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Zeros of Polynomials in Several Variables and Fractional Order Differences of Their Coefficients

B. Mond* and O. Shisha*

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A classical result of Eneström (if $c_0, c_1, \ldots, c_n$ ($n \geqslant 1$) are not all zero and if they satisfy $c_0 \geqslant c_1 \ldots \geqslant c_n \geqslant 0$, then every zero ζ of $\sum_{k=0}^{n} c_k z^k$ satisfies $|\zeta| \geqslant 1$) is generalized to polynomials in several variables. A result in the same direction, involving fractional order differences of coefficients, is then established.

1. A Generalization of Eneström's Theorem

We begin with the following

THEOREM 1. *Consider a polynomial* $E(z_1, z_2, \ldots, z_p) \equiv \sum_{h_p=0}^{n_p} \ldots \sum_{h_1=0}^{n_1} c_{h_1 \ldots h_p} z_1^{h_1} \ldots z_p^{h_p}$ $(\not\equiv 0)$ *in the complex variables* $z_1, \ldots, z_p$. *Set* $c_{h_1 \ldots h_p} = 0$ *whenever the* h_ν *are integers but some* h_j *does not satisfy* $0 \leqslant h_j \leqslant n_j$.[1] *Suppose*

$$\nabla c_{h_1 \ldots h_p} = \sum_{j=1}^{p} c_{h_1 \ldots h_p} - c_{h_1 \ldots h_{j-1} h_j - 1 h_{j+1} \ldots h_p} = p c_{h_1 \ldots h_p} - \sum_{j=1}^{p} c_{h_1 \ldots h_{j-1} h_j - 1 h_{j+1} \ldots h_p} \leqslant 0 \quad (1)$$

$(h_\nu = 0, 1, \ldots, n_\nu + 1;\ \nu = 1, 2, \ldots, p;\ (h_1 \ldots h_p) \neq (0, \ldots 0))$. *If* $E(\zeta_1, \ldots, \zeta_p) = 0$, *then at least one* $|\zeta_j|$ is $\geqslant 1$.[2]

Proof. A straightforward computation yields

$$\left(p - \sum_{j=1}^{p} z_j\right) E(z_1, \ldots, z_p) \equiv \sum_{h_p=0}^{n_p+1} \ldots \sum_{h_1=0}^{n_1+1} (p c_{h_1 \ldots h_p} - c_{h_1 - 1 h_2 \ldots h_p} - \ldots$$

$$- c_{h_1 \ldots h_{p-1} h_p - 1}) z_1^{h_1} \ldots z_p^{h_p} \equiv p c_{0 \ldots 0} + \sum_{(h_1, \ldots, h_p) \epsilon \sigma} (\nabla c_{h_1 \ldots h_p}) z_1^{h_1} \ldots z_p^{h_p},$$

where σ is the set of all sequences of integers $(h_1, \ldots, h_p)$ with $0 \leqslant h_\nu \leqslant n_\nu + 1$, $\nu = 1, 2, \ldots, p$, $(h_1, \ldots h_p) \neq (0, \ldots 0)$. Setting $z_1 = z_2 = \ldots = z_p = 1$, we have $p c_{0 \ldots 0} = -\sum_{(h_1, \ldots, h_p) \epsilon \sigma} (\nabla c_{h_1 \ldots h_p})$, and thus $\left(p - \sum_{j=1}^{p} z_j\right) E(z_1, \ldots, z_p) \equiv \sum_{(h_1, \ldots, h_p) \epsilon \sigma} (\nabla c_{h_1 \ldots h_p})(z_1^{h_1} \ldots z_p^{h_p} - 1)$, from which we infer that $\nabla c_{h_1 \ldots h_p} \neq 0$ for at least one $(h_1 \ldots h_p) \epsilon \sigma$. If $|z_\nu| < 1$ for $\nu = 1, 2, \ldots, p$ then

*Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.
[1] Below, instead of repeating the last sentence, we shall merely place the sign * at the appropriate place.
[2] Thus, if, for example, $p = 2$, the curve $E(x, y) = 0$ in the x, y plane does not intersect the square $-1 < x < 1, -1 < y < 1$.

$\operatorname{Re}\left[\left(p-\sum_{j=1}^{p} z_j\right)E(z_1,\ldots,z_p)\right]=\sum_{(h_1,\ldots,h_p)\in\sigma}(\nabla c_{h_1\ldots h_p})\operatorname{Re}(z_1^{h_1}\ldots z_p^{h_p}-1)>0$, and so $E(z_1,\ldots,z_p)\neq 0$.

When $p=1$, Theorem 1 reduces to a result of Eneström [1][3] stating that if $c_0, c_1, \ldots, c_n (n\geqslant 1)$ are not all zero and if they satisfy $c_0\geqslant c_1\ldots\geqslant c_n\geqslant 0$, then every zero ζ of $\sum_{k=0}^{n} c_k z^k$ satisfies $|\zeta|\geqslant 1$.

2. Fractional Order Differences

The main purpose of this paper is to generalize a result of Cargo and Shisha [2, Theorem 1 and the end of IV] relating the zeros of a polynomial (in one variable) to fractional order differences of its coefficients.

Consider a multisequence $a_{h_1\ldots h_p}$ ($p\geqslant 1$ fixed, $h_\nu=\ldots-2,-1,0,1,2,\ldots$; $\nu=1,2,\ldots,p$). Let $\mathbf{I}a_{h_1\ldots h_p}\equiv a_{h_1\ldots h_p}$, and for $j=1,2,\ldots,p$, let $\mathbf{E}_j a_{h_1\ldots h_p}\equiv a_{h_1\ldots h_{j-1}h_j+1h_{j+1}\ldots h_p}$. Let α be a complex number. From (1), one is led to the following symbolic equalities

$$\nabla^\alpha=\left(p\mathbf{I}-\sum_{j=1}^{p}\mathbf{E}_j^{-1}\right)^\alpha=p^\alpha\left(\mathbf{I}-p^{-1}\sum_{j=1}^{p}\mathbf{E}_j^{-1}\right)^\alpha=p^\alpha\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}p^{-m}\left(\sum_{j=1}^{p}\mathbf{E}_j^{-1}\right)^m$$

$$=\sum_{m=0}^{\infty}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}\mathbf{E}_1^{-i_1}\ldots\mathbf{E}_p^{-i_p}.$$

Thus, we define

$$\nabla^\alpha a_{h_1\ldots h_p}=\sum_{m=0}^{\infty}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}a_{h_1-i_1\ldots h_p-i_p}$$

for every $(h_1,\ldots,h_p)$ for which the last infinite sum converges. If $a_{\nu_1\ldots\nu_p}=0$ whenever some ν_j is <0, and if $h_j\geqslant 0, j=1,2,\ldots,p$, then

$$\nabla^\alpha a_{h_1\ldots h_p}=\sum_{m=0}^{h_1+\ldots+h_p}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}a_{h_1-i_1\ldots h_p-i_p}.$$

THEOREM 2. *Consider a polynomial* $E(z_1,\ldots,z_p)\equiv\sum_{h_p=0}^{n_p}\ldots\sum_{h_1=0}^{n_1}c_{h_1\ldots h_p}z_1^{h_1}\ldots z_p^{h_p}\not\equiv 0)$ *in the complex variables* $z_1,\ldots,z_p$.* *Let* $0<\alpha<1$, *and suppose that* $c_{h_1\ldots h_p}\geqslant 0$ $(h_\nu=0,1,\ldots,n_\nu;\nu=1,2,\ldots,p)$, $\nabla^\alpha c_{h_1\ldots h_p}\leqslant 0$ $(h_\nu=0,1,\ldots n_\nu;\nu=1,2,\ldots,p;(h_1,\ldots,h_p)\neq(0,\ldots,0))$. *If* $E(\zeta_1,\ldots,\zeta_p)=0$, *then at least one* $|\zeta_j|$ *is* >1.

Proof. For $\left|\sum_{j=1}^{p} z_j\right|\leqslant p$ we have [4]

$$\left(1-p^{-1}\sum_{j=1}^{p}z_j\right)^\alpha E(z_1,\ldots,z_p)=\left[\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j p^{-j}\left(\sum_{\nu=1}^{p}z_\nu\right)^j\right]\sum_{j=0}^{\infty}\sum_{\substack{h_k\geqslant 0\\ \Sigma h_k=j}}c_{h_1\ldots h_p}z_1^{h_1}\ldots z_p^{h_p}$$

$$=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\left\{\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{-m}[i_1!\ldots i_p!]^{-1}z_1^{i_1}\ldots z_p^{i_p}\right\}\left\{\sum_{\substack{h_k\geqslant 0\\ \Sigma h_k=j-m}}c_{h_1\ldots h_p}z_1^{h_1}\ldots z_p^{h_p}\right\}$$

[3] Figures in brackets indicate the literature references at the end of this paper.

[4] By z^α, for a complex z, we mean the principal value of that power (0 if $z=0$).

$$=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m\\ q_k\geqslant i_k\\ \Sigma q_k=j}}(-1)^m m!\binom{\alpha}{m}p^{-m}[i_1!\ \ldots\ i_p!]^{-1}c_{q_1-i_1\ldots q_p-i_p}z_1^{q_1}\ \ldots\ z_p^{q_p}$$

$$=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{\substack{q_k\geqslant 0\\ \Sigma q_k=j}}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{-m}[i_1!\ \ldots\ i_p!]^{-1}c_{q_1-i_1\ldots q_p-i_p}z_1^{q_1}\ \ldots\ z_p^{q_p}$$

$$=\sum_{j=0}^{\infty}\sum_{\substack{q_k\geqslant 0\\ \Sigma q_k=j}}\sum_{m=0}^{\infty}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{-m}[i_1!\ \ldots\ i_p!]^{-1}c_{q_1-i_1\ldots q_p-i_p}z_1^{q_1}\ \ldots\ z_p^{q_p}$$

$$=\sum_{j=0}^{\infty}\sum_{\substack{q_k\geqslant 0\\ \Sigma q_k=j}}p^{-\alpha}(\nabla^{\alpha}c_{q_1\ldots q_p})z_1^{q_1}\ \ldots\ z_p^{q_p}.$$

Setting $z_1=\ldots=z_p=1$, we get $0=p^{-\alpha}\nabla^{\alpha}c_{0\ldots 0}+\sum_{j=1}^{\infty}\sum_{\substack{q_k\geqslant 0\\ \Sigma q_k=j}}p^{-\alpha}\nabla^{\alpha}c_{q_1\ldots q_p}$.

Thus, for $|\sum_{j=1}^{p}z_j|\leqslant p$ we have

$$\left(p-\sum_{j=1}^{p}z_j\right)^{\alpha}E(z_1,\ldots,z_p)=\sum_{j=1}^{\infty}\sum_{\substack{q_k\geqslant 0\\ \Sigma q_k=j}}(\nabla^{\alpha}c_{q_1\ldots q_p})(z_1^{q_1}\ldots z_p^{q_p}-1). \tag{2}$$

Let $q_1,\ldots,q_p$ be arbitrary integers $\geqslant 0$ (not all zero). Then $\nabla^{\alpha}c_{q_1\ldots q_p}\leqslant 0$. To see this we may assume that some q_j is $>m_j$. But then, noting that $(-1)^m\binom{\alpha}{m}<0$ for $m=1,2,\ldots$, we have

$$\nabla^{\alpha}c_{q_1\ldots q_p}=\sum_{m=1}^{q_1+\ldots+q_p}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ \ldots\ i_p!]^{-1}c_{q_1-i_1\ \ldots\ q_p-i_p}\leqslant 0.$$

From (2) we obtain, for $\left|\sum_{j=1}^{p}z_j\right|\leqslant p$,

$$\mathrm{Re}\left\{\left(p-\sum_{j=1}^{p}z_j\right)^{\alpha}E(z_1,\ \ldots,\ z_p)\right\}=\sum_{j=1}^{\infty}\sum_{\substack{q_k\geqslant 0\\ \Sigma q_k=j}}(\nabla^{\alpha}c_{q_1\ldots q_p})\,\mathrm{Re}\,(z_1^{q_1}\ \ldots\ z_p^{q_p}-1). \tag{3}$$

Suppose $E(\zeta_1,\ldots,\zeta_p)=0$, where all $|\zeta_j|$ are $\leqslant 1$. If $q_1,\ldots,q_p$ are non-negative integers (not all zero), then $\mathrm{Re}\,(\zeta_1^{q_1}\ \ldots\ \zeta_p^{q_p}-1)\leqslant 0$, and equality must hold if $\nabla^{\alpha}c_{q_1\ldots q_p}<0$. Let $j(1\leqslant j\leqslant p)$ be an integer. Since some $c_{h_1\ldots h_p}$ must be positive, we have

$$\nabla^{\alpha}c_{n_1\ldots n_{j-1}n_j+1n_{j+1}\ldots n_p}$$
$$=\sum_{m=1}^{n_1+\ldots+n_p+1}\sum_{\substack{i_k\geqslant 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}c_{n_1-i_1\ldots n_{j-1}-i_{j-1}\,n_j+1-i_j\,n_{j+1}-i_{j+1}\ \ldots\ n_p-i_p}<0,$$

and similarly $\nabla^{\alpha}c_{n_1\ldots n_{j-1}\ n_j+2n_{j+1}\ldots n_p}<0$. Thus

$\operatorname{Re}(\zeta_1^{n_1} \ldots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+1}\zeta_{j+1}^{n_{j+1}} \ldots \zeta_p^{n_p}-1)=\operatorname{Re}(\zeta_1^{n_1} \ldots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+2}\zeta_{j+1}^{n_{j+1}} \ldots \zeta_p^{n_p}-1)=0$, and so

$$\zeta_1^{n_1} \ldots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+1}\zeta_{j+1}^{n_{j+1}} \ldots \zeta_p^{n_p}=1=\zeta_1^{n_1} \ldots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+2}\zeta_{j+1}^{n_{j+1}} \ldots \zeta_p^{n_p},$$

which implies that $\zeta_j=1$. Thus $\zeta_1=\zeta_2=\ldots=\zeta_p=1$, and so $0=E(1,1,\ldots,1)$ $=\sum_{h_p=0}^{n_p} \ldots \sum_{h_1=0}^{n_1} c_{h_1 \ldots h_p}>0$, which proves the Theorem.

3. Remarks

We consider a polynomial $E(z_1,\ldots,z_p)\equiv\sum_{h_p=0}^{n_p}\ldots\sum_{h_1=0}^{n_1}\alpha_{h_1\ldots h_p}z_1^{h_1}\ldots z_p^{h_p}$ $(\not\equiv 0)$ in the complex variables $z_1,\ldots,z_p$.

A. Suppose the $\alpha_{h_1\ldots h_p}$ are complex. Let $r_1,\ldots,r_p$ be complex, non-zero numbers, and set $c_{h_1\ldots h_p}=\alpha_{h_1\ldots h_p}r_1^{h_1}\ldots r_p^{h_p}(h_\nu=0,1,\ldots,n_\nu;\ \nu=1,2,\ldots,p)$.* Assume $\nabla c_{h_1\ldots h_p}\leq 0$ $(h_\nu=0,1,\ldots,n_\nu+1;\ \nu=1,2,\ldots,p;\ (h_1,\ldots,h_p)\neq(0,\ldots 0))$. Suppose $E(\zeta_1,\ldots,\zeta_p)=0$. Then at least one $|\zeta_j|$ is $\geq|r_j|$. Indeed,

$$E(z_1,\ldots,z_p)\equiv\sum_{h_p=0}^{n_p}\ldots\sum_{h_1=0}^{n_1}c_{h_1\ldots h_p}\left(\frac{z_1}{r_1}\right)^{h_1}\ldots\left(\frac{z_p}{r_p}\right)^{h_p},$$

and therefore by Theorem 1, at least one $|\zeta_j/r_j|$ is ≥ 1.

As a simple example, take $p=1$, $n_1=n(\geq 1)$, and assume the α_ν are alternating in sign: $\alpha_\nu=(-1)^\nu|\alpha_\nu|(\nu=0,1,\ldots,n)$ and $|\alpha_0|\geq|\alpha_1|\ldots\geq|\alpha_n|$.

By taking $r_1=-1$, we obtain that every zero of the polynomial $\sum_{\nu=0}^{n}\alpha_\nu z^\nu$ has a modulus ≥ 1.

B. Suppose that $E(z_1,\ldots,z_p)$ is not a constant and that the $\alpha_{h_1\ldots h_p}$ are ≥ 0. Let

$$r=\min\left[\left(\sum_{\substack{j=1\\h_j\geq 1}}^{p}\alpha_{h_1\ldots h_{j-1}h_j-1h_{j+1}\ldots h_p}\right)\Big/p\alpha_{h_1\ldots h_p}\right]$$

where the minimum is taken over all sequences $(h_1,\ldots,h_p)$ with $0\leq h_\nu\leq n_\nu(\nu=1,2,\ldots,p)$, $(h_1,\ldots,h_p)\neq(0,\ldots,0)$, $\alpha_{h_1\ldots h_p}\neq 0$. Suppose $E(\zeta_1,\ldots,\zeta_p)=0$. Then at least one $|\zeta_j|$ is $\geq r$. Indeed, we may suppose $r>0$. Let $c_{h_1\ldots h_p}=\alpha_{h_1\ldots h_p}r^{h_1+\ldots+h_p}(h_\nu=0,1,\ldots,n_\nu;\ \nu=1,2,\ldots,p)$.* Let $0\leq h_\nu\leq n_\nu+1$, $\nu=1,2,\ldots,p$; $(h_1,\ldots h_p)\neq(0,\ldots,0)$. We shall show that $\nabla c_{h_1\ldots h_p}\leq 0$. We may clearly assume that all h_ν are $\leq n_\nu$ and that $\alpha_{h_1\ldots h_p}\neq 0$.

Then $\nabla c_{h_1\ldots h_p}=r^{h_1+\ldots+h_p-1}\left[rp\alpha_{h_1\ldots h_p}-\sum_{\substack{j=1\\h_j\geq 1}}^{p}\alpha_{h_1\ldots h_{j-1}h_j-1h_{j+1}\ldots h_p}\right]\leq 0$. By A, at least one $|\zeta_j|$ is $\geq r$.

4. References

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[2] Cargo, G. T., and Shisha, O., Zeros of polynomials and fractional order differences of their coefficients. J. Math. Analysis and Applications **7,** 176–182 (1963).

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