MAGNETIC RELUCTIVITY RELATIONSHIP

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ABSTRACT

An experimental and theoretical study of the magnetic properties of pure iron near saturation has led to the conclusion that the so-called reluctivity relationship of Kennelly does not truly represent the properties of pure and homogeneous materials, and that the constants in the reluctivity equation are without physical significance. The Kennelly formula and one recently proposed by Gokhale have practical value, each within its own range, for interpolation and extrapolation.

The conclusion that the constants in the reluctivity formula are without physical significance makes the problem of correlating the magnetic properties of materials with their other physical properties more difficult. A new basis for correlation must be sought.

CONTENTS

I. INTRODUCTION

In a recent paper [1] presented before the American Institute of Electrical Engineers Gokhale discusses the magnetic reluctivity relationship generally known as Kennelly's law and arrives at the conclusion that it has neither a sound theoretical foundation nor experimental justification. This question is of considerable importance, not only from the standpoint of magnetic theory but also from the point of view of the correlation of the magnetic properties of ferromagnetic materials with their other physical properties. The connection between these points of view is fairly close, for if the magnetic and other physical properties are as closely related as it now appears, any satisfactory theory of magnetism must take this relationship into account.

According to our present ideas of the effect of inhomogeneity on magnetic properties, Gokhale's experimental evidence does not appear to be conclusive. Moreover, his theoretical development is

[1] Gokhale, J. Am. Inst. Elec. Engs., **45**, p. 846; 1926.

based upon a fundamental assumption having no better physical foundation than that from which the Kennelly law is derived. In view of the importance of the question and the far-reaching effects on the problem of correlation, in case it should be necessary to discard the Kennelly law as a way of expressing the magnetic properties of a material in the region near saturation, it appeared to be desirable to make a further study of the subject. The results of such a study are here presented.

II. RELUCTIVITY RELATIONSHIP

The reluctivity relationship generally ascribed to Kennelly was developed independently by Kennelly[2] and by Fleming[3] from the assumption that "the permeability is proportional to the magnetizability"; that is,

$$\mu = a(S-B).$$

Kennelly recognized the fact that it is not the total induction B, but the intrinsic or ferric induction ($\beta = B-H$) which approaches a saturation value (S) and developed the reluctivity relationship on that basis.[4]

Starting, then, from the equation

$$\mu = a(S-\beta) \tag{1}$$

it is possible by simple algebraic transformation to throw the equation into various forms.

Substituting $\frac{\beta}{H}$ for μ and solving for β,

$$\beta = \frac{HaS}{1+aH} \tag{2}$$

If we put $\sigma = \frac{1}{S}$ and $\alpha = \frac{1}{aS}$ and substitute in equation (2)

$$\beta = \frac{H}{\alpha + \sigma H} \tag{3}$$

Since $\beta = \frac{H}{\rho}$ we can see at once that

$$\rho = \alpha + \sigma H \tag{4}$$

This is Kennelly's law as usually stated. σ is the reciprocal of the saturation value and α is a constant which determines the rate of approach toward saturation. The convenience of such an equation is obvious. It expresses the magnetic properties, in the neighbor-

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[2] Kennelly, Am. Inst. Elec. Eng. Trans., **8**, p. 485; 1891.

[3] Fleming, J. Inst. Elec. Eng., **15**, p. 570; 1886.

[4] The ferric induction β, or $B-H$, is 4π times what is generally called intensity of magnetization or magnetic polarization. The symbol μ is here used for ferric permeability or $\frac{\beta}{H}$ and is 4π times the magnetic susceptibility.

hood of saturation, in terms of two constants which are independent of the degree of magnetization, and, if the law should have any true physical significance, the constant α, which has been termed the coefficient of magnetic hardness, should have some relationship with the mechanical properties.

Gokhale clearly demonstrated the insensitive nature of the reluctivity curve and pointed out that, on account of this insensitivity, erroneous conclusions might earily be drawn.

If in equation (1) we solve for β

$$\beta = S - \frac{\mu}{a} \tag{5}$$

or, since

$$a = \frac{1}{\alpha S}$$

$$\beta = S(1 - \alpha\mu) \tag{6}$$

If the relationship holds, therefore, we should get a straight line with a negative slope by plotting β against μ. As Gokhale has

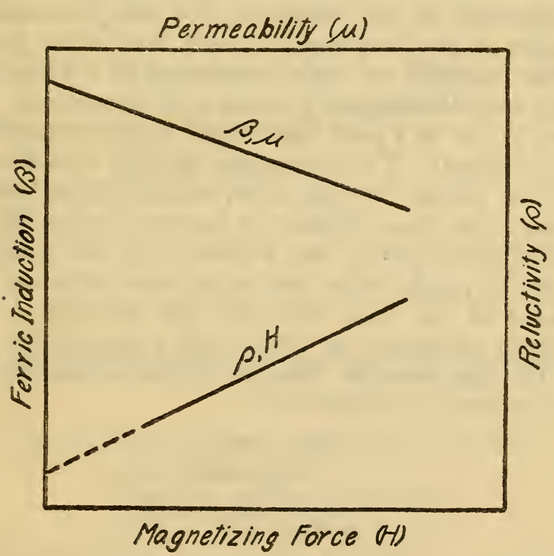

FIG. 1.—*Graphical representation of the reluctivity relationship*

pointed out, this is a much more sensitive test for the validity of the reluctivity relationship than the corresponding ρH curve.

The relationships represented by equations (4) and (6) are shown graphically in Figure 1. In view of the more sensitive nature of the $\beta\mu$ curve, this will be used in what follows rather than the corresponding ρH curve.

III. EFFECT OF INHOMOGENEITY ON RELUCTIVITY

If the Kennelly law is valid at all, it should hold only for magnetically homogeneous materials; for it is obvious that a material composed of two or more magnetically distinct components should not be expected to follow the simple law. It is not necessary that the different components be chemically distinct, because, as is well known, mechanical strain has a marked influence on magnetic properties, and therefore various portions even of a chemically homogeneous material which are in different conditions as regards mechanical strain might very well constitute separate and distinct magnetic components.

As might be expected from our knowledge of the nature of magnetic materials in general, experimentally determined $\beta\mu$ curves are seldom straight. It was with this point in mind that the statement was made that the experimental evidence presented by Gokhale was not conclusive, for with the possible exception of the pure iron none of his samples is even chemically homogeneous in the sense that it has only one magnetically distinct component. And, although the iron samples were carefully annealed, it is generally acknowledged that annealed materials are not necessarily free from mechanical strain.[5]

In the light of the foregoing it is clear that, if the properties of homogeneous materials are truly represented by a linear reluctivity relationship and inhomogeneity causes a deviation from the linear, the $\beta\mu$ curve should be a good index of the degree of magnetic homogeneity of a material. This conception has been used with apparent success by the author [6] for the interpretation of the results of an investigation of the effect of stress on magnetic properties.

In developing this idea it was assumed that the individual components were in parallel with each other, were subjected to a magnetizing force of the same intensity, and contributed to the resultant flux in proportion to their relative permeability and cross sections. On this basis the formula for the reluctivity of a two-component system is

$$\rho = \frac{\rho_1 \rho_2}{A_1\rho_2 + A_2\rho_1}$$

in which

$$\rho_1 = \alpha_1 + \rho_1 H$$

and

$$\rho_2 = \alpha_2 + \rho_2 H$$

represent the reluctivities of the components and A_1 and A_2 their relative cross-sectional areas. The resultant ρH curve is a hyperbola

[5] As a matter of fact the bar of electrolytic iron tested at the Bureau of Standards for which Gokhale gives data was examined for magnetic homogeneity along its length and was found to be markedly inhomogeneous. This is characteristic of materials which are magnetically very soft.

[6] Sanford, B. S. Sci. Paper No. 496; 1924.

approaching its asymptote from below and is consequently concave downward. The corresponding $\beta\mu$ curve is concave upward.

The curves experimentally obtained for hard materials are most frequently of this type. Those obtained for magnetically soft materials, however, are usually more complex, having a point of inflection, the $\beta\mu$ curve being first concave downward and then concave upward. The point of inflection generally comes at an induction slightly greater than 90 per cent of the saturation value. $\beta\mu$ curves for three different materials are given in Figure 2.

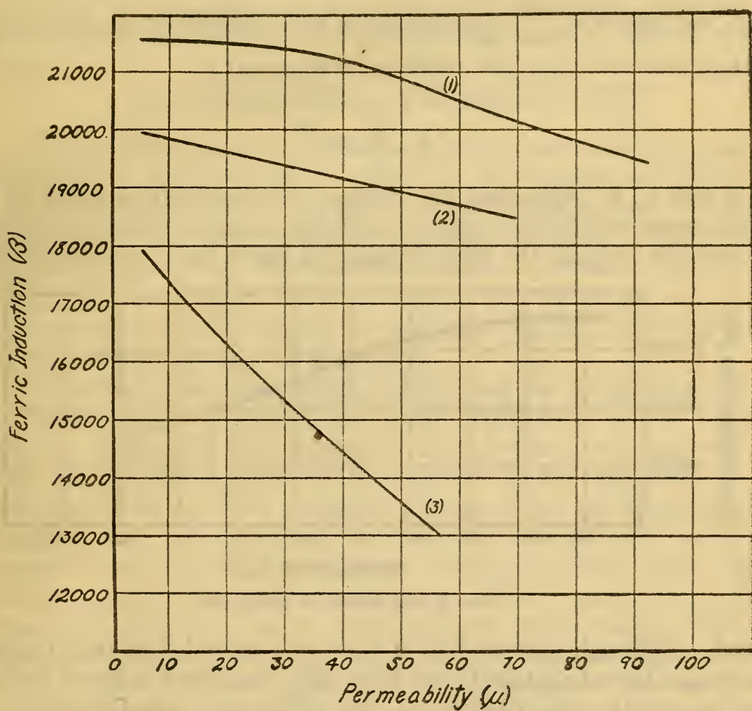

FIG. 2.—*$\beta\mu$ curves for three different types of material*

1. Pure iron, annealed.
2. High-carbon steel, quenched and drawn.
3. High-carbon steel, quenched.

An attempt was made to obtain by calculation a curve having the inflection observed for soft materials by assuming that some of the crystals in the interior of the material were in a state of compression and surrounded by other material in tension. In such a case we would have crystals of one component imbedded in another component having a different magnetic permeability. Under this condition the magnetizing force acting on the imbedded component is not equal to the impressed magnetizing force. If we assume as a first approximation that the imbedded component is in the form of spheres not touching each other, the magnetizing force is

$$H' = H\frac{3\rho_2}{2\rho_2 + \rho_1}$$

where ρ_1 is the reluctivity of the outer component and ρ_2 is the true reluctivity of the imbedded component.

Since we can not measure H' but only the impressed field H, the apparent reluctivity will be different from the true value.

If we take $R = \frac{3\rho_2}{2\rho_2 + \rho_1}$ as a variable ratio depending on H, the apparent reluctivity for the imbedded component is

$$\rho'_2 = \frac{\alpha 2}{R} + \rho_2 H$$

If $\alpha_1 > \alpha_2$, R approaches the limiting value unity from below as H is increased indefinitely and the resulting apparent reluctivity curve is concave upward, the corresponding $\beta\mu$ curve being concave down-

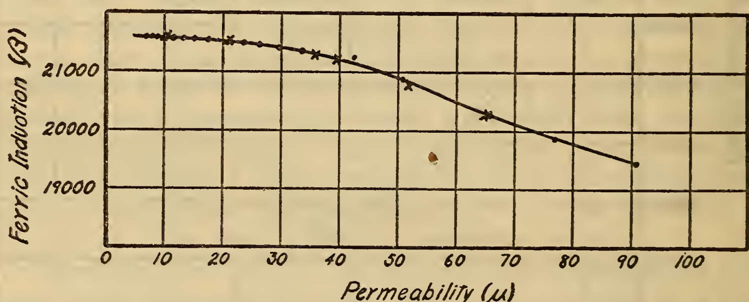

FIG 3.—$\beta\mu$ *curve for pure iron*

ward. Although a great deal of time was expended in the attempt to obtain by calculation based upon this conception a curve having the form of the experimental one, the results were unsatisfactory.

The question as to the possible influence of internal strain was further investigated experimentally. A sample of vacuum-fused iron of exceptional purity, having 0.018 per cent carbon and negligible amounts of other impurities, was annealed from 900° C. in vacuo and the $\beta\mu$ curve determined. The results are indicated by the dots in Figure 3. This curve is characteristic of magnetically soft materials. If the downward concavity and inflection are caused by internal strain, it should be possible to modify the curve by suitable heat treatment. The sample was, therefore, subjected to various heat treatments and the magnetic measurements repeated after each treatment as follows: (1) 100° C. for 18 hours; (2) 250° C. for 30 minutes, cooled in the furnace; (3) quenched in water from

900° C.; (4) reheated to 500° C., held 1 hour at temperature and cooled in air; and (5) reheated to 775° C., held one-half hour at temperature and cooled in air. The $\beta\mu$ curve obtained after each of these treatments was identical with the original curve within the experimental error. The crosses in Figure 3 indicate the results for the quenched condition. Since it is well known that mechanical strain affects the magnetic properties, it seems safe to conclude that this material was originally free from internal strain and that the downward concavity is not due to strain. If this be true the Kennelly law can not represent the magnetic properties of homogeneous and strain-free materials and the constant has no physical significance.

IV. THEORETICAL CONSIDERATIONS

It is not within the scope of this paper to discuss in detail the various theories of ferromagnetism. There is no general agreement at present on this subject. Moreover, no theory has yet been developed from either the mathematical or the physical point of view which agrees quantitatively with observed data. Certain principles are fairly well established, however, and it may be of interest to see whether these can be utilized in predicting the form of the $\beta\mu$ curve in the region very near to saturation.

In the first place, it is generally agreed that the magnetic properties of ferromagnetic materials are to be attributed to the presence within the material of what may be termed elementary magnets, and that the process of magnetization consists in the orientation of the magnetic axes of these elementary magnets by the influence of a magnetic field. In accordance with modern theory these elementary magnets are composed of electrons rotating in fixed orbits which, therefore, may be termed current rings. These current rings, then, constitute ampere turns or magnetomotive force which, when the rings are properly oriented, unite with the impressed field in producing in space the condition recognized as magnetization. In other words, the process of magnetization is merely the making effective of the magnetomotive force already inherent in the material.

These current rings are probably concentrated into groups to a greater or lesser extent, according to the kind of material. In the neutral or unmagnetized condition the current rings in each group are considered to be oriented in such a fashion as to constitute in effect closed magnetic circuits having no external influence. With the application of a magnetizing field the current rings are oriented in such a way as to have a resultant in the direction of the applied field. It will be convenient to express the magnetization as the projection of the areas of the rings upon a surface perpendicular to the direction of the applied field with due regard for sign. The magnetization, β, is then the maximum or saturation value S, multiplied by the average sine of the angle between the planes of the current rings and the direction of the applied field.

$$\beta = S \sin \theta \tag{8}$$

The angle θ may be called the orientation angle.

The process of magnetization takes place in three more or less well-defined stages. In the first stage the bond between the rings in each group is relatively strong and the orientation angle increases at a comparatively slow rate. During this stage the orientation angle of approximately half of the rings is negative, the average of all the rings, however, having a small positive value. As the magnetizing field is increased the rings approach a condition of unstable equilibrium, until the rings in some of the groups begin to flop suddenly from negative to positive angles. At this point the second stage has begun. During the second stage group after group "flops" from part negative to all positive angles, these sudden and continuous changes probably giving rise to the well-known Barkhausen [7] effect.

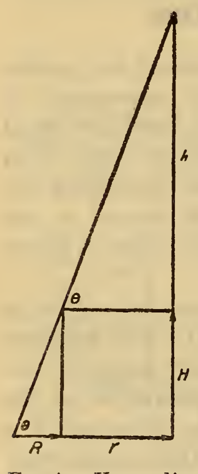

FIG. 4.—*Vector diagram showing relation of the components of the magnetic field acting upon an elementary magnet*

It is evident that within this range of instability the magnetization will be determined not only by the intensity of the magnetizing field, but also by previous magnetic history. Burrows [8] in the course of his investigation on demagnetization found that there is a critical point on the magnetization curve above which previous magnetic history has a negligible effect. It is probable that this critical value is the point at which the last group has "flopped" to all positive angles and that here the third stage of magnetization begins. In this stage the condition of unstable equilibrium has been passed and the application of a given magnetizing force results in a definite value of magnetization regardless of previous magnetic history.

It is difficult to treat quantitatively the first and second stages of magnetization, because the form of the curve is greatly modified by magnetic hysteresis for which no adequate theoretical explanation has been given. Within the third stage, however, the effect of hysteresis is relatively unimportant and it may be possible to predict the form of the magnetization curve.

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[7] Barkhausen, Phys. Zeitschr., **20**, p. 401; 1919.

[8] Burrows, B. S. Sci. Paper No. 78; 1908.

Let us consider the case of a single current ring whose orientation angle is equal to the average angle for all the rings and whose condition, therefore, represents the average for the material. This representative current ring will be oriented in such a direction that its axis is in the direction of the resultant field at that point. This resultant field is made up of two parts, the applied field H and the field resulting from the influence of the other current rings. These may be termed the applied and internal field, respectively. The internal field has two components, one parallel and the other at right angles to the external field. This condition is represented in the diagram of Figure 4, in which H represents the applied field, h the component of the internal field parallel to H, and $R+r$ represents the component of the internal field perpendicular to H. For convenience we may divide the perpendicular component into the two parts R and r, as indicated. θ is the orientation angle. From the similar right triangles thus formed the following relationships are evident:

$$(H+h)\cos\theta=(R+r)\sin\theta$$
$$h\cos\theta=r\sin\theta$$
$$H\cos\theta=R\sin\theta$$

R represents the part of the restoring field $(R+r)$ whose torque is not balanced by that of the component h of the internal field.

$$\frac{H}{R}=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

$$H=R\tan\theta$$

Since

$$\beta=S\sin\theta \tag{8}$$

$$\mu=\frac{\beta}{H}=\frac{\beta}{R\tan\theta}=\frac{S\cos\theta}{R}$$

or

$$R=\frac{S\cos\theta}{\mu}=\frac{S\,H\cos\theta}{\beta} \tag{9}$$

If we know S, β, and H, then R can be calculated. This is the effective restoring field tending to reduce β if the applied field H is removed. It is a complex function of β and H which is as yet undetermined.

If we should make the assumption that R is practically constant very near to saturation, where the angle θ changes very little as H

increases indefinitely, the $\beta\mu$ curve would take the form of S sin θ plotted against $\frac{S}{R}$ cos θ as shown in Figure 5, and should be concave downward and not straight as called for by the Kennelly law. If R should increase as the angle θ decreases, the curvature would be even more pronounced. It seems reasonable to believe that this may be the case, since at saturation all of the elementary magnets have an orientation angle of 90°, their fields being in the direction of H, and therefore have no perpendicular component. R would therefore be zero at saturation and increase as θ decreases.

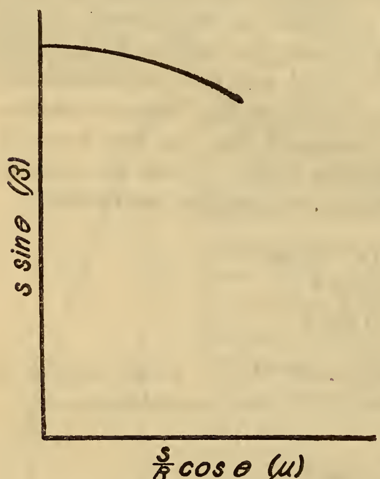

Fig. 5.—*Theoretical $\beta\mu$ curve based upon the assumption of a constant restoring field*

Although the value of R for any given value of β can not be predicted from theory,[9] it can be calculated from the relationship

$$R=\frac{S\cos\theta}{\mu}=\frac{S\,H\cos\theta}{\beta} \tag{9}$$

or, since

$$\cos\theta=\sqrt{1-\left(\frac{B}{S}\right)^2}=\frac{\sqrt{S^2-\beta^2}}{S}$$

$$R=\frac{\sqrt{S^2-\beta^2}}{\mu} \tag{10}$$

This has been done from the data of a number of tests on pure iron samples. It is interesting to note that the curve obtained is practically identical for a number of samples of pure iron tested by different methods and different investigators, as shown in Figure 6. We still await the discovery of the functional relationship between β and H before the form of the $R\beta$ curve can be predicted by theory.

It must be remembered that the $\beta\mu$ curves for materials having more than one magnetically distinct component must be combinations of the curves characteristic of the independent components. Since we have concluded that these characteristic curves can not have a linear form, the problem of correlating magnetic properties of materials with their other physical properties becomes more complicated than would be the case if magnetic constants independent of the degree of magnetization could be determined.

[9] Some modern theories would indicate that there are thermal or electrostatic influences as well as magnetic, but for the present purpose they may be all considered together as a single restoring force.

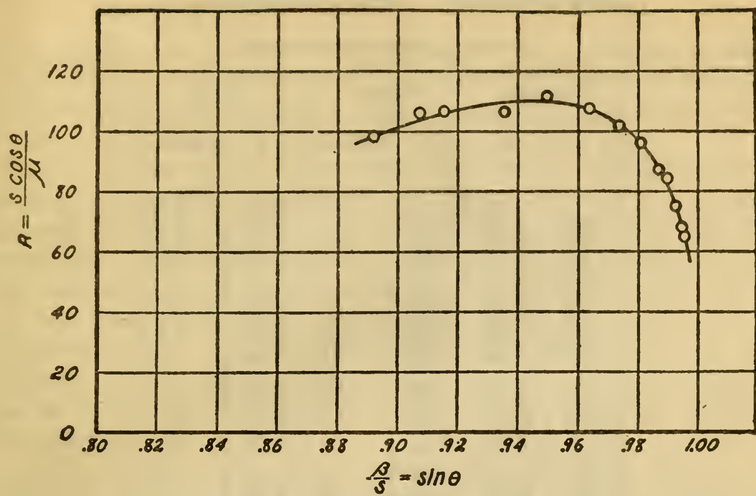

FIG. 6.—*Relation between effective restoring field and degree of magnetization*

V. PRACTICAL VALUE OF EMPIRICAL FORMULAS

It should not be concluded from the foregoing that the Kennelly formula has no practical value, but only that the constants have no definite physical significance. The constants of the Gokhale formula[10] are likewise devoid of physical meaning, but either formula may be used for interpolation to good advantage. A comparison between the two formulas is given in the data of Table 1 and Table 2. These results are fairly indicative of the variation to be expected in the range of materials from very soft to very hard. The choice of a formula will depend upon the conditions and desired results. The Kennelly formula is more convenient to handle and gives results within 1 or 2 per cent over a relatively wide range of magnetization when used for interpolation. It is not so good for extrapolation, however, as the error in the determination of the saturation value may be as great as 5 per cent or more.

The Gokhale formula, though not so convenient, is particularly good for the higher values of magnetization, the differences between observed and calculated values for magnetizing forces above 500 gilberts per centimeter being well within the experimental error. For extrapolation the formula is very good if data above 500 in H are available, and probably gives results well within the experimental error.

[10] The Gokhale formula is as follows: $\beta = S\,(1 - b\,e^{-aH})$, or in the more practical form $\log\,(S-\beta) = f - gH$ For detailed discussion and development refer to the original paper, J. Am. Inst. Elec. Eng., **45**, p. 846; 1926.

TABLE 1.—*Comparison of Kennelly and Gokhale formulas*

[Pure iron (C, 0.018 per cent) annealed]

Observation		Kennelly		Gokhale	
H	β	β	Difference	β	Difference
			Per cent		*Per cent*
99	18,020	17,250	4.3	21,020	−16.7
128	18,440	18,130	1.7	21,050	−14.2
157	18,810	18,720	.5	21,080	−12.1
214	19,480	19,460	.1	21,140	−8.5
258	19,860	19,820	.2	21,180	−6.6
307	20,280	20,120	.8	21,200	−4.5
412	20,890	20,530	1.2	21,290	−1.9
503	21,260	20,760	2.4	21,340	−.4
630	21,370	20,950	2.0	21,400	−.1
720	21,440	21,080	1.7	21,440	0
809	21,460	21,140	1.5	21,460	0
910	21,490	21,220	1.3	21,490	0
1,031	21,510	21,300	1.0	21,510	0
1,217	21,540	21,380	.7	21,540	0
1,411	21,570	21,450	.6	21,560	0
1,622	21,580	21,490	.4	21,570	0
1,831	21,570	21,530	.2	21,580	0
2,027	21,560	21,560	.0	21,590	−.1
2,227	21,560	21,590	−.1	21,590	−.1
2,413	21,590	21,590	0	21,600	0
2,621	21,600	21,600	0	21,600	0
2,801	21,570	21,610	−.2	21,600	−.1
3,014	21,570	21,630	−.3	21,600	−.1

Kennelly........... $\rho=0.00120+0.0000458H$, $S=21,830$
Gokhale..log $(S-\beta)=2.850 \quad -0.000880H$, $S=21,600$
Difference = 1.1 per cent.

TABLE 2.—*Comparison of Kennelly and Gokhale formulas*

[Hardened high-carbon steel (C, 0.85 per cent) quenched in water from 800°C.]

Observation		Kennelly		Gokhale	
H	β	β	Difference	β	Difference
			Per cent		*Per cent*
232	13,020	12,690	2.5	13,700	−5.2
318	14,080	13,850	1.6	14,260	−1.3
430	14,890	14,800	.6	14,880	+.1
535	15,430	15,400	.2	15,360	+.5
685	15,980	15,970	.1	15,920	+.4
860	16,440	16,420	.1	16,380	+.4
1,090	16,810	16,790	.1	16,920	−.7
1,380	17,190	17,130	.3	17,170	+.1
1,710	17,390	17,370	−.1	17,410	−.1
2,000	17,520	17,500	.1	17,520	+.0
2,420	17,730	17,650	.5	17,620	+.6
2,630	17,810	17,710	.6	17,640	+.9

Kennelly........... $\rho=0.00570+0.0000543\,H$, $S=18,420$
Gokhale..log $(S-\beta)=3.780 \quad -0.000767H$, $S=17,700$
Difference=4 per cent.

VI. SUMMARY AND CONCLUSIONS

To recapitulate briefly: The problem of the correlation between the magnetic and other physical properties would be much simplified if a formula were available for expressing the magnetic properties in terms of constants having a definite physical significance.

It has for some time been considered that the Kennelly law of magnetic reluctivity represented the magnetic properties of magnetically homogeneous materials, and it was hoped that the constants might have some definite physical significance. Gokhale has called this view into question and proposed a substitute formula. While his experimental evidence did not seem to be conclusive and his proposed formula has no better theoretical foundation than the Kennelly law, it appeared worth while to make a further study of the subject. Such a study was carried out with the following conclusions:

1. The characteristic curve between magnetization and permeability for pure materials near saturation has a point of inflection and double curvature, and is not straight as called for by the Kennelly relationship. This view is upheld by experimental evidence and theoretical considerations.

2. Since the reluctivity curve for pure and homogeneous materials is not straight, but only apparently so, on account of the insensitive nature of the curve, the true saturation value is not indicated by the reciprocal of the slope of the curve and the value of the intercept has no physical significance.

3. While the Gokhale formula fits the observed curve near saturation better than the Kennelly formula, it is developed from a basic assumption having no better physical foundation and it is improbable that its constants have a physical significance.

4. Both formulas are useful, each in its own range, for interpolation. The Gokhale formula is better for extrapolation, provided data are available for values of H greater than 500 gilberts per centimeter.

5. In the light of these conclusions the reluctivity relationship has a limited value, if any, in connection with the problem of the correlation between the magnetic and other physical properties of ferromagnetic materials, and a new basis of correlation must be sought.

WASHINGTON, October 12, 1926.