Supplementary Material : "On the Marčenko-Pastur law for linear time series"

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S1. Technical lemmas. We state a number of key technical lemmas before providing the details of the proofs of the main results. The following two Lemmas are restatements of Corollary A.41 and Theorem A.44 of Bai and Silverstein [6], respectively.

LEMMA S.1. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are two* $p \times p$ *normal matrices with ESDs* $F^{\mathbf{A}}$ *and* $F^{\mathbf{B}}$*, respectively. Then,*

$$L^3(F^{\mathbf{A}}, F^{\mathbf{B}}) \leq \frac{1}{p} tr[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^*],$$

where $L(F,G)$ *denotes the Lévy distance between two probability distributions* F *and* G *defined on the real line.*

LEMMA S.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two* $p \times n$ *complex matrices and let* $\mathbf{C}$ *and* $\mathbf{D}$ *be Hermitian matrices of order* $p \times p$ *and* $n \times n$*, respectively. Then*

$$\sup_\sigma |F^{\mathbf{C}+\mathbf{A}\mathbf{D}\mathbf{A}^*}(\sigma) - F^{\mathbf{C}+\mathbf{B}\mathbf{D}\mathbf{B}^*}(\sigma)| \leq \frac{1}{p} rank(\mathbf{A}-\mathbf{B}).$$

The following result (Theorem 1 of Geronimo and Hill [16]) characterizes the weak convergence of probability distributions in terms of convergence of Stieltjes transforms.

LEMMA S.3. *Suppose that* $\{P_n\}$ *is a sequence of Borel probability measures with corresponding Stieltjes transforms* $\{s_n\}$*. If* $\lim_{n\to\infty} s_n(z) = s(z)$ *for all* $z \in \mathbb{C}^+$*, then there exists a Borel probability measure* P *with Stieltjes transform* $s_P = s$ *if and only if* $\lim_{v\to\infty} ivs(iv) = -1$*, in which case* P_n *converges to* P *in distribution.*

An additional lemma is stated and proved. It is essential at various places in the arguments that follow. The lemma is a generalization of part (a) of Lemma 2.3 in Bai and Silverstein [4]. There, in addition to being Hermitian, the involved matrices are also assumed to be nonnegative definite. This assumption, however, fails to be true for the symmetrized autocovariance matrices under consideration here and the following extension becomes crucial in verifying the main results.

LEMMA S.4. *Suppose that $s_F(\cdot)$ is the Stieltjes transform of a distribution function F with total mass M and that $\mathbf{B}$ is an $n\times n$ Hermitian matrix. Then, for $z = w + iv \in \mathbb{C}^+$,*

$$\left\|(s_F(z)\mathbf{B}+\mathbf{I})^{-1}\right\| \le \max\left\{\frac{4M\|\mathbf{B}\|}{v}, 2\right\}.$$

PROOF. Without loss of generality, assume that $M = 1$. Note that,

$$\begin{aligned}
s_F(z) &= \int_{-\infty}^{\infty} \frac{1}{\sigma - z} dF(\sigma) \\
&= \int_{-\infty}^{\infty} \frac{\sigma - w}{(\sigma-w)^2+v^2} dF(\sigma) + i\int_{-\infty}^{\infty} \frac{v}{(\sigma-w)^2+v^2} dF(\sigma)
\end{aligned}$$

and therefore, applying the Cauchy–Schwarz inequality,

$$\begin{aligned}
\frac{\operatorname{Im}(s_F(z))}{v} &= \int_{-\infty}^{\infty} \frac{1}{(\sigma-w)^2+v^2} dF(\sigma)\int_{-\infty}^{\infty} \frac{(\sigma-w)^2+v^2}{(\sigma-w)^2+v^2} dF(\sigma) \\
&\ge \int_{-\infty}^{\infty} \frac{1}{(\sigma-w)^2+v^2} dF(\sigma)\int_{-\infty}^{\infty} \frac{(\sigma-w)^2}{(\sigma-w)^2+v^2} dF(\sigma) \\
&\ge \left(\int_{-\infty}^{\infty} \frac{\sigma-w}{(\sigma-w)^2+v^2} dF(\sigma)\right)^2 \\
&= |\operatorname{Re}(s_F(z))|^2.
\end{aligned}$$

Letting $r = \operatorname{Re}(s_F(z))$, it follows that, for any b,

$$|s_F(z)b+1|^2 = (\operatorname{Re}(s_F(z))b+1)^2 + (\operatorname{Im}(s_F(z)))^2 b^2 \ge (rb+1)^2 + \frac{v^2(rb)^4}{b^2} .$$

If $rb > -1/2$, then $|s_F(z)b+1|^2 \ge 1/4$. If $rb < -1/2$, then $|s_F(z)b+1|^2 > v^2(16b^2)^{-1}$. Combining the latter two statements, it follows that

$$\left|\frac{1}{1+s_F(z)b}\right| \le \max\left\{\frac{4|b|}{v}, 2\right\},$$

which implies the assertion of the lemma. □

S1.1. *Completing the proof of Theorem 7.1.*

LEMMA S.5. *Let $\epsilon > 0$. If the assumptions of Theorem 7.1 are satisfied, then $\mathbb{P}(\max_{t\le n}|d_{\tau,t}^{(1)}| > \epsilon)$ tends to zero at a rate faster than n^{-1}.*

PROOF. Since the trace is invariant under cyclical permutations, it follows that

$$|d_{\tau,t}^{(1)}| = \left|\frac{\tilde{\beta}_{\tau,t}\gamma_{\tau,t}}{zp}\tilde{Z}_t^*\boldsymbol{\psi}_t^*\Big(\big(\mathbf{I}+\mathbf{H}(z)\big)^{-1} - \big(\mathbf{I}+\mathbf{H}_t(z)\big)^{-1}\Big)\mathbf{D}\tilde{\mathbf{R}}_t(z)\boldsymbol{\psi}_t\tilde{Z}_t\right|$$

$$\leq \bar{\lambda}_{\mathbf{A}}^2\bar{\lambda}_{\mathbf{D}}\left|\frac{\tilde{\beta}_{\tau,t}\gamma_{\tau,t}}{zp}\right|\|\tilde{Z}_t\|^2\left\|\big(\mathbf{I}+\mathbf{H}(z)\big)^{-1} - \big(\mathbf{I}+\mathbf{H}_t(z)\big)^{-1}\right\|\|\tilde{\mathbf{R}}_t(z)\|, \tag{S.1}$$

where $\bar{\lambda}_{\mathbf{A}}$ and $\bar{\lambda}_{\mathbf{D}}$ are the bounds for the eigenvalues of $\mathbf{A}$ and $\mathbf{D}$ specified in Assumption 2.1 and Theorem 7.1, respectively. In the following, the terms on the right-hand side of the latter inequality will be bounded.

Step 1: Write

$$\begin{aligned}\big\|\big(\mathbf{I}+\mathbf{H}(z)\big)^{-1} - \big(\mathbf{I}+\mathbf{H}_t(z)\big)^{-1}\big\| \\ = \big\|\big(\mathbf{I}+\mathbf{H}_t(z)\big)^{-1}\big(\mathbf{H}_t(z)-\mathbf{H}(z)\big)\big(\mathbf{I}+\mathbf{H}(z)\big)^{-1}\big\|.\end{aligned} \tag{S.2}$$

Recall the definition of $\mathbf{H}(z)$ in (6.3) and note that $n^{-1}\mathrm{tr}[\tilde{\mathbf{R}}(z)\mathcal{H}_t]$, as a Stieltjes transform, has positive imaginary part. It follows that

$$-\gamma_{\tau,t}\beta_{\tau,t} = -\gamma_{\tau,t}\left(1+\frac{1}{n}\gamma_{\tau,t}\mathrm{tr}[\tilde{\mathbf{R}}(z)\mathcal{H}_t]\right)^{-1},$$

where $\beta_{\tau,t}$ is defined in (7.4), has nonnegative imaginary part irrespective of the sign of $\gamma_{\tau,t}$, and strictly positive imaginary part if $\gamma_{\tau,t}\neq 0$. Observe next that $\mathcal{H}_t$ is a positive semidefinite matrix with the same set of eigenvectors as $(\mathbf{I}+\mathbf{H}(z))^{-1}$. Arguing for each eigenvalue individually, we need to bound

$$z\left(z-\frac{\sigma}{n}\sum_{t=1}^n\gamma_{\tau,t}\beta_{\tau,t}\right)^{-1},$$

where σ denotes an eigenvalue of $\mathcal{H}_t$. Since the term $(\ldots)^{-1}$ has an imaginary part not larger than v, with $z=u+iv$, it follows that $\|(\mathbf{I}+\mathbf{H}(z))^{-1}\|\leq |z|v^{-1}$. In the same way, one verifies also that $\|(\mathbf{I}+\mathbf{H}_t(z))^{-1}\|\leq |z|v^{-1}$. Next, write

$$\begin{aligned}\mathbf{H}_t(z)-\mathbf{H}(z) &= \frac{1}{zn}\sum_{t=1}^n\gamma_{\tau,t}(\beta_{\tau,t}-\bar{\beta}_{\tau,t})\mathcal{H}_t \\ &= \frac{1}{zn^2}\sum_{t=1}^n\gamma_{\tau,t}^2\beta_{\tau,t}\bar{\beta}_{\tau,t}\mathrm{tr}\big[(\tilde{\mathbf{R}}_t(z)-\tilde{\mathbf{R}}(z))\mathcal{H}_t\big]\mathcal{H}_t,\end{aligned}$$

where $\bar{\beta}_{\tau,t}$ is as in (7.5). First, suppose that $\mathbf{V}$ is a unitary matrix such that $\mathbf{V}^*\tilde{\mathbf{C}}_\tau\mathbf{V} = \Sigma$, the diagonal matrix of eigenvalues of $\tilde{\mathbf{C}}_\tau$. Define $Y = n^{-1/2}\mathbf{V}^*\tilde{X}_t$ and observe that

$$
\begin{aligned}
(\text{S.3}) \qquad & \left|\mathrm{tr}\big[(\tilde{\mathbf{R}}_t(z) - \tilde{\mathbf{R}}(z))\mathcal{H}_t\big]\right| \\
&= \left|\frac{\gamma_{\tau,t} n^{-1}\tilde{X}_t^*\tilde{\mathbf{R}}_t(z)\mathcal{H}_t\tilde{\mathbf{R}}_t(z)\tilde{X}_t}{1+\gamma_{\tau,t} n^{-1}\tilde{X}_t^*\tilde{\mathbf{R}}_t(z)\tilde{X}_t}\right| \\
&= \left|\frac{\gamma_{\tau,t} Y^*(\Sigma - zI)^{-1}\mathbf{V}^*\mathcal{H}_t\mathbf{V}(\Sigma - zI)^{-1}Y}{1+\gamma_{\tau,t} Y^*(\Sigma - zI)^{-1}Y}\right| \\
&\le \|\mathcal{H}_t\| \frac{\sum_{j=1}^p (|y_j|/|\sigma_j - z|)^2}{\mathrm{Im}(1/\gamma_{\tau,t} + Y^*(\Sigma - zI)^{-1}Y)}\mathbb{I}(\gamma_{\tau,t}\neq 0) \\
&\le \bar{\lambda}_{\mathbf{A}}^2 \frac{\sum_{j=1}^p (|y_j|/|\sigma_j - z|)^2}{v\sum_{j=1}^p (|y_j|/|\sigma_j - z|)^2}\mathbb{I}(\gamma_{\tau,t}\neq 0) \ \le \bar{\lambda}_{\mathbf{A}}^2\frac{1}{v}.
\end{aligned}
$$

The terms $\beta_{\tau,t}$ and $\bar{\beta}_{\tau,t}$ can be bounded in the same fashion. Observe next that $n^{-1}\mathrm{tr}[\tilde{\mathbf{R}}(z)\mathcal{H}_t]$ and $n^{-1}\mathrm{tr}[\tilde{\mathbf{R}}_t(z)\mathcal{H}_t]$ are Stieltjes transforms on $\mathbb{R}_+$, the positive real numbers, with mass not bigger than $c_n\bar{\lambda}_{\mathbf{A}}^2$, since $\|\mathcal{H}_t\| \le \bar{\lambda}_{\mathbf{A}}^2$. Without loss of generality, assume that $c_n < \bar{c}$ for all n. Since $|\gamma_{\tau,t}| \le 1$, by Lemma S.4, it follows that $\max\{|\beta_{\tau,t}|, |\bar{\beta}_{\tau,t}|\} \le C_1(z)$, where $C_1(z) = \max\{4\bar{c}\bar{\lambda}_{\mathbf{A}}^2 v^{-1}, 2\}$. It follows that $\|\mathbf{H}_t(z) - \mathbf{H}(z)\| \le Cn^{-1}$ with an appropriate constant $C > 0$ which may depend on z. In the remainder of this proof, C denotes a generic positive constant that may vary from line to line. Combining the previous estimates and going back to (S.2), it has been established that

$$\left\|\big(\mathbf{I} + \mathbf{H}(z)\big)^{-1} - \big(\mathbf{I} + \mathbf{H}_t(z)\big)^{-1}\right\| \le \frac{C}{n}.$$

Step 2: In the following, bounds for the remaining terms in (S.1) will be given. First, notice that Lemma S.4 implies $|\tilde{\beta}_{\tau,t}| \le \max\{4\bar{\lambda}_{\mathbf{A}}^2\|\tilde{Z}_t\|^2(nv)^{-1}, 2\}$. Note also that $\|\tilde{\mathbf{R}}_t(z)\| \le v^{-1}$. Combining these estimates with the ones obtained in Step 1 of the proof implies then that

$$|d_{\tau,t}^{(1)}| \le \frac{C_1}{pn^2}\|\tilde{Z}_t\|^4 + \frac{C_2}{pn}\|\tilde{Z}_t\|^2,$$

with appropriately chosen positive constants C_1 and C_2. Then, using the formula for moments of Gaussian random variables, we have $\mathbb{E}[\|\tilde{Z}_t\|^{2m}] \le C'_m n^m$ for some constant $C'_m > 0$. Therefore, for any $\epsilon > 0$,

$$P\Big(\max_{t\le n}|d_{\tau,t}^{(1)}| > \epsilon\Big) \le \sum_{t=1}^n \mathbb{P}\big(|d_{\tau,t}^{(1)}| > \epsilon\big)$$

$$
\begin{aligned}
&= n\mathbb{P}\Big(\|\tilde{Z}_t\|^4 > \frac{pn^2\epsilon}{C_1}\Big) + n\mathbb{P}\Big(\|\tilde{Z}_t\|^2 > \frac{pn\epsilon}{C_2}\Big) \\
&\leq n\Big(\frac{C_1}{pn^2\epsilon}\Big)^m \mathbb{E}\big[\|\tilde{Z}_t\|^{4m}\big] + n\Big(\frac{C_2}{pn\epsilon}\Big)^m \mathbb{E}\big[\|\tilde{Z}_t\|^{2m}\big] \\
&\leq Cn^{1-m}.
\end{aligned}
$$

Taking $m > 2$, it can be seen that the assertion of the lemma follows. □

LEMMA S.6. *Let $\epsilon > 0$. If the assumptions of Theorem 7.1 are satisfied, then $\mathbb{P}(\max_{t\leq n}|d^{(2)}_{\tau,t}| > \epsilon)$ tends to zero at a rate faster than $\frac{1}{n}$.*

PROOF. Let $\epsilon > 0$ and define $\mathbf{G}_t = \boldsymbol{\psi}_t^*(\mathbf{I}+\mathbf{H}_t(z))^{-1}\mathbf{D}\tilde{\mathbf{R}}_t(z)\boldsymbol{\psi}_t$. Using the invariance of the trace under cyclic permutations, it follows that

$$
\mathbb{P}\big(|d^{(2)}_{\tau,t}| > \epsilon\big) \leq \mathbb{P}\big(|\tilde{\beta}_{\tau,t}| > \sqrt{\epsilon}vp^{1/3}\big) + \mathbb{P}\Big(\frac{1}{p^{2/3}}\big|\tilde{Z}_t^*\mathbf{G}_t\tilde{Z}_t - \mathrm{tr}[\mathbf{G}_t]\big| \geq \sqrt{\epsilon}\Big).
$$

Using the bounds on $\tilde{\beta}_{\tau,t}$ and $\mathbb{E}[\|\tilde{Z}_t\|^{2m}]$ in the proof of Lemma S.5, for large enough p, the first term on the right-hand side of the latter display is bounded by

$$
\mathbb{P}\big(\|\tilde{Z}_t\|^2 > C_1\sqrt{\epsilon}np^{1/3}\big) \leq \Big(\frac{1}{C_1\sqrt{\epsilon}np^{1/3}}\Big)^{2m}\mathbb{E}\big[\|\tilde{Z}_t\|^{4m}\big] \leq C_2 n^{-2m/3},
$$

which tends to zero faster than n^{-2} if $m > 3$. For the second term, notice that, using arguments from the proof of the previous lemma, $\|\mathbf{G}_t\| \leq |z|v^{-2}\bar{\lambda}^2_{\mathbf{A}}\bar{\lambda}_{\mathbf{D}}$ and that therefore $\mathrm{tr}[\mathbf{G}\mathbf{G}^*] \leq p\|\mathbf{G}_t\|^2 \leq Cn^{-1}$. Now, an application of Lemma B.26 in [6] implies that

$$
\begin{aligned}
\mathbb{P}\Big(\frac{1}{p^{2/3}}\big|\tilde{Z}_t^*\mathbf{G}_t\tilde{Z}_t - \mathrm{tr}[\mathbf{G}_t]\big| \geq \sqrt{\epsilon}\Big) &\leq \frac{1}{\epsilon^m p^{4m/3}}\mathbb{E}\big[\big|\tilde{Z}_t^*\mathbf{G}_t\tilde{Z}_t - \mathrm{tr}[\mathbf{G}_t]\big|\big] \\
&\leq Cn^{-m/3},
\end{aligned}
$$

which converges to zero faster than n^{-2} if $m > 6$. □

LEMMA S.7. *Let $\epsilon > 0$. If the assumptions of Theorem 7.1 are satisfied, then $\mathbb{P}(\max_{t\leq n}|d^{(3)}_{\tau,t}| > \epsilon)$ tends to zero at a rate faster than $\frac{1}{n}$.*

PROOF. Note first that, for some positive constant C,

$$
\mathrm{tr}\big[\tilde{\mathbf{R}}_t(z)\mathcal{H}_t\big((\mathbf{I}+\mathbf{H}_t(z))^{-1} - (\mathbf{I}+\mathbf{H}(z))^{-1}\big)\mathbf{D}\big]
$$

$$\leq \; p\big\|\tilde{\mathbf{R}}_t(z)\mathcal{H}_t\big((\mathbf{I}+\mathbf{H}_t(z))^{-1}-(\mathbf{I}+\mathbf{H}(z))^{-1}\big)\mathbf{D}\big\| \; \leq \; C,$$

using estimates obtained in the proofs of Lemmas S.5 and S.6. Therefore, $|d_{\tau,t}^{(3)}| \leq C|\tilde{\beta}_{\tau,t}|n^{-1}$. To show the assertion of this lemma, it suffices then to show that $n^{-2}\mathbb{P}(|\tilde{\beta}_{\tau,t}| > n\epsilon) \to 0$. The arguments are similar to the ones already established in Lemma S.6, where it was shown that $n^{-2}\mathbb{P}(|\tilde{\beta}_{\tau,t}| > n^{1/3}\epsilon) \to 0$. Details are hence omitted. □

LEMMA S.8. *Let $\epsilon > 0$. If the assumptions of Theorem 7.1 are satisfied, then $\mathbb{P}(\max_{t\leq n}|d_{\tau,t}^{(4)}| > \epsilon)$ tends to zero at a rate faster than $\frac{1}{n}$.*

PROOF. Recognizing that $\mathrm{tr}\big[\big(\tilde{\mathbf{R}}_t(z)-\tilde{\mathbf{R}}(z)\big)\mathcal{H}_t(\mathbf{I}+\mathbf{H}(z))^{-1}\mathbf{D}\big] \leq C$ for some $C > 0$, depending on z, the proof is similar to the one of Lemma S.7. □

LEMMA S.9. *Let $\epsilon > 0$. If the assumptions of Theorem 7.1 are satisfied, then $\mathbb{P}(\max_{t\leq n}|d_{\tau,t}^{(5)}| > \epsilon)$ tends to zero at a rate faster than $\frac{1}{n}$.*

PROOF. Recall that $\bar{\beta}_{\tau,t} = (1+n^{-1}\gamma_{\tau,t}\mathrm{tr}[\tilde{\mathbf{R}}_t(z)\mathcal{H}_t])^{-1}$. Write

$$d_{\tau,t}^{(5)} = \frac{1}{zp}\gamma_{\tau,t}\big([\tilde{\beta}_{\tau,t}-\bar{\beta}_{\tau,t}]+[\bar{\beta}_{\tau,t}-\beta_{\tau,t}]\big)\mathrm{tr}\big[\tilde{\mathbf{R}}(z)\mathcal{H}_t(\mathbf{I}+\mathbf{H}(z))^{-1}\mathbf{D}\big].$$

Observe first that $p^{-1}\mathrm{tr}[\tilde{\mathbf{R}}(z)\mathcal{H}_t(\mathbf{I}+\mathbf{H}(z))^{-1}\mathbf{D}] \leq \|\tilde{\mathbf{R}}(z)\mathcal{H}_t(\mathbf{I}+\mathbf{H}(z))^{-1}\mathbf{D}\| \leq C$ for some $C > 0$. Let $\mathbf{G}_t = \boldsymbol{\psi}_t^*\tilde{\mathbf{R}}_t(z)\boldsymbol{\psi}_t$. Then,

$$\begin{aligned}|\tilde{\beta}_{\tau,t}-\bar{\beta}_{\tau,t}| &= \Big|\tilde{\beta}_{\tau,t}\bar{\beta}_{\tau,t}\gamma_{\tau,t}\frac{1}{n}\big(\mathrm{tr}[\mathbf{G}_t]-\tilde{Z}_t^*\mathbf{G}_t\tilde{Z}_t\big)\Big| \\ &\leq C|\tilde{\beta}_{\tau,t}|\frac{1}{n}\big|\tilde{Z}_t^*\mathbf{G}_t\tilde{Z}_t-\mathrm{tr}[\mathbf{G}_t]\big|,\end{aligned}$$

where the right-hand side can now be estimated along the lines of the arguments used in the proof of Lemma S.6, noting that $\|\mathbf{G}_t\|$ is uniformly bounded. The term involving $|\bar{\beta}_{\tau,t}-\beta_{\tau,t}|$ can be handled similarly. □

S1.2. *Completing the proofs of Theorems 7.2 and 7.3.*

LEMMA S.10. *Let the assumptions of Theorem 7.2 be satisfied. Then, for any $\nu_1,\nu_2 \in [0,2\pi]$,*

$$\|\mathcal{H}(\mathbf{A},\nu_1)-\mathcal{H}(\mathbf{A},\nu_2)\| \leq 2\sqrt{2}\bar{\lambda}_{\mathbf{A}}\bar{\lambda}'_{\mathbf{A}}|\nu_1-\nu_2|.$$

For a fixed $z \in \mathbb{C}^+$, the two function families $\tilde{K}_\tau(z,\nu,\omega)$ and $\bar{d}(z,\nu,\omega)$ are uniformly equicontinuous in ν (viewing p and ω as parameters).

PROOF. Let $\mathcal{H}_i = \mathcal{H}(\mathbf{A}, \nu_i)$ and $\boldsymbol{\psi}_i = \boldsymbol{\psi}(\mathbf{A}, \nu_i)$, $i = 1, 2$. Since $\mathcal{H}_i = \boldsymbol{\psi}_i \boldsymbol{\psi}_i^*$, it follows that

$$\|\mathcal{H}_1 - \mathcal{H}_2\| \le \|\boldsymbol{\psi}_1(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2)^*\| + \|(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2)\boldsymbol{\psi}_2^*\|.$$

Therein, $\|\boldsymbol{\psi}_i\| \le \bar{\lambda}_{\mathbf{A}}$ and

$$\begin{aligned}\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\| &= \Big\| \sum_{\ell=0}^{q} (e^{i\ell\nu_1} - e^{i\ell\nu_2}) \mathbf{A}_\ell \Big\| \le \sqrt{2} |\nu_1 - \nu_2| \sum_{\ell=0}^{q} \ell \|\mathbf{A}_\ell\| \\ &\le \sqrt{2} |\nu_1 - \nu_2| \sum_{\ell=0}^{q} \ell \bar{\lambda}_{\mathbf{A}} \le \sqrt{2} |\nu_1 - \nu_2| \bar{\lambda}'_{\mathbf{A}}.\end{aligned}$$

It follows that $\|\mathcal{H}_1 - \mathcal{H}_2\| \le 2\sqrt{2}\bar{\lambda}_{\mathbf{A}}\bar{\lambda}'_{\mathbf{A}}|\nu_1 - \nu_2|$, which is the first assertion of the lemma. For the second, note that

$$\begin{aligned}\big|\tilde{K}_\tau(z, \nu_1, \omega) - \tilde{K}_\tau(z, \nu_2, \omega)\big| &= \frac{1}{p}\Big|\operatorname{tr}\big[\tilde{\mathbf{R}}(z, \omega)(\mathcal{H}_1 - \mathcal{H}_2)\big]\Big| \\ &\le \|\tilde{\mathbf{R}}(z, \omega)(\mathcal{H}_1 - \mathcal{H}_2)\| \\ &\le C(v)\|\nu_1 - \nu_2\|,\end{aligned}$$

where $C(v) = 2\sqrt{2}\bar{\lambda}_{\mathbf{A}}\bar{\lambda}'_{\mathbf{A}} v^{-1}$. The third assertion of the lemma follows in the same fashion using the bound $\|(\mathbf{I} + \mathbf{H}(z))^{-1}\| \le |z| v^{-1}$. □

LEMMA S.11. *Let the assumptions of Theorem 7.2 be satisfied. The following statements hold true.*

(a) For any $\omega \in \Omega_0$ and $z \in \mathbb{C}^+_{\mathbb{Q}}$, $\bar{d}(z, \nu, \omega) \to 0$ uniformly in ν.

(b) For any $\omega \in \Omega_0$ and $z \in \mathbb{C}^+$, there is a subsequence $\{p_\ell\}$ such that $\tilde{K}_\tau^{(p_\ell)}(z, \nu, \omega)$ converges uniformly to a continuous function of ν. Moreover, the limit of every uniformly convergent subsequence of $\tilde{K}_\tau(z, \nu, \omega)$ satisfies (2.5).

PROOF. (a) Observe that $\bar{d}(z, \nu, \omega) \to 0$ pointwise as a function of $\nu \in [0, 2\pi]_{\mathbb{Q}}$ holding $\omega \in \Omega_0$ and $z \in \mathbb{C}^+_{\mathbb{Q}}$ fixed. Further, Lemma S.10 implies that $\bar{d}(z, \nu, \omega)$ are uniformly equicontinuous in ν. Therefore, $\bar{d}(z, \nu, \omega) \to 0$ uniformly on $[0, 2\pi]$.

(b) Fix $\omega \in \Omega_0$ and $z \in \mathbb{C}^+$. Then $|\tilde{K}_\tau(z, \nu, \omega)| \le \bar{\lambda}^2_{\mathbf{A}} v^{-1}$. Lemma S.10 and the Arzela–Ascoli theorem imply that there is a subsequence $\{p_\ell\}$ along which $\tilde{K}_\tau(z, \nu, \omega)$ converges uniformly to a function continuous in ν. This is the first part of the assertion. For the second, let $\{p_\ell\}$ be a subsequence along which $\tilde{K}_\tau(z, \nu, \omega)$ converges to a limit, say, $\mathfrak{K}_\tau(z, \nu, \omega)$ uniformly on $[0, 2\pi]$.

Note that the dependence of $\mathfrak{K}_\tau(z,\nu,\omega)$ on the subsequence is implicit to keep the notation simpler. In what follows, it will be shown that $\mathfrak{K}_\tau(z,\nu,\omega)$ satisfies (2.5), that is,

$$\mathfrak{K}_\tau(z,\nu,\omega) = \int [\mathfrak{U}_\tau(z,\lambda,\omega) - z]^{-1} h(\lambda,\omega) dF^{\mathbf{A}}(\lambda), \tag{S.4}$$

where $\mathfrak{U}_\tau(z,\lambda,\omega) = (2\pi)^{-1}\int_0^{2\pi} \beta_\tau(z,\nu)\gamma_\tau(\nu)h(\lambda,\nu)d\nu$. To this end, $\bar{d}(z,\nu,\omega)$ along the subsequence $\{p_\ell\}$ is utilized (with dependence on $\{p_\ell\}$ again implicit). Let $\nu_{t\ell} = 2\pi t n_\ell^{-1}$, with n_ℓ such that $p_\ell = p(n_\ell)$, and write

$$\bar{d}(z,\nu,\omega) = \tilde{K}_\tau^{(p_\ell)}(z,\nu,\omega) - \int [U_\tau^{(p_\ell)}(z,\lambda,\omega) - z]^{-1} h(\lambda,\nu) dF_{p_\ell}^{\mathbf{A}}(\lambda),$$

with $U_\tau^{(p_\ell)}(z,\lambda,\omega) = n_\ell^{-1}\sum_{t=1}^{n_\ell} \beta_{\tau,\ell}(z,\nu_{t\ell})\gamma_\tau(\nu_{t\ell})h(\lambda,\nu_{t\ell})$ and $\beta_{\tau,\ell}(z,\nu) = (1 + c_{p_\ell}\gamma_\tau(\nu)\tilde{K}_\tau^{(p_\ell)}(z,\nu,\omega))^{-1}$. Since $\bar{d}(z,\nu,\omega) \to 0$ uniformly on $[0,2\pi]$ it can be seen that the latter equation is a finite-sample analog of (S.4).

It will be shown next that $U_\tau^{(p_\ell)}(z,\lambda,\omega) \to \mathfrak{U}_\tau(z,\lambda,\omega)$ uniformly in λ as $p_\ell \to \infty$. Write

$$\begin{aligned}
U_\tau^{(p_\ell)}&(z,\lambda,\omega) - \mathfrak{U}_\tau(z,\lambda,\omega) \\
&= \frac{1}{n_\ell}\sum_{t=1}^{n_\ell} \big(\beta_{\tau,\ell}(z,\nu_{t\ell}) - \beta_\tau(z,\nu_{t\ell})\big)\gamma_\tau(\nu_{t\ell})h(\lambda,\nu_{t\ell}) \\
&\quad + \frac{1}{n_\ell}\sum_{t=1}^{n_\ell} \beta_\tau(z,\nu_{t\ell})\gamma_\tau(\nu_{t\ell})h(\lambda,\nu_{t\ell}) \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi} \beta_\tau(z,\nu)\gamma_\tau(\nu)h(\lambda,\nu)d\nu \\
&= \frac{1}{n_\ell}\sum_{t=1}^{n_\ell} \beta_{\tau,\ell}(z,\nu_{t\ell})\beta_\tau(z,\nu_{t\ell})\gamma_\tau^2(\nu_{t\ell}) \\
&\qquad \times \big(c\mathfrak{K}_\tau(z,\nu_{t\ell},\omega) - c_{p_\ell}\tilde{K}_\tau^{(p_\ell)}(z,\nu_{t\ell},\omega)\big)h(\lambda,\nu_{t\ell}) \\
&\quad + \frac{1}{n_\ell}\sum_{t=1}^{n_\ell} \beta_\tau(z,\nu_{t\ell})\gamma_\tau(\nu_{t\ell})h(\lambda,\nu_{t\ell}) \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi} \beta_\tau(z,\nu)\gamma_\tau(\nu)h(\lambda,\nu)d\nu.
\end{aligned} \tag{S.5}$$

As in the proof of Lemma S.5, it can be verified that $|\beta_{\tau,\ell}(z,\nu_{t\ell})\beta_\tau(z,\nu_{t\ell})| \le C$ for some $C > 0$ depending on z. Further,

$$\big|c\mathfrak{K}_\tau(z,\nu_{t\ell},\omega) - c_{p_\ell}\tilde{K}_\tau^{(p_\ell)}(z,\nu_{t\ell},\omega)\big|$$

$$\leq \quad \big|(c - c_{p_\ell})\mathfrak{K}_\tau(z,\nu_{t\ell},\omega)\big| + c_{p\ell}\big|\mathfrak{K}_\tau(z,\nu_{t\ell},\omega) - \tilde{K}_\tau^{(p_\ell)}(z,\nu_{t\ell},\omega)\big|,$$

where the right-hand side converges to zero uniformly in λ. Using $|h(\lambda,\nu)| \leq \bar{\lambda}_{\mathbf{A}}^2$, the first difference on the right-hand side of (S.5) goes to zero uniformly in λ. Fix $\nu_1,\nu_2 \in [0,2\pi]$. Then,

$$\begin{aligned}
&\beta_\tau(z,\nu_1)\gamma_\tau(\nu_1)h(\lambda,\nu_1) - \beta_\tau(z,\nu_2)\gamma_\tau(\nu_2)h(\lambda,\nu_2)\\
&\qquad = \beta_\tau(z,\nu_1)\beta_\tau(z,\nu_2)\big(\gamma_\tau(\nu_1)h(\lambda,\nu_1) - \gamma_\tau(\nu_2)h(\lambda,\nu_2)\big)\\
&\qquad\quad + c\beta_\tau(z,\nu_1)\beta_\tau(z,\nu_2)\gamma_\tau(\nu_1)\gamma_\tau(\nu_2)\\
&\qquad\qquad \times \big(\mathfrak{K}_\tau(z,\nu_2,\omega)h(\lambda,\nu_1) - \mathfrak{K}_\tau(z,\nu_1,\omega)h(\lambda,\nu_2)\big).
\end{aligned}$$

Note again that $|\beta_{\tau,\ell}(z,\nu_{t\ell})\beta_\tau(z,\nu_{t\ell})| \leq C$. Since both $h(\lambda,\nu)$ and $\tilde{K}_\tau(z,\nu,\omega)$ are uniformly bounded and equicontinuous in ν (with λ, p and ω as parameters), the right-hand side of the latter equation can be bounded by $\tilde{C}|\nu_1-\nu_2|$ for some suitable $\tilde{C} > 0$. It follows that the second difference of the right-hand side in (S.5) also converges to zero uniformly in λ.

Utilizing that $|(U_\tau^{(p_\ell)}(z,\lambda,\omega) - z)^{-1}|, |(\mathfrak{U}_\tau(z,\lambda,\omega) - z)^{-1}| < v^{-1}$, that $h(\lambda,\nu)$ is a bounded continuous function, and the weak convergence of $F_{p_\ell}^{\mathbf{A}}(\lambda)$ to $F^{\mathbf{A}}(\lambda)$, it follows that

$$\int [U_\tau^{(p_\ell)}(z,\lambda,\omega) - z]^{-1}h(\lambda,\nu)dF_{p_\ell}^{\mathbf{A}} \to \int [\mathfrak{U}_\tau(z,\lambda,\omega) - z]^{-1}h(\lambda,\nu)dF^{\mathbf{A}}$$

under (2.2) and $\mathfrak{K}_\tau(z,\nu,\omega)$ satisfies (2.5). The proof is complete. □

LEMMA S.12. *Let the assumptions of Theorem 7.2 be satisfied. Then, the ESDs of $\tilde{\mathbf{C}}_\tau$ is tight on Ω_0 as sequences in p.*

PROOF. This is proved by showing that as $n \to \infty$, $\|\tilde{\mathbf{C}}_\tau\|$ is almost surely bounded. Let $\mathbf{D}_\tau = (\tilde{\mathbf{L}}_\tau + \tilde{\mathbf{L}}_\tau^*)/2$. Then,

$$\begin{aligned}
\|\tilde{\mathbf{C}}_\tau\| &= \frac{1}{n}\bigg\|\bigg(\mathbf{Z} + \sum_{\ell=1}^q \mathbf{A}_\ell\mathbf{Z}\tilde{\mathbf{L}}^\ell\bigg)\mathbf{D}_\tau\bigg(\mathbf{Z} + \sum_{\ell=1}^q \mathbf{A}_\ell\mathbf{Z}\tilde{\mathbf{L}}^\ell\bigg)^*\bigg\|\\
&\leq \frac{1}{n}\bigg\|\bigg(\mathbf{Z} + \sum_{\ell=1}^q \mathbf{A}_\ell\mathbf{Z}\tilde{\mathbf{L}}^\ell\bigg)\bigg\|^2 \leq \bigg\|\frac{1}{n}\mathbf{Z}\mathbf{Z}^*\bigg\|\bigg(\sum_{\ell=0}^q \bar{\lambda}_{\mathbf{A}_\ell}\bigg)^2,
\end{aligned}$$

which follows from the fact that $\|\mathbf{L}^\ell\| = 1$ for each $\ell \geq 1$. Now since the Z_{jt}'s are i.i.d. random variables with zero mean, unit variance and finite fourth moment, it follows that, as $n,p \to \infty$, $\|n^{-1}\mathbf{Z}\mathbf{Z}^*\|$ is bounded almost surely by $(1+\sqrt{c})^2 + \delta$ for any given $\delta > 0$. □

LEMMA S.13. *Suppose that $\{\mu_p\}$ is a tight sequence of Borel probability measures and that $\{s_p(z)\}$ is the corresponding sequence of Stieltjes transforms. If $s_p(z) \to s(z)$ for all $z = w + iv \in \mathbb{C}^+$, then $\lim_{v\to\infty} ivs(iv) = -1$. Therefore $s(z)$ is the Stieltjes transform of a Borel probability measure.*

PROOF. To verify that $ivs(iv) \to -1$ as $v \to \infty$, it is sufficient to verify that, for all $\epsilon > 0$, there is v_0 such that for all $v > v_0$ and for all p, $|ivs_p(iv)| \leq \epsilon$. To this end, note that, for $\lambda_0 > 0$,

$$\begin{aligned} ivs_p(iv) + 1 &= \int_{\{|\lambda|\leq\lambda_0\}} \frac{\lambda}{\lambda - iv} d\mu_p(\lambda) + \int_{\{|\lambda|>\lambda_0\}} \frac{\lambda}{\lambda - iv} d\mu_p(\lambda) \\ &\leq \frac{\lambda_0}{\lambda_0 + v} + \mu_p(\{|\lambda| > \lambda_0\}). \end{aligned}$$

Since μ_p is a sequence of Borel measures, λ_0 can be chosen so large that $\mu_p(\{|\lambda| > \lambda_0\}) < \epsilon/2$. Choosing v_0 such that $\lambda_0(\lambda_0+v)^{-1} < \epsilon/2$, shows that $ivs(iv) \to -1$ as $v \to \infty$. That $s(z)$ is a Stieltjes transform is a consequence of Lemma S.3. □

S1.3. *Bounds for partial derivatives.*

LEMMA S.14. *If the assumptions of Theorem 2.1 are satisfied, then (9.7) and (9.8) imply that (9.6) converges to zero under (2.2).*

PROOF. Elementary computations show that

$$\begin{aligned} &\partial^{(3)}_{j,t,1}\mathbf{R}(z) \\ &= -6\mathbf{R}(z)\big[\partial^{(1)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z)\big[\partial^{(1)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z)\big[\partial^{(1)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z)\big[\partial^{(1)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z) \\ &\quad + 3\mathbf{R}(z)\big[\partial^{(2)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z)\big[\partial^{(1)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z) \\ &\quad + 3\mathbf{R}(z)\big[\partial^{(1)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z)\big[\partial^{(2)}_{j,t,1}\mathbf{C}_\tau\big]\mathbf{R}(z), \end{aligned}$$

where

$$\begin{aligned} \partial^{(1)}_{j,t,1}\mathbf{C}_\tau &= \frac{1}{2n}\sum_{\ell\in\mathcal{I}_+(t)} \big(\mathbf{A}_\ell e_j X_{t+\tau+\ell} + X_{t+\tau+\ell} e_j^* \mathbf{A}_\ell\big) \\ &\qquad + \frac{1}{2n}\sum_{\ell\in\mathcal{I}_-(t)} \big(\mathbf{A}_\ell e_j X_{t-\tau+\ell} + X_{t-\tau+\ell} e_j^* \mathbf{A}_\ell\big), \\ \partial^{(2)}_{j,t,1}\mathbf{C}_\tau &= \frac{1}{n}\sum_{\ell\in\mathcal{G}_+(t)} \big(\mathbf{A}_\ell e_j e_j^* \mathbf{A}_{\ell+\tau}\big) + \frac{1}{n}\sum_{\ell\in\mathcal{G}_-(t)} \big(\mathbf{A}_{\ell-\tau} e_j e_j^* \mathbf{A}_\ell\big), \end{aligned}$$

with $\mathcal{I}_\pm(t)$ defined in Section 9.1 and $\mathcal{G}_\pm(t) = \mathcal{I}_\pm(t) \cap \{\ell\colon 0 \le \ell \pm \tau \le q\}$. Define $\mathbf{C}_\tau^{(k)} = \partial_{j,t,1}^{(k)} \mathbf{C}_\tau$. Then,

$$\begin{aligned}\left|\operatorname{tr}\left[\partial_{j,t,1}^{(3)}\mathbf{R}(z)\right]\right| \le & 6\left|\operatorname{tr}\left[\mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)\right]\right| \\ &+ 3\left|\operatorname{tr}\left[\mathbf{R}(z)\mathbf{C}_\tau^{(2)}\mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)\right]\right| \\ &+ 3\left|\operatorname{tr}\left[\mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)\mathbf{C}_\tau^{(2)}\mathbf{R}(z)\right]\right| \\ =& 6T_{\tau,1}(z) + 3T_{\tau,2}(z) + 3T_{\tau,3}(z).\end{aligned}$$

Let $\mathbf{Q}_{\tau,1}(z) = \mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)$ and $\mathbf{Q}_{\tau,2}(z) = \mathbf{C}_\tau^{(1)}\mathbf{R}(z)\mathbf{C}_\tau^{(1)}\mathbf{R}(z)$. Since we have that $\operatorname{rank}(\mathbf{C}_\tau^{(k)}) \le 4k^{-1}(q+1)$ for $k = 1, 2$, it follows that

$$\begin{aligned}T_{\tau,1} &\le \|\mathbf{Q}_{\tau,1}(z)\|\|\mathbf{Q}_{\tau,2}(z)\| \min\{\operatorname{rank}(\mathbf{Q}_{\tau,1}(z)), \operatorname{rank}(\mathbf{Q}_{\tau,2}(z))\} \\ &\le \frac{4(q+1)}{v^4}\|\mathbf{C}_\tau^{(1)}\|^3 \\ &\le \frac{4(q+1)}{v^4 n^3}\left(\sum_{\ell\in\mathcal{I}_+(t)}\|\mathbf{A}_\ell\|\|X_{t+\tau+\ell}\| + \sum_{\ell\in\mathcal{I}_-(t)}\|\mathbf{A}_\ell\|\|X_{t-\tau+\ell}\|\right).\end{aligned}$$

By similar arguments, for $k = 2, 3$,

$$\begin{aligned}T_{\tau,k}(z) \le \frac{2(q+1)}{v^3 n^2}&\left(\sum_{\ell\in\mathcal{I}_+(t)}\|\mathbf{A}_\ell\|\|X_{t+\tau+\ell}\| + \sum_{\ell\in\mathcal{I}_-(t)}\|\mathbf{A}_\ell\|\|X_{t-\tau+\ell}\|\right) \\ &\times\left(\sum_{\ell\in\mathcal{G}_+(t)}\|\mathbf{A}_\ell\|\|\mathbf{A}_{\ell+\tau}\| + \sum_{\ell\in\mathcal{G}_-(t)}\|\mathbf{A}_\ell\|\|\mathbf{A}_{\ell-\tau}\|\right).\end{aligned}$$

From this, the assertion of the lemma follows. □

LEMMA S.15. *If the assumptions of Theorem 2.1 are satisfied, then*

$$\frac{q+1}{n^2}\mathbb{E}\left[|Z_{jt}^R|^3\bar{\lambda}_{\mathbf{A}}\left(\sum_{\ell=-q,\ell\ne\mp\tau}^{q}\bar{\lambda}_\ell^{\mathrm{rem}}\|Z_{t\pm\tau+\ell}\| + \bar{\lambda}_{\pm\tau}^{\mathrm{rem}}\left(\|V_t^0\| + |Z_{jt}^R|\right)\right)^3\right] \to 0,$$

under (2.2), *where* $\bar{\lambda}_\ell^{\mathrm{rem}} = (\sum_{\ell'=|\ell|}^q \bar{\lambda}_{\mathbf{A}_{\ell'}}^2)^{1/2}$. *This statement holds even when* $q = q(p) \to \infty$ *such that* $q = O(p^{1/3})$.

PROOF. First, observe that $\bar{\lambda}_\ell^{\mathrm{rem}} \leq 2(1+|\ell|)^{-1}(\sum_{\ell'=|\ell|}^{q} \ell' \bar{\lambda}_{\mathbf{A}_{\ell'}})$ for all $|\ell| \leq q$. Next, since $(A+B)^3 \leq 3A^3 + 3B^3$, it holds that

$$\begin{aligned}
&\Bigg(\sum_{\ell=-q,\ell\neq\mp\tau}^{q} \bar{\lambda}_\ell^{\mathrm{rem}} \|Z_{t\pm\tau+\ell}\| + \bar{\lambda}_{\pm\tau}^{\mathrm{rem}} \left(\|V_t^0\| + |Z_{jt}^R| \right) \Bigg)^3 \\
&\qquad \leq 3 \Bigg(\sum_{\ell=-q,\ell\neq\mp\tau}^{q} \bar{\lambda}_\ell^{\mathrm{rem}} \|Z_{t\pm\tau+\ell}\| \Bigg)^3 + 9 \left(\bar{\lambda}_{\pm\tau}^{\mathrm{rem}} \|V_t^0\| \right)^3 + 9 \left(\bar{\lambda}_{\pm\tau}^{\mathrm{rem}} |Z_{jt}^R| \right)^3 .
\end{aligned} \tag{S.6}$$

Let $a = \max\{\mathbb{E}[|Z_{11}|^4], \mathbb{E}[|W_{11}^R + iZ_{11}^I|^4]\}$. Using that $\mathbb{E}[|Z_{jt}^R|^6]$ grows at most at the same rate as $p\mathbb{E}[|Z_{jt}^R|^4] \leq pa$, it follows that

$$\frac{q+1}{n^2} \mathbb{E}\Big[|Z_{jt}^R|^3 \left(\bar{\lambda}_{\pm\tau}^{\mathrm{rem}} |Z_{jt}^R| \right)^3 \Big] = O\left(\frac{qp}{n^2} \right) \to 0$$

under (2.2), where the $O(\cdots)$ term is independent of j and t, and the third term in (S.6) vanishes in the limit. Since $\mathbb{E}[|Z_{jt}^R|^3] \leq (\mathbb{E}[|Z_{jt}^R|^4])^{3/4} \leq a$, and $\mathbb{E}[\|V_t^0\|^3] \leq (\mathbb{E}[\|V_t^0\|^4])^{3/4} \leq ap^{3/2}$, for positive constants a that are independent of j and t but may differ from one application to another, by independence, it follows that

$$\frac{q+1}{n^2} \mathbb{E}\Big[|Z_{jt}^R|^3 \left(\bar{\lambda}_{\pm\tau}^{\mathrm{rem}} \|V_t^0\| \right)^3 \Big] = O\left(\frac{qp^{3/2}}{n^2} \right) \to 0$$

under (2.2). Thus the second term in (S.6) is asymptotically negligible. For the first term, note that

$$\begin{aligned}
&\Bigg(\sum_{\ell=-q,\ell\neq\mp\tau}^{q} \bar{\lambda}_\ell^{\mathrm{rem}} \|Z_{t\pm\tau+\ell}\| \Bigg)^3 \\
&= \sum_{\ell=-q,\ell\neq\mp\tau}^{q} \left(\bar{\lambda}_\ell^{\mathrm{rem}} \right)^3 \|Z_{t\pm\tau+\ell}\|^3 \\
&\quad + 3 \sum_{\ell=-q,\ell\neq\mp\tau}^{q} \sum_{\ell'=-q,\ell'\neq\mp\tau,\ell'\neq\ell}^{q} \left(\bar{\lambda}_\ell^{\mathrm{rem}} \right)^2 \|Z_{t\pm\tau+\ell}\|^2 \bar{\lambda}_{\ell'}^{\mathrm{rem}} \|Z_{t\pm\tau+\ell'}\| \\
&\quad + 3 \sum_{\ell=-q,\ell\neq\mp\tau}^{q} \sum_{\ell'=-q,\ell'\neq\mp\tau,\ell'\neq\ell}^{q} \bar{\lambda}_\ell^{\mathrm{rem}} \|Z_{t\pm\tau+\ell}\| \left(\bar{\lambda}_{\ell'}^{\mathrm{rem}} \right)^2 \|Z_{t\pm\tau+\ell'}\|^2 \\
&\quad + 6 \sum_{\ell=-q,\ell\neq\mp\tau}^{q} \sum_{\ell'=-q,\ell'\neq\mp\tau\ell'\neq\ell}^{q}
\end{aligned}$$

$$\times \sum_{\ell''=-q,\ell''\neq\mp\tau,\ell''\neq\ell,\ell''\neq\ell'}^{q} \bar{\lambda}_{\ell}^{\mathrm{rem}}\|Z_{t\pm\tau+\ell}\|\bar{\lambda}_{\ell'}^{\mathrm{rem}}\|Z_{t\pm\tau+\ell'}\|\bar{\lambda}_{\ell''}^{\mathrm{rem}}\|Z_{t\pm\tau+\ell''}\|.$$

Applying the relations $\mathbb{E}[\|Z_t\|^3] \leq (\mathbb{E}[\|Z_t\|^4])^{3/4} \leq ap^{3/2}$, $\mathbb{E}[\|Z_t\|^2] \leq ap$ and $\mathbb{E}[\|Z_t\|] \leq (\mathbb{E}[\|Z_t\|^2])^{1/2} \leq ap^{1/2}$ for some constant $a > 0$, independence of terms in the summand and the Z_{jt}'s, and $\bar{\lambda}_{\ell}^{\mathrm{rem}} \leq 2(1+|\ell|)^{-1}(\sum_{\ell'=|\ell|}^{q} \ell' \bar{\lambda}_{\mathbf{A}_{\ell'}}) \leq 2(1+|\ell|)^{-1}\bar{\lambda}'_{\mathbf{A}}$, it follows that

$$\frac{q+1}{n^2}\mathbb{E}\left[|Z_{jt}^R|^3\left(\sum_{\ell=-q,\ell\neq\mp\tau}^{q} \bar{\lambda}_{\ell}^{\mathrm{rem}}\|Z_{t\pm\tau+\ell}\|\right)^3\right] = O\left(\frac{(\log q)^3 qp^{3/2}}{n^2}\right) \to 0$$

under (2.2). This completes the proof. □

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