SUPPLEMENT TO "SEMIPARAMETRIC GEE ANALYSIS IN PARTIALLY LINEAR SINGLE-INDEX MODELS FOR LONGITUDINAL DATA"

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APPENDIX C: SOME AUXILIARY LEMMAS AND PROOF OF THEOREM 3

This appendix first gives some technical lemmas which have been used to prove the main results in Appendix B. It then gives the proof of Theorem 3 in Section 4. As in Appendix B, let C denote a generic positive constant whose value may change from line to line. Define

$$V_{ij}(u,\boldsymbol{\theta},\kappa)=\frac{1}{h}\Big(\frac{\mathbf{X}_{ij}^{\top}\boldsymbol{\theta}-u}{h}\Big)^{\kappa}K\Big(\frac{\mathbf{X}_{ij}^{\top}\boldsymbol{\theta}-u}{h}\Big),\ \kappa=0,1,2,\ldots,$$

for $i=1,\ldots,n$ and $j=1,\ldots,m_i$.

C.1. Some lemmas. We next give the uniform consistency results of the weighted nonparametric kernel-based estimators for the longitudinal data, which are of independent interest.

LEMMA 1. *Suppose that Assumptions 1, 2(ii) and 3(i) in Appendix A are satisfied and*

$$h\to 0,\quad \frac{n^2}{N_n(h)\log n}\to\infty,\quad \frac{\log n}{h^2N_n(h)}=O(1), \tag{C.1}$$

where $N_n(h)=\sum_{i=1}^n 1/(m_ih)$. Then we have, for any integer $\kappa\geq 0$ and as $n\to\infty$,

$$\sup_{(u,\boldsymbol{\theta}^{\top})^{\top}\in\mathcal{U}(\Theta)}\Big|\frac{1}{n}\sum_{i=1}^n\frac{1}{m_i}\sum_{j=1}^{m_i}V_{ij}(u,\boldsymbol{\theta},\kappa)-f_{\boldsymbol{\theta}}(u)\mu_\kappa\Big|=O_P\Big(h^{\tau_\kappa}+\frac{\sqrt{N_n(h)\log n}}{n}\Big), \tag{C.2}$$

where $\mathcal{U}(\Theta)=\big\{(u,\boldsymbol{\theta}^{\top})^{\top}:u\in\mathcal{U},\ \boldsymbol{\theta}\in\Theta\big\}$, $\mathcal{U}$ and $f_{\boldsymbol{\theta}}(\cdot)$ were defined in Assumption 3(i), Θ is a compact parameter space, $\mu_\kappa=\int v^\kappa K(v)dv$, $\tau_\kappa=1$ if κ is odd, and $\tau_\kappa=2$ if κ is even.

PROOF. For simplicity, let $\epsilon_n = \frac{\sqrt{N_n(h)\log n}}{n}$. To prove (C.2), it suffices to show that

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \Big| \frac{1}{n}\sum_{i=1}^n \frac{1}{m_i}\sum_{j=1}^{m_i}\Big\{V_{ij}(u,\boldsymbol{\theta},\kappa) - \mathrm{E}[V_{ij}(u,\boldsymbol{\theta},\kappa)]\Big\}\Big| = O_P(\epsilon_n) \tag{C.3}$$

and

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \Big|\mathrm{E}[V_{ij}(u,\boldsymbol{\theta},\kappa)] - f_{\boldsymbol{\theta}}(u)\mu_\kappa\Big| = O(h^{\tau_\kappa}). \tag{C.4}$$

By Assumptions 1, 2(ii) and 3(i) in Appendix A, we have

$$\begin{aligned} \mathrm{E}[V_{ij}(u,\boldsymbol{\theta},\kappa)] &= \frac{1}{h}\int \Big(\frac{u_1-u}{h}\Big)^\kappa K\Big(\frac{u_1-u}{h}\Big) f_{\boldsymbol{\theta}}(u_1)du_1 \\ &= \int v^\kappa K(v) f_{\boldsymbol{\theta}}(u+hv)dv \\ &= f_{\boldsymbol{\theta}}(u)\mu_\kappa + \dot{f}_{\boldsymbol{\theta}}(u)\mu_{\kappa+1}h + O(h^2) \end{aligned}$$

uniformly for $(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)$, where $\dot{f}_{\boldsymbol{\theta}}(\cdot)$ is the first-order derivative of $f_{\boldsymbol{\theta}}(\cdot)$. Hence, we can prove that (C.4) holds.

Let us now turn to the proof of (C.3). The main idea is to consider covering the set $\mathcal{U}(\Theta)$ by a finite number of subsets $S(k)$, which are centered at $s_k^\top \equiv \big(u_k, \boldsymbol{\theta}_k^\top\big)$ with radius $r = o(h^2)$. Let $\mathcal{N}_n$ be the total number of such subsets, $S(k)$, $k = 1, 2, \ldots, \mathcal{N}_n$. Then $\mathcal{N}_n = O(r^{-(p+1)})$. It is easy to show that

$$\begin{aligned} &\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \Big| \frac{1}{n}\sum_{i=1}^n \frac{1}{m_i}\sum_{j=1}^{m_i}\Big\{V_{ij}(u,\boldsymbol{\theta},\kappa) - \mathrm{E}[V_{ij}(u,\boldsymbol{\theta},\kappa)]\Big\}\Big| \\ &\le \max_{1\le k\le \mathcal{N}_n} \Big| \frac{1}{n}\sum_{i=1}^n \frac{1}{m_i}\sum_{j=1}^{m_i}\Big\{V_{ij}(s_k,\kappa) - \mathrm{E}[V_{ij}(s_k,\kappa)]\Big\}\Big| \\ &\quad + \max_{1\le k\le \mathcal{N}_n} \sup_{(u,\boldsymbol{\theta}^\top)^\top \in S(k)} \Big| \frac{1}{n}\sum_{i=1}^n \frac{1}{m_i}\sum_{j=1}^{m_i}\Big[V_{ij}(u,\boldsymbol{\theta},\kappa) - V_{ij}(s_k,\kappa)\Big]\Big| \\ &\quad + \max_{1\le k\le \mathcal{N}_n} \sup_{(u,\boldsymbol{\theta}^\top)^\top \in S(k)} \Big| \frac{1}{n}\sum_{i=1}^n \frac{1}{m_i}\sum_{j=1}^{m_i}\Big\{\mathrm{E}[V_{ij}(u,\boldsymbol{\theta},\kappa)] - \mathrm{E}[V_{ij}(s_k,\kappa)]\Big\}\Big| \\ &\equiv \Pi_{n1} + \Pi_{n2} + \Pi_{n3}, \end{aligned} \tag{C.5}$$

where $V_{ij}(s_k,\kappa) = V_{ij}(u_k,\boldsymbol{\theta}_k,\kappa)$.

Noting that $K(\cdot)$ is Lipschitz continuous by Assumption 1 and taking $r = C\epsilon_n h^2$ for some positive constant C, we have

$$\Pi_{n2} = O_P\Big(\frac{r}{h^2}\Big) = O_P(\epsilon_n), \quad \Pi_{n3} = O(\epsilon_n). \tag{C.6}$$

For Π_{n1}, we apply the Bernstein inequality for i.i.d. random variables (see, for example, Lemma 2.2.9 in van der Vaart and Wellner, 1996) to obtain the convergence rate. Note that by Assumptions 1, 2(ii) and 3(i),

$$\frac{1}{m_i}\sum_{j=1}^{m_i}\Big|V_{ij}(s_k,\kappa)-\mathrm{E}[V_{ij}(s_k,\kappa)]\Big|\leq\frac{C}{h}\quad\text{for some } C>0, \tag{C.7}$$

and

$$\mathrm{Var}\Big[\frac{1}{m_i}\sum_{i=1}^{m_i}V_{ij}(s_k,\kappa)\Big]=\frac{1}{m_i^2}\cdot\mathrm{Var}\big[\sum_{i=1}^{m_i}V_{ij}(s_k,\kappa)\big]\leq\frac{C}{m_ih}. \tag{C.8}$$

By (C.7), (C.8), Assumption 2(ii) and the Bernstein inequality, we have, for some sufficiently large positive constant C_ϵ,

$$\begin{aligned}
\mathrm{P}(\Pi_{n1}>C_\epsilon\epsilon_n) &\leq \mathcal{N}_n\exp\Big\{\frac{-n^2C_\epsilon^2\epsilon_n^2}{\big(2CN_n(h)+\frac{2CC_\epsilon n\epsilon_n}{3h}\big)}\Big\}\\
&\leq \mathcal{N}_n\exp\Big\{\frac{-n^2C_\epsilon^2\epsilon_n^2}{C_\epsilon^{3/2}N_n(h)}\Big\}\\
&\leq \mathcal{N}_n\exp\{-C_\epsilon^{1/2}\log n\}=o(1),
\end{aligned} \tag{C.9}$$

which implies that

$$\Pi_{n1}=O_P(\epsilon_n). \tag{C.10}$$

In view of (C.5), (C.6) and (C.10), we have shown (C.3), completing the proof of Lemma 1.

□

LEMMA 2. *Suppose that Assumptions 1, 2(ii), 3 and 5 are satisfied. Then we have*

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top\in\mathcal{U}(\Theta)}\Big|\sum_{i=1}^n\mathbf{s}_i(u|\boldsymbol{\theta})\mathbf{Z}_i-\rho_{\mathbf{Z}}(u|\boldsymbol{\theta})\Big|=O_P\left(h^2+\epsilon_n\right), \tag{C.11}$$

where $\mathbf{s}_i(u|\boldsymbol{\theta})$ *was defined in Section 2,* $\rho_{\mathbf{Z}}(u|\boldsymbol{\theta})=\mathrm{E}\big[\mathbf{Z}_{ij}|\mathbf{X}_{ij}^\top\boldsymbol{\theta}=u\big]$, $\epsilon_n=\frac{\sqrt{N_n(h)\log n}}{n}$ *and* $N_n(h)$ *was defined in Lemma 1.*

PROOF. It is easy to show that the bandwidth conditions in Assumption 5 imply that the bandwidth conditions in (C.1) are satisfied. Hence, by letting $H=\mathrm{diag}(1,h)$ and Lemma 1 we have

$$H^{-1}\Big[\frac{1}{n}\sum_{i=1}^n\overline{\mathbf{X}}_i^\top(u|\boldsymbol{\theta})K_i(u|\boldsymbol{\theta})\overline{\mathbf{X}}_i(u|\boldsymbol{\theta})\Big]H^{-1}=f_{\boldsymbol{\theta}}(u)\ \mathrm{diag}(1,\mu_2)+o_P(1), \tag{C.12}$$

uniformly for $(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)$, where $\overline{\mathbf{X}}_i(u|\boldsymbol{\theta})$ and $K_i(u|\boldsymbol{\theta})$ were defined in Section 2.

We then use arguments similar to those in the proof of Lemma 1 to show that

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \Big| \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i} V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa) - f_{\boldsymbol{\theta}}(u)\mu_\kappa \rho_{\mathbf{Z}}(u|\boldsymbol{\theta})\Big| = O_P(h^{\tau_\kappa}+\epsilon_n), \tag{C.13}$$

where

$$V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa) = \frac{\mathbf{Z}_{ij}}{h}\Big(\frac{\mathbf{X}_{ij}^\top\boldsymbol{\theta}-u}{h}\Big)^\kappa K\Big(\frac{\mathbf{X}_{ij}^\top\boldsymbol{\theta}-u}{h}\Big), \quad \kappa = 0,1,\ldots.$$

To prove (C.13), we need only to show that

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i} \Big\{V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa) - \mathrm{E}[V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa)]\Big\} = O_P(\epsilon_n) \tag{C.14}$$

and

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \Big| \mathrm{E}[V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa)] - f_{\boldsymbol{\theta}}(u)\mu_\kappa\rho_{\mathbf{Z}}(u|\boldsymbol{\theta})\Big| = O_P\big(h^{\tau_\kappa}\big). \tag{C.15}$$

By Assumptions 1, 2(ii) and 3(ii), we have

$$\begin{aligned}
\mathrm{E}[V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa)] &= \frac{1}{h}\int \Big(\frac{u_1-u}{h}\Big)^\kappa K\Big(\frac{u_1-u}{h}\Big) f_{\boldsymbol{\theta}}(u_1)\rho_{\mathbf{Z}}(u_1|\boldsymbol{\theta})du_1 \\
&= \int v^\kappa K(v) f_{\boldsymbol{\theta}}(u+hv)\rho_{\mathbf{Z}}(u+hv|\boldsymbol{\theta})dv \\
&= f_{\boldsymbol{\theta}}(u)\rho_{\mathbf{Z}}(u|\boldsymbol{\theta})\mu_\kappa + \dot f_{\boldsymbol{\theta}}(u|\boldsymbol{\theta})\rho_{\mathbf{Z}}(u|\boldsymbol{\theta})\mu_{\kappa+1}h \\
&\quad + f_{\boldsymbol{\theta}}(u)\dot\rho_{\mathbf{Z}}(u|\boldsymbol{\theta})\mu_{\kappa+1}h + O(h^2)
\end{aligned}$$

uniformly in $(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)$, which implies (C.15).

As in the proof of Lemma 1, the main idea in proving (C.14) is to consider covering the set $\mathcal{U}(\Theta)$ by a finite number of subsets $S(k)$ centered at s_k with radius $r = o(h^2)$. Letting s_k and $\mathcal{N}_n$ be defined as in the proof of Lemma 1, it is easy to show that

$$\begin{aligned}
&\sup_{(u,\boldsymbol{\theta}^\top)^\top \in \mathcal{U}(\Theta)} \Big| \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i} \Big\{V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa) - \mathrm{E}[V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa)]\Big\}\Big| \\
&\quad \le \max_{1\le k\le \mathcal{N}_n} \Big| \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i} \Big\{V_{ij,\mathbf{Z}}(s_k,\kappa) - \mathrm{E}[V_{ij,\mathbf{Z}}(s_k,\kappa)]\Big\}\Big|
\end{aligned}$$

$$
\begin{aligned}
& + \max_{1\le k\le \mathcal{N}_n} \sup_{(u,\boldsymbol{\theta}^\top)^\top\in S(k)} \Big|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i}\Big[V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa)-V_{ij,\mathbf{Z}}(s_k,\kappa)\Big]\Big| \\
& + \max_{1\le k\le \mathcal{N}_n} \sup_{(u,\boldsymbol{\theta}^\top)^\top\in S(k)} \Big|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i}\Big\{\mathrm{E}[V_{ij,\mathbf{Z}}(u,\boldsymbol{\theta},\kappa)]-\mathrm{E}[V_{ij,\mathbf{Z}}(s_k,\kappa)]\Big\}\Big| \\
\text{(C.16)} \quad & \equiv \Pi_{n4}+\Pi_{n5}+\Pi_{n6},
\end{aligned}
$$

where $V_{ij,\mathbf{Z}}(s_k,\kappa)=V_{ij,\mathbf{Z}}(u_k,\boldsymbol{\theta}_k,\kappa)$.

Similarly to the proof of (C.6) above, taking $r=O\left(\epsilon_n h^2\right)$, we have

$$
\text{(C.17)} \qquad \Pi_{n5}+\Pi_{n6}=O_P\left(\frac{r}{h^2}\right)=O_P(\epsilon_n).
$$

We next obtain the convergence rate for Π_{n4}, which is slightly more complicated than its counterpart in the proof of Lemma 1. As $\mathbf{Z}_{ij}$ may be unbounded, we apply a truncation method. For this purpose, we define

$$
\overline{V}_{ij,\mathbf{Z}}(k)=V_{ij,\mathbf{Z}}(s_k,\kappa)I\big\{\|\mathbf{Z}_{ij}\|\le MT_n^{\frac{1}{2+\delta}}\big\}
$$

and

$$
\widetilde{V}_{ij,\mathbf{Z}}(k)=V_{ij,\mathbf{Z}}(s_k,\kappa)-\overline{V}_{ij,\mathbf{Z}}(k),
$$

where $I\{\cdot\}$ is an indicator function, $T_n=\sum_{i=1}^n m_i$, and M is a positive constant which will be specified later. It is easy to show that

$$
\begin{aligned}
\Pi_{n4} \quad \le \quad & \max_{1\le k\le \mathcal{N}_n}\Big|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i}(\overline{V}_{ij,\mathbf{Z}}(k)-\mathrm{E}[\overline{V}_{ij,\mathbf{Z}}(k)])\Big| \\
& + \max_{1\le k\le \mathcal{N}_n}\Big|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m_i}\sum_{j=1}^{m_i}(\widetilde{V}_{ij,\mathbf{Z}}(k)-\mathrm{E}[\widetilde{V}_{ij,\mathbf{Z}}(k)])\Big| \\
\text{(C.18)} \qquad \equiv \quad & \Pi_{n4,1}+\Pi_{n4,2}.
\end{aligned}
$$

Note that for $C_*>0$ and any $\varepsilon>0$,

$$
\begin{aligned}
\mathrm{P}\Big(\Pi_{n4,2}>C_*\epsilon_n\Big) \quad & \le \quad \mathrm{P}\Big(\max_{1\le k\le \mathcal{N}_n}\max_{1\le i\le n,1\le j\le m_i}|\widetilde{V}_{ij,\mathbf{Z}}(k)|>0\Big) \\
& \le \quad \sum_{i=1}^{n}\sum_{j=1}^{m_i}\mathrm{P}\Big(\|\mathbf{Z}_{ij}\|>MT_n^{\frac{1}{2+\delta}}\Big) \\
& \le \quad M^{-(2+\delta)}\mathrm{E}\big[\|\mathbf{Z}_{ij}\|^{2+\delta}\big]<\varepsilon,
\end{aligned}
$$

if we choose $M>\mathrm{E}\big[\|\mathbf{Z}_{ij}\|^{2+\delta}\big]^{1/(2+\delta)}\varepsilon^{-1/(2+\delta)}$. Then by letting ε be arbitrarily small, we have shown that

$$
\text{(C.19)} \qquad \Pi_{n4,2}=O_P(\epsilon_n).
$$

We next use the Bernstein inequality to deal with the convergence of $\Pi_{n4,1}$. Note that for any k, we have

$$\frac{1}{m_i}\sum_{j=1}^{m_i}\Big|\overline{V}_{ij,\mathbf{Z}}(k)-\mathrm{E}[\overline{V}_{ij,\mathbf{Z}}(k)]\Big| \le \frac{CT_n^{\frac{1}{2+\delta}}}{h}, \tag{C.20}$$

$$\mathrm{Var}\Big[\frac{1}{m_i}\sum_{j=1}^{m_i}\overline{V}_{ij,\mathbf{Z}}(k)\Big] \le \frac{C}{m_i h}, \tag{C.21}$$

where C is a positive constant which is independent of k. By (C.20), (C.21), Assumptions 2(ii), 5 and the Bernstein inequality for i.i.d. random variables, we have, for $C_* > 0$ sufficiently large,

$$\begin{aligned}
\mathrm{P}\Big(\Pi_{n4,1} > C_*\epsilon_n\Big) &\le \mathcal{N}_n \exp\Big\{\frac{-C_*^2 n^2\epsilon_n^2}{2CN_n(h)+\frac{2CC_* n\epsilon_n T_n^{\frac{1}{2+\delta}}}{3h}}\Big\} \\
&\le \mathcal{N}_n \exp\big\{-C_*^{1/2}\log(n)\big\} \\
&= O(\mathcal{N}_n n^{-\sqrt{C_*}}) = o(1). \qquad \text{(C.22)}
\end{aligned}$$

Hence, we have

$$\Pi_{n4,1} = O_P(\epsilon_n). \tag{C.23}$$

By (C.16)–(C.19) and (C.23), we know that (C.14) holds, which, together with (C.15), implies that (C.13) holds. In view of (C.12) and (C.13) as well as the definition of $\mathbf{s}_i(u|\boldsymbol{\theta})$, (C.11) is readily seen.

□

LEMMA 3. *Let*

$$\widetilde{\boldsymbol{\eta}}(\mathbf{X}_i,\boldsymbol{\theta}) = \Big(\dot{\eta}(\mathbf{X}_{i1}^\top\boldsymbol{\theta})\mathbf{X}_{i1},\ldots,\dot{\eta}(\mathbf{X}_{im_i}^\top\boldsymbol{\theta})\mathbf{X}_{im_i}\Big)^\top,$$

and suppose that the conditions in Lemma 2 are satisfied. Then we have

$$\sup_{(u,\boldsymbol{\theta}^\top)^\top\in\mathcal{U}(\Theta)}\Big|\sum_{i=1}^n \mathbf{s}_i(u|\boldsymbol{\theta})\widetilde{\boldsymbol{\eta}}(\mathbf{X}_i,\boldsymbol{\theta}) - \dot{\eta}(u)\rho_{\mathbf{X}}(u|\boldsymbol{\theta})\Big| = O_P\left(h^2+\epsilon_n\right), \tag{C.24}$$

where $\rho_{\mathbf{X}}(u|\boldsymbol{\theta}) = \mathrm{E}\big[\mathbf{X}_{ij}|\mathbf{X}_{ij}^\top\boldsymbol{\theta}=u\big]$.

PROOF. The proof is similar to those of Lemmas 1 and 2 given above. We thus omit the details.

□

We next give the proof of Theorem 3, whose main idea is analogous to that of the proof of Theorem 1 in Chen *et al.* (2009) in the time series context.

C.2. Proof of Theorem 3. Note that

$$
\begin{aligned}
\widehat{\sigma}^2(t)-\sigma^2(t) &= \frac{\exp\{\widehat{\sigma}_\diamond^2(t)\}}{\widehat{\tau}}-\frac{\exp\{\sigma_\diamond^2(t)\}}{\tau} \\
&= \Big[\frac{\exp\{\widehat{\sigma}_\diamond^2(t)\}}{\widehat{\tau}}-\frac{\exp\{\widehat{\sigma}_\diamond^2(t)\}}{\tau}\Big]+\Big[\frac{\exp\{\widehat{\sigma}_\diamond^2(t)\}}{\tau}-\frac{\exp\{\sigma_\diamond^2(t)\}}{\tau}\Big] \\
&\equiv \Xi_{n1}+\Xi_{n2}. \qquad \text{(C.25)}
\end{aligned}
$$

We first consider Ξ_{n2}. By a first-order Taylor expansion and some standard techniques in local linear estimation, we can show that

$$
\begin{aligned}
\Xi_{n2} &\overset{P}{\sim} \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau}\big[\widehat{\sigma}_\diamond^2(t)-\sigma_\diamond^2(t)\big] \overset{P}{\sim} \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1} \\
&\quad\times\Big\{\sum_{i=1}^n w_i\sum_{j=1}^{m_i}\big[\log(\widehat{r}_{ij}+\zeta_n)-\sigma_\diamond^2(t)-\dot{\sigma}_\diamond^2(t)(t_{ij}-t)\big]K_1\big(\frac{t_{ij}-t}{h_1}\big)\Big\} \\
&= \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\sum_{i=1}^n w_i\sum_{j=1}^{m_i}\big[\sigma_\diamond^2(t_{ij})-\sigma_\diamond^2(t)-\dot{\sigma}_\diamond^2(t)(t_{ij}-t)\big]K_1\big(\frac{t_{ij}-t}{h_1}\big)\Big\} \\
&\quad+\frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\sum_{i=1}^n w_i\sum_{j=1}^{m_i}\big[\log(\widehat{r}_{ij}+\zeta_n)-\log(r_{ij})\big]K_1\big(\frac{t_{ij}-t}{h_1}\big)\Big\} \\
&\quad+\frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\sum_{i=1}^n w_i\sum_{j=1}^{m_i}\xi_\diamond(t_{ij})K_1\big(\frac{t_{ij}-t}{h_1}\big)\Big\} \\
&\equiv \Xi_{n2,1}+\Xi_{n2,2}+\Xi_{n2,3}, \qquad \text{(C.26)}
\end{aligned}
$$

where $a_n \overset{P}{\sim} b_n$ denotes $a_n=b_n(1+o_P(1))$.

Noting that $\mathrm{E}[\xi_\diamond(t_{ij})]=0$, by (4.7) and the central limit theorem, it is readily proven that

$$
\varphi_{n\diamond}^{1/2}(h_1)\cdot\Xi_{n2,3}\xrightarrow{d}\mathrm{N}\left(0,\frac{\exp\{2\sigma_\diamond^2(t)\}}{\tau^2 f_T(t)}\sigma_\diamond^2\right), \qquad \text{(C.27)}
$$

where $\frac{\exp\{2\sigma_\diamond^2(t)\}}{\tau^2}=\sigma^4(t)$ by the relevant definition in (4.2).

By Assumption 8 and a second-order Taylor expansion of $\sigma_\diamond^2(\cdot)$, we can show that

(C.28)

$$
\Xi_{n2,1}=\frac{\exp\{\sigma_\diamond^2(t)\}}{2\tau}\ddot{\sigma}_\diamond^2(t)h_1^2\int v^2K_1(v)dv+o_P(h_1^2)=h_1^2b_{\sigma1}(t)+o_P(h_1^2).
$$

We next prove that $\Xi_{n2,2}$ is asymptotically negligible (compared with $\Xi_{n2,3}$). Let $\chi_n=\log^{-2}(T_n)\varphi_{n\diamond}^{-1/2}(h_1)$ and $\xi(t_{ij})$ be defined as in Section 4 such that $r(t_{ij}) = \sigma^2(t_{ij})\xi^2(t_{ij})$ and $\mathrm{E}[\xi^2(t_{ij})|t_{ij}] = 1$ with probability 1. Note that

$$\begin{aligned}\Xi_{n2,2} = {} & \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\sum_{i=1}^n w_i \sum_{j=1}^{m_i}\big[\log(\widehat{r}_{ij}+\zeta_n)-\log(r_{ij})\big] \\ & \times K_1\big(\frac{t_{ij}-t}{h_1}\big) I\{\xi^2(t_{ij})\le \chi_n\}\Big\} \\ & + \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\sum_{i=1}^n w_i \sum_{j=1}^{m_i}\big[\log(\widehat{r}_{ij}+\zeta_n)-\log(r_{ij})\big] \\ & \times K_1\big(\frac{t_{ij}-t}{h_1}\big) I\{\xi^2(t_{ij}) > \chi_n\}\Big\} \\ \equiv {} & \Xi_{n2,21}+\Xi_{n2,22}. \end{aligned} \tag{C.29}$$

Recalling that $\zeta_n = 1/T_n$, by Assumption 7 and the definitions of $\widehat{r}_{ij}$ and r_{ij}, we have

$$\begin{aligned}|\Xi_{n2,21}| \le {} & \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\sum_{i=1}^n w_i \sum_{j=1}^{m_i}\big|\log(\sigma^2(t_{ij})\xi^2(t_{ij}))\big| K_1\big(\frac{t_{ij}-t}{h_1}\big) I(\xi^2(t_{ij})\le\chi_n)\Big\} \\ & + \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\log(T_n)\sum_{i=1}^n w_i \sum_{j=1}^{m_i} K_1\big(\frac{t_{ij}-t}{h_1}\big) I(\xi^2(t_{ij})\le\chi_n)\Big\} \\ \le {} & \frac{\exp\{\sigma_\diamond^2(t)\}}{\tau f_T(t)h_1}\Big\{\big|\log\chi_n\big|\sum_{i=1}^n w_i \sum_{j=1}^{m_i} K_1\big(\frac{t_{ij}-t}{h_1}\big) I(\xi^2(t_{ij})\le\chi_n)\Big\} + O_P(\chi_n\log T_n) \\ = {} & O_P(\chi_n|\log\chi_n| + \chi_n\log T_n) = o_P(\varphi_n^{-1/2}(h_1)). \end{aligned} \tag{C.30}$$

In a way similar to Fan and Yao (1998) and Chen *et al.* (2009), we can show that $\Xi_{n2,22} = o_P(\varphi_n^{-1/2}(h_1))$, which, in combination with (C.30), implies

$$\Xi_{n2,2} = o_P(\varphi_n^{-1/2}(h_1)). \tag{C.31}$$

We next consider Ξ_{n1}. Following the proofs of Lemmas 2 and 3 above, we can similarly prove the uniform convergence rate for the local linear estimator of the link function. Then following the proof of Theorem 1 and (4.5), we can show that

$$\begin{aligned}\frac{1}{\widehat{\tau}}-\frac{1}{\tau} \overset{P}{\sim} {} & \Big[\frac{1}{T_n}\sum_{i=1}^n\sum_{j=1}^{m_i} r_{ij}\big(\exp\{-\widehat{\sigma}_\diamond^2(t_{ij})\}-\exp\{-\sigma_\diamond^2(t_{ij})\}\big)\Big] \\ & + \Big[\frac{1}{T_n}\sum_{i=1}^n\sum_{j=1}^{m_i}(\widehat{r}_{ij}-r_{ij})\exp\{-\widehat{\sigma}_\diamond^2(t_{ij})\}\Big]\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{T_n}\sum_{i=1}^{n}\sum_{j=1}^{m_i} r_{ij}\exp\{-\sigma_\diamond^2(t_{ij})\}\big(\exp\{-\widehat{\sigma}_\diamond^2(t_{ij})+\sigma_\diamond^2(t_{ij})\}-1\big)\\
&+o_P(\varphi^{-1/2}(h_1)+h_1^2)\\
=&-\frac{1}{T_n}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\frac{\xi^2(t_{ij})}{2\tau}\ddot{\sigma}_\diamond^2(t_{ij})h_1^2\int v^2K_1(v)dv+o_P(\varphi^{-1/2}(h_1)+h_1^2)\\
=&-\frac{h_1^2}{2\tau}\mathrm{E}\big[\ddot{\sigma}_\diamond^2(t_{ij})\big]\mu_2+o_P(\varphi^{-1/2}(h_1)+h_1^2),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Xi_{n1}=&-\frac{\exp\{\sigma_\diamond^2(t)\}h_1^2}{2\tau}\mathrm{E}\big[\ddot{\sigma}_\diamond^2(t_{ij})\big]\mu_2+o_P(\varphi^{-1/2}(h_1)+h_1^2)\\
=&-h_1^2 b_{\sigma 2}(t)+o_P(\varphi^{-1/2}(h_1)+h_1^2).
\end{aligned}
\tag{C.32}
$$

In view of (C.25)–(C.32), we have completed the proof of Theorem 3. □

APPENDIX D: SOME ADDITIONAL SIMULATION RESULTS

We present below the average angles between the true single-index parameter vector and its estimates over additional 100 replications for the simulated examples in Section 5. Case 1 represents the scenario where an AR(1) within-subject error correlation is correctly specified in the error variance-covariance matrix estimation, Case 2 the scenario where an ARMA(1,1) correlation structure is correctly specified, Case 3 the scenario where an ARMA(1,1) correlation structure is misspecified as an AR(1) structure, and Case 4 the scenario where the covariates Z have a discrete distribution and the AR(1) error correlation structure is correctly specified. The numbers in the parentheses are the standard errors of the angles over the 100 replications. These results indicate that SGEE has smaller average angles than PULS in all the cases considered.

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TABLE 1
Average angles between true and estimated single-index parameter vectors

	$\overline{m}$ \ n	30		50	
		PULS	SGEE	PULS	SGEE
Case 1	10	0.0470(0.0254)	0.0268(0.0350)	0.0335(0.0168)	0.0162(0.0150)
	30	0.0325(0.0164)	0.0196(0.0108)	0.0462(0.0097)	0.0238(0.0111)
Case 2	10	0.0445(0.0256)	0.0291(0.0163)	0.0325(0.0164)	0.0196(0.0108)
	30	0.0529(0.0256)	0.0299(0.0167)	0.0376(0.0199)	0.0202(0.0099)
Case 3	10	0.0418(0.0248)	0.0337(0.0448)	0.0322(0.0187)	0.0184(0.0125)
	30	0.0423(0.0235)	0.0329(0.0146)	0.0376(0.0227)	0.0293(0.0118)
Case 4	10	0.0434(0.0287)	0.0213(0.0246)	0.0315(0.0174)	0.0149(0.0209)
	30	0.0589(0.0332)	0.0307(0.0430)	0.0275(0.0127)	0.0117(0.0054)

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