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On Involutions

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Two methods are described of constructing real functions over the reals which are one-to-one, assume every real value and are their own inverses, and several examples are given. It is also shown that such a function, if everywhere continuous, is either the function $f(x) \equiv x$ or else is strictly decreasing.

Key Words: Inverses, involutions, real functions.

1. We shall consider real functions f whose domain is the set of real numbers, which take on every real value, are one-to-one, and satisfy for every real x, $f^{-1}(x)=f(x)$, where f^{-1} is the inverse function of f. We denote by I the set of all such functions. Recall that functions which are their own inverses are called *involutions.*

Suppose that a real function f has as its domain the set of real numbers. Then it belongs to I if and only if

$$f(f(x))=x \text{ for every real } x. \tag{1}$$

Indeed, if $f\epsilon I$, then for every real x, $f(f(x))=f(f^{-1}(x))=x$. Conversely, if (1) holds, then f takes on every real value, is one-to-one (for $f(x_1)=f(x_2)$ implies $x_1=f(f(x_1))=f(f(x_2))=x_2$), and for every real x, $f^{-1}(x)=f(x)$.

Note that the graph of every f in I is symmetric in the line $y=x$.

Conversely, if G is a set in the x, y plane, symmetric in the line $y=x$ and containing, for every real x, a unique point whose abscissa is x, then G is the graph of a function belonging to I.

2. One way of obtaining functions in I is the following. Start with a real function $g(x, y)$ whose domain is the set of all ordered pairs of real numbers, and which is such that $g(x, y)=0$ implies $g(y, x)=0$. (This property holds, e.g., if g is symmetric, i.e., if for every real x, y, we have $g(y, x)=g(x, y)$.) Suppose that for every real x, there is a unique real y (to be denoted $f(x)$) such that $g(x, y)=0$. Then f (with domain the set of reals) belongs to I. Indeed, for every real x,

$$g(f(x), x)=g(x, f(x))=0,$$

and consequently $f(f(x))=x$.

EXAMPLE 1. Let $g(x, y)\equiv x+y-c$, c being an arbitrary real constant. We obtain from it the function $f(x)\equiv c-x$ belonging to I.

EXAMPLE 2. Let $g(x, y)\equiv x-y$. The corresponding $f\epsilon I$ is $f(x)\equiv x$.

EXAMPLE 3. Let $g(x, y)\equiv x^3+y^3-c$, c being an arbitrary real constant. We get from it the function $f(x)\equiv \sqrt[3]{c-x^3}\epsilon I$.

3. Another method of obtaining functions in I is based on the last paragraph of section 1. We illustrate this method by the following

EXAMPLE 4. In the X, Y plane consider the hyperbola $X^2-\frac{1}{2}Y^2=1$. Let R, L denote, respectively, its right-hand and left-hand branches. Consider now a new coordinate system, x, y, obtained from the X, Y system by a clockwise rotation of 45°. In the new coordinate system the equation of R (which is symmetric in the line $y=x$) is

$$y=-3x+2(2x^2+1)^{1/2}.$$

Thus, $f(x)\equiv -3x+2(2x^2+1)^{1/2}$ belongs to I. Similarly, the equation of L in the new coordinate system is

$$y=-3x-2(2x^2+1)^{1/2},$$

and consequently, $f(x)\equiv -3x-2(2x^2+1)^{1/2}$ belongs to I.

4. Consider the functions in I which are everywhere continuous. Since such a function takes on every real value exactly once, it must be, throughout the real line, either strictly increasing or strictly decreasing. For example, $f(x)\equiv x$ is a function in I which is strictly increasing. It is interesting to note that *all other everywhere continuous functions in I are strictly decreasing.* Indeed, let $F(x)(\not\equiv x)$ be an everywhere continuous function in I. Then its graph contains two points which do not lie on the line $y=x$, but which are symmetric in this line. Let (x_1, y_1), (x_2, y_2) (with $x_1<x_2$) be such points. Then $y_1>y_2$ (draw a figure!). So $F(x_1)>F(x_2)$, and consequently, F is strictly decreasing.

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5. Let us examine the smoothness of the various examples we have of functions belonging to I. The functions $c-x$ and x of Examples 1 and 2 are differentiable throughout the real line; in fact they are analytic at each real point. The function $f(x) \equiv \sqrt[3]{c-x^3}$ of Example 3 is everywhere continuous. If $c=0$, it reduces to $-x$. Otherwise, it is everywhere differentiable except at the point $x=\sqrt[3]{c}$, where it is not.

Let us now look at the functions f of Example 4. The function $2z^2+1$ of the complex variable z vanishes at $2^{-1/2}i$, $-2^{-1/2}i$ and nowhere else. Consequently, the real functions $-3x+2(2x^2+1)^{1/2}$, $-3x-2(2x^2+1)^{1/2}$ of Example 4, are analytic at every point of the x axis.

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