]\!]} |z_{i, j}| < R' / 2. \tag{4.3}$$

By translation, (4.2) is reduced to $t_0 = 0$ with $g \in \mathcal{C}^{0, \omega}([0, r - t_0] \times \Omega \times \mathcal{O}; \mathbb{C}^N)$ that is to say to a system like (1.1). As above, there exist $s' \in]0, r - t_0]$ and an open neighbourhood $\Omega' \subset \Omega$ of the origin in $\mathbb{C}_x^n$ such that $\mathcal{U}_1$ and $\mathcal{U}_2$ coincide on $[0, s'] \times \Omega'$, hence U_1 and U_2 coincide on $[t_0, t_0 + s'] \times \Omega'$, i.e., on $[t_0, t_0 + s'] \times \Omega$. This allows us to conclude that $t_0 = r$. □

5. Proof of Theorem 1.2

Suppose $U = (u_1, \ldots, u_N)$ is a function satisfying Theorem 1.2. Let $i \in [\![1, N]\!] \setminus \mathcal{I}$, we define uniquely $\widetilde{u}_i \in \mathcal{C}^{1, \omega}(]0, r] \times \Omega)$ by the relation

$$u_i(t, x) = w_i(x) + t^{1 - a_i} \widetilde{u}_i(t, x). \tag{5.1}$$

Let us show that $\widetilde{u}_i$ belongs necessarily to $\mathcal{C}^{0, \omega}([0, r] \times \Omega)$. As $a_i \in [0, 1[$, we observe that

$$u_i(t, x) - u_i(0, x) = \int_0^t \tau^{-a_i} v_i(\tau, x) d\tau, \quad \text{where} \quad v_i = t^{a_i} D_t u_i \in \mathcal{C}^{0, \omega}([0, r] \times \Omega),$$

therefore,

$$u_i(t, x) = w_i(x) + t^{1 - a_i} \int_0^1 \sigma^{-a_i} v_i(\sigma t, x) d\sigma. \tag{5.2}$$

If Δ is an open polydisk such that $\overline{\Delta} \subset \Omega$, the mapping $(\sigma, t, x) \in]0, 1] \times [0, r] \times \Delta \mapsto \sigma^{-a_i} v(\sigma t, x) \in \mathbb{C}$ is continuous, holomorphic with respect to x and bounded by $M\sigma^{-a_i}$, where $M = \max_{[0, r] \times \overline{\Delta}} |v_i|$. It follows that the last integral belongs to $\mathcal{C}^{0, \omega}([0, r] \times \Omega)$; ultimately we have $\widetilde{u}_i \in \mathcal{C}^{1, \omega}(]0, r] \times \Omega) \cap \mathcal{C}^{0, \omega}([0, r] \times \Omega)$.

Denoting by f_i the i-th component of f, system (1.1) is also written in the form

$$t^{a_i} D_t u_i(t, x) = \sum_{j=1}^{N} \lambda_{ij}(t, x) u_j(t, x) + f_i(t, x, (\zeta U)(t, x), t^A D_x U(t, x)),$$

$$i \in [\![1, N]\!]. \quad (5.3)$$

Injecting (5.1), we obtain

$$t^{a_i} D_t u_i = \sum_{j \in \mathcal{I}} \lambda_{ij} u_j + \sum_{j \notin \mathcal{I}} \lambda_{ij}(w_j + t^{1-a_j} \widetilde{u}_j) + f_i(t, x, \zeta U, t^A D_x U), \text{ for } i \in \mathcal{I}, \quad (5.4)$$

and

$$t D_t \widetilde{u}_i = (a_i - 1)\widetilde{u}_i + \sum_{j \in \mathcal{I}} \lambda_{ij} u_j + \sum_{j \notin \mathcal{I}} \lambda_{ij}(w_j + t^{1-a_j} \widetilde{u}_j) + f_i(t, x, \zeta U, t^A D_x U),$$

$$\text{when } i \notin \mathcal{I}. \quad (5.5)$$

Now we set

$$\widetilde{A} = \operatorname{diag}(\widetilde{a}_1, \ldots, \widetilde{a}_N), \quad \text{where} \quad \widetilde{a}_i = \begin{cases} a_i & \text{if } i \in \mathcal{I}, \\ 1 & \text{if not}, \end{cases}$$

and $\widetilde{U} = (\widetilde{u}_1, \ldots, \widetilde{u}_N)$ with $\widetilde{u}_i \equiv u_i$ for $i \in \mathcal{I}$. Point out that $\widetilde{U} \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$. We denote by $\mathcal{M} = (\widetilde{\lambda}_{ij})_{1 \le i, j \le N}$ the square matrix of order N, where the $\widetilde{\lambda}_{ij}$ are functions of $(t, x) \in \mathbb{R} \times \mathbb{C}^n$ defined by

$$\widetilde{\lambda}_{ij}(t, x) = \begin{cases} \lambda_{ij}(t, x) & \text{if } j \in \mathcal{I}, \\ a_i - 1 & \text{if } j \notin \mathcal{I} \text{ and } j = i, \\ 0 & \text{if } j \notin \mathcal{I} \text{ and } j \neq i. \end{cases} \quad (5.6)$$

Using a matrix representation and expanding along columns, we note that

$$\det(\lambda I - \mathcal{M}(0, 0)) = \pm\widetilde{P}(\lambda)\prod_{i\notin\mathcal{I}}[\lambda - (a_i - 1)]. \tag{5.7}$$

Let

$$g = (g_i)_{i\in[\![1, N]\!]}, \quad \text{where} \quad g_i(x, y) = \sum_{j\notin\mathcal{I}}\lambda_{ij}(w_j(x) + y_j),$$

and

$$\delta = (\delta_i)_{i\in[\![1, N]\!]}, \quad \text{where} \quad \delta_i(t) = \begin{cases} 0 & \text{if } i \in \mathcal{I}, \\ t^{1-a_i} & \text{if not.} \end{cases}$$

Equations (5.4) and (5.5) can then be written as the following system:

$$t^{\widetilde{A}}D_t\widetilde{U} = \mathcal{M}\widetilde{U} + g(x, \delta\widetilde{U}) + f(t, x, \zeta U, t^A D_x U).$$

Regarding f, by putting $W = (w_1, \dots, w_N)$ with $w_i \equiv 0$ for $i \in \mathcal{I}$, one has

$$f(t, x, \zeta U, t^A D_x U) = f(t, x, \zeta W + \epsilon\widetilde{U}, t^A D_x W + t^{\widetilde{A}} D_x\widetilde{U}) \equiv h(t, x, \epsilon\widetilde{U}, t^{\widetilde{A}} D_x\widetilde{U}),$$

where

$$\epsilon = (\epsilon_i)_{i\in[\![1, N]\!]}, \quad \epsilon_i(t, x) = \begin{cases} 0 & \text{if } i \in \mathcal{I}, \\ t^{1-a_i} & \text{if not,} \end{cases}$$

and, since ζW and $t^A D_x W$ vanish at the origin of $\mathbb{R}\times\mathbb{C}^n$ thanks to (1.2) and (1.4), there exist $r_0' > 0$ and an open neighbourhood $\Omega_0' \subset \Omega_0$ (resp., $\mathcal{O}_0' \subset \mathcal{O}_0$) of the origin in $\mathbb{C}_x^n$ (resp., $\mathbb{C}_y^N \times \mathbb{C}_z^{nN}$) such that $h \in \mathcal{C}^{0,\omega}([0, r_0'] \times (\Omega_0' \times \mathcal{O}_0'); \mathbb{C}^N)$. After all, letting $\widetilde{f}(t, x, y, \overline{y}, z) = g(x, y) + h(t, x, \overline{y}, z)$, $\widetilde{U}$ satisfies

$$t^{\widetilde{A}}D_t\widetilde{U} = \mathcal{M}\widetilde{U} + \widetilde{f}(t, x, \delta\widetilde{U}, \epsilon\widetilde{U}, t^{\widetilde{A}} D_x\widetilde{U}).$$

Considering the proof of Theorem 1.1, it can also be written for such a $\widetilde{f}$, hence we have existence and uniqueness for $\widetilde{U} \in \mathcal{C}^{1,\omega}(]0, r] \times \Omega; \mathbb{C}^N) \cap \mathcal{C}^{0,\omega}([0, r] \times \Omega; \mathbb{C}^N)$ which completes the proof of Theorem 1.2. □

6. Proof of Theorem 1.4

As explained in Section 2, matrix Λ can be considered constant. Since the diagonal matrix $t^a I_N$ commutes with any matrix of order N, a fortiori, with an invertible one, it follows, after changing the notations, that it is enough to study system (1.1) for an upper triangular constant matrix $\Lambda \in \mathcal{T}_N^+(\mathbb{C})$. By applying Theorem 1.2 for such a matrix and for a_i all equal to a, we achieve our expected result. □

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