OPERATIONS RESEARCH AND DECISIONS

No. 1 2019

DOI: 10.5277/ord190102

Arnt O. HOPLAND[1]
Sturla F. KVAMSDAL[2]
Leif K. SANDAL[1]

AN ANALYSIS OF MAINTENANCE SCHEDULES FOR PUBLIC FACILITIES

We present a flexible, formal framework for maintenance scheduling for public facilities. Key features of the model include an accelerating deterioration scheme, a general utility measure, and real estate market effects in the salvage function. The model is rich enough to capture a range of stylistic scenarios pertaining to public facilities while remaining simple enough to allow formal analysis of the optimal maintenance schedule. Based on our analysis, we draw a phase diagram that classifies the generic behavior of the optimal solution. We illustrate our analysis in numerical examples that highlight essential trade-offs and the time dependence of the facility maintenance problem. Under simplifying assumptions, we also derive the basics of an exact solution.

Keywords: *maintenance, public facilities, optimization*

1. Introduction

The analysis of maintenance schedules for public facilities involves the deterioration-maintenance relation, a measure of utility with respect to building state, intertemporal substitution effects, a real-estate market, and presents an applied and complex decision problem. Maintenance of public facilities is important because facilities are significant inputs in the production of public services and public facilities represent vast amounts of capital. Importance notwithstanding, maintenance tends to lose budgetary battles against current service provision and new investments and may suffer dispropor-

[1]NHH Norwegian School of Economics, Helleveien 30, N-5045 Bergen, Norway.

[2]SNF – Centre for Applied Research at NHH, Helleveien 30, N-5045 Bergen, Norway, corresponding author, e-mail address: sturla.kvamsdal@snf.no

tionally when policy makers must cut public budgets. Public budget decisions are complex matters subject to a host of factors. In our view, the following are key factors related to maintenance: Facilities deterioration is slow and difficult to observe in the short run, say, within election cycles, and the understanding of optimal maintenance schedules is somewhat underdeveloped, such that effects of postponement or reduction in maintenance are misrepresented. Both factors underpin the lack of political – and perhaps managerial – will to prioritize maintenance.

Maintenance models and systems are broad issues in the engineering and management literatures. Common themes are the importance of maintenance to secure reliability and availability, and direct and indirect costs of poor maintenance. Direct costs include expedited and accelerated deterioration of physical capital; indirect costs include unnecessary or expedited replacement costs and loss of services and business. Fraser [8] provides an extensive and fairly recent review of maintenance models; Fraser et al. [9] reviews empirical studies. Much of the literature is concerned with maintenance in manufacturing and processing, where maintenance has become a strategic concern. But Fraser [8] notes that maintenance has become more strategically important also with regard to buildings and facilities. With technological development and innovative ideas about maintenance planning and organization, the extensive issue of maintenance has increased in complexity and strategic importance [5, 6].

We establish a simplified but flexible analytical framework to analyze optimal maintenance schedules for public facilities. We consider a governmental agent who produces a welfare service, for example schooling or elderly care, in a public purpose building. The quality or level of this service depends on the state of the facility in a simple way; generally such that a better state may provide better or more public service. The government seeks to maximize the utility provided to a representative voter, who gains positive utility from the welfare service and suffers negative utility from the cost of the service and maintenance costs. Further, we assume the nature of the deterioration process to be such that it accelerates the further the facility has deteriorated.

The need for purpose buildings and public facilities changes over time. Facilities may become obsolete with regard to public service provision, regardless of their technical state, for example because of new technical requirements or demographic changes. Facilities soon to become obsolete may not defend maintenance spending unless the service provision is critically dependent upon the state of the facility. Thus, the age or datedness of a facility are significant for the maintenance schedule. Further, decommissioned public facilities may be of interest to private real-estate developers who may or may not value the state of the facility. These factors, obsoleteness and the real-estate market, provides strategic motifs for the government in their maintenance scheduling and decommissioning of public facilities.

Our formal analysis enables us to draw a state-space phase diagram that describes general properties of the optimal maintenance schedule and the resulting development of the facility state. Broadly, the optimal maintenance schedule has a moderate level of maintenance effort when the facility is new. This effort is then scheduled to increase to prevent deterioration to accelerate too fast. Thus, the facility deteriorates relatively slowly. At some point however, as deterioration has started to take hold and the state of the facility is such that it is less valuable in the service production, the maintenance effort is scheduled to decrease and the deterioration is allowed to accelerate toward the point where the facility is decommissioned. Thus, we have two periods; the first signified by increasing maintenance efforts and the second signified by decreasing efforts. However, these findings are sensitive to various assumptions, in particular with regard to the real-estate market, but also to what we call the rate relationship (involving the discount rate and the rate of deterioration acceleration) and the shape of the service production function. Under different assumptions, the optimal maintenance effort may be increasing throughout the life cycle of the facility, or the facility may be kept in a constant state for parts of its service time. We explore some of these dynamics in numerical examples.

2. Literature review

Kamien and Schwartz [16], Bensoussan and Sethi [2], and others analyze machine maintenance, and our analysis has common features with this literature. Kamien and Schwartz [16] address the stochastic problem of optimal maintenance and sale age for a machine subject to stochastic failure by assuming that the failure probability is governed by a deterministic equation, and are thus able to formulate the problem as a deterministic problem. The probability of failure is the state variable, while maintenance and sale age are the control variables. The generic approach is subsequently applied to a range of problems; see Bensoussan and Sethi [2] for a brief review. Bensoussan and Sethi [2], rather than reducing the problem to a deterministic problem, address the underlying stochastic problem that potentially has a wider area of application than the original approach. A further review of the machine maintenance literature leads astray from our focus on maintenance of public facilities, thus we think it suffices to point out key features of interest in the forthcoming analysis.

As mentioned above, the engineering and management literatures on maintenance are extensive. The review by Fraser [8] highlights the four most popular maintenance models: total productive maintenance, reliability-centered maintenance, condition-based maintenance, and condition monitoring. Notably, in most of this literature, the notion of model is different than the formal notion we promote below. Most of the literature considers theoretical or principal issues, and the empirical evidence on applied maintenance models is more limited [9]. A common observation (see, for example, Cooke [6] and Fraser [8]) is that maintenance has become more complex and demanding with technological development, automation, and sophistication of equipment and procedures. More comprehensive health and safety regulations have added further to resource needs in maintenance. At the same time, ideas about maintenance systems and their organization, and further about coordination between maintenance and the overall business strategy, have progressed. Broadly, maintenance strategies have moved from reactive and preventive to proactive schemes [6]. Finally, that maintenance needs are somewhat unpredictable and difficult to budget adds complexity in its management [5]. All these developments have contributed towards maintenance becoming regarded as a strategic business issue in many sectors [9].

There exists a small but relevant literature on local public facilities. Borge and Hopland [3, 4] and Hopland and Kvamsdal [13] study maintenance and building conditions in Norwegian local governments and find that poor fiscal conditions and myopic political leadership are important predictors for low levels of maintenance and poor building conditions. Borge and Hopland [4] further provide a useful analogy between maintenance backlogs and debt: Both insufficient maintenance and debt mean that costs are postponed. If the deterioration process is accelerating, as we assume in the present analysis, postponing maintenance leads to higher maintenance requirements in the future. That is, a maintenance backlog incurs interests, not unlike debt. Hopland and Kvamsdal [13] observe negative correlations in building conditions over a relatively short period, and worry that this, together with further evidence, suggest that local governments spend too little on maintenance. A related study [14] report on a survey among public facility managers. The survey reveal concerns over weak economic conditions and lack of political will to prioritize facility management.

A common theme in many of the studies on local public facilities is that myopic behavior on behalf of politicians at least partly explains low levels of maintenance spending. Yet, studies that take a game-theoretic approach show that low levels of maintenance can be fully consistent with rational decision making in the local governments. Hopland [11] argues that local governments can postpone maintenance strategically in order to extract additional funds (bailouts) in the future. He thus argues that decay of public buildings is not necessarily a consequence of myopia or irrationality, but can result from rational behavior. Gauteplass and Hopland [10], however, contend that low capital expenditures in local governments could reflect that local governments do not fully take into account value of the services produced in their buildings because there are positive but unacknowledged spillover effects. Central grants to stimulate local capital expenditures may remedy the situation [10].

Hopland and Kvamsdal [12] promotes a model for cost-minimization of maintenance of local public buildings. The model assumes that service production is unaffected by building condition as long as the condition is above a given minimal threshold. The model further has no real-estate market, and the objective function is linear in the control. The simplified framework notwithstanding, Hopland and Kvamsdal [12] identify two key mechanisms of public maintenance. First, savings yield interests, providing incentives to postpone maintenance expenditures. Second, the accelerating deterioration process inflates an eventual maintenance backlog, which may offset the savings interests effect. This rate relationship, the relationship between the rate of discount (the interest rate) and the rate of accelerating deterioration, is central to maintenance scheduling.

While we are not aware of other studies taking the micro approach to optimal public maintenance, several theoretical studies have discussed the importance of maintenance at a macro level. Analyzing endogenous growth models Rioja [19, 20], Kalaitzidakis and Kalyvitis [15], and Agénor [1] emphasize the distinction between maintenance and the building of new infrastructure. These are similar to our study in the sense that they derive optimal policies based on rational actors. The policy purposes, on the other hand, are very different. Whereas the macro studies give guidance about optimal spending rules that a central government can take into account in national budgets, micro-level studies give guidance for how to schedule maintenance of existing buildings. The results show that fiscal variables are important determinants for both expenditures, and that maintenance seems to be somewhat more sensitive, at least to short-run fluctuations than investment. Further, whereas political fragmentation is associated with low levels of maintenance, it does not seem to affect investments.

Whereas maintenance of public facilities has received limited attention in the international literature, many have raised concerns about the more general issue of low levels of public investment. For example, the experiences in the OECD countries during the 1980s have received much attention; see, among others, Oxley and Martin [18], De Haan et al. [7] and Sturm [21]. Hopland and Kvamsdal [14] discuss the literature on public investments further.

3. A facility maintenance model

We formulate our model in continuous time. Let $x \geq 0$ denote the state of a given facility. We have $x = 1$ for a new facility, and this "as new" state is the highest possible facility state ($x \leq 1$). The facility deteriorates naturally according to a known function, $g(x) \geq 0$ but the deterioration can be reduced, halted, or reversed with maintenance, denoted $m \geq 0$. Following Hopland and Kvamsdal [12], we assume

$$g'(x) \leq 0, \quad x \in [0, 1] \tag{1}$$

That is, deterioration accelerates as x falls away from 1. In effect, the worse the state of a facility, the more maintenance is needed to keep it from further deterioration. We further have $g(1) > 0$, that is, a new facilitate naturally deteriorates without maintenance. To maintain $x \geq 0$, we have $g(0) = 0$. Deterioration and maintenance are the only variables affecting the facility state, such that

$$\dot{x} = m - g(x) \tag{2}$$

The dot-notation is shorthand for the time derivative. Maintenance, m, is the control or decision variable.

The facility is used for public service provision, and the quality (q) of the service depends on the state of the facility such that a better facility may provide a higher quality service. We write $q(x) \geq 0$ and $q'(x) \geq 0$. Maintenance costs, $c(m)$, depend on the level of maintenance and are strictly convex, $c'(m) > 0$, $c''(m) > 0$. There are certain fixed running costs, $c_F > 0$, such that $c(m) \geq c_F$ (thus, costs cannot go to zero). The objective of the governmental agent running the facility is to maximize service quality less of maintenance costs, thus we consider $Q(x, m) = q(x) - c(m)$ as the objective function in the optimization problem.

The facility has a salvage value that is obtained when the facility is taken out of use. The salvage value depends on the state, but also on time, and we write $S(x, t) \geq 0$. The salvage value increases with x such that $S(x, t) \geq 0$; the subscript denotes the partial derivative. The time dependence reflects a general appreciation of real estate values and can also represent that the facility gets outdated.

The facility is in public use from time 0 to T, where T is decided by the governmental agent. In practice, there is an upper limit on T because of changing technical requirements and limited durability of structural components. In our formal analysis, we abstract from any upper limit on the decommissioning time, however, the agent maximizes discounted returns (service quality less maintenance costs) plus salvage value. Thus, we consider the following criterion

$$\int_0^T \mathrm{e}^{-rt} Q(x, m) dt + \mathrm{e}^{-rt} S(X, T) \tag{3}$$

In Equation (3), r is the rate of discount and $X = x(T)$. The criterion in Eq. (3) is to be maximized with respect to stoppage time, $T > 0$, and the maintenance schedule, $m(t)$, $0 \leq t \leq T$. The maximization is subject to the dynamics, Eq. (2), the initial condition, $x(0) = x_0$, and further assumptions made above regarding $g(x)$, $Q(x, m)$, and $S(x, t)$. We also assume all involved functions to be continuous and sufficiently smooth. For a brand new facility, we have $x_0 = 1$, but we may consider any $0 < x_0 \leq 1$, if, for example, the government obtains an existing facility, wants to change the maintenance regime for an existing facility where the state is not "as new", or if the facility is built for a different usage than the one we model with $q(x)$. In our examples below, we consider both cases with $x_0 = 1$ and with $x_0 < 1$.

4. Analysis

As mentioned earlier, our model is related to that of Kamien and Schwartz [16] and subsequent research into machine maintenance. Thus, before we proceed with our main analysis, we develop an expression for the optimal decommissioning time T in the same manner as Kamien and Schwartz [16]. Let $V(T)$ denote the maximum of Eq. (3) if T is the time of decommissioning. We differentiate, obtaining $V'(T)$, and consider $V'(T) = 0$. After some manipulations, we get

$$Q(X, M) - rS(X, T) = -\left(S_t(X, T) + S_x(X, T)\, f(X, M)\right) \tag{4}$$

In Equation (4), $M = m(T)$ and $f(x, m) = m - g(x)$. Collected on the left hand side are gains and losses from postponing decommissioning slightly: Gained is utility from the quality of service less maintenance costs; lost is interest on the salvage value. Collected on the right hand side are the changes to the salvage value from a slight change in the time of decommissioning: Change simply from the passage of time (datedness, real estate market valuation), and change from changes to the state of the facility. Equation (4) shows that at the optimal decommissioning time T, gains and losses from postponing decommissioning just balance changes in the salvage value. The last term on the right in Eq. (4) is additional to the expression derived in Kamien and Schwartz [16] because in the classical machine model, the salvage value did not depend on x.

If, for example, the only time dependence of the salvage value is real-estate appreciation, we can write $S(x, t) = \mathrm{e}^{\alpha T}\sigma(x)$, where α is the rate of appreciation in real-estate values and $\sigma(x)$ is the relationship between salvage value and state. Then, Eq. (4) can be written as follows

$$\mathrm{e}^{\alpha T} = \frac{Q(X, M)}{(r-\alpha)\,\sigma(X) - \sigma'(X)\, f(X, M)}$$

If $r > \alpha$, that is, if the rate of discount is larger than the rate of appreciation in real-estate values, the optimal decommissioning time T is earlier than if $r < \alpha$. If $\sigma(x) = 0$, that is, there is no salvage value, the above equation yields either that the terminal (decommissioning) time T is infinitely large and where x approaches a steady state, or $Q(X, M) = 0$. Intuitively, decommissioning means that no further gains are possible when there is no salvage value. Thus, as long as there is a steady state with a positive utility flow, indefinite postponement of decommissioning is optimal. An effect of datedness in either the service quality or maintenance costs would likely yield a finite optimal decommission time. It is worth noting that cases with a constant real salvage value have equivalent analyses to the zero salvage case because a constant term can be scaled to zero. Such cases may be important as public facilities may have insignificant market values in themselves, for example because they often are highly specialized, but the land that they stand on may have a value that is constant in real terms.

It appears as somewhat of a paradox that indefinite postponement of decommissioning (that is, infinitely large T) is optimal for both small (near zero) and large (larger than r) values for the rate of appreciation in real-estate, while the optimal date for decommissioning is indeed finite for a mid-range of α-values. This apparent paradox can however be intuitively understood. The latter case is straightforward: when real estate appreciates fast (large α), it will at all times be optimal to postpone decommissioning, in particular if the facility state can be kept constant with a positive utility flow. Such steady states do indeed exist. The first case is in practice identical to a zero salvage value as discounting withers away any reasonable, initial salvage value. The zero salvage value case was discussed above.

A strong case can be made for $\alpha \leq r$ being reasonable because the discount rate r reflects alternative investments that include other real estate prospects.

The facility maintenance model poses a typical optimal control problem, and our analysis applies basic optimal control theory [17]. To solve for the optimal maintenance schedule, the optimal policy, we define the current value Hamiltonian

$$H\left(x, m, \lambda\right) = Q\left(x, m\right) + \lambda\left(m - g(x)\right) \tag{5}$$

where λ is the (current value) costate or shadow value of the state x. The optimal policy m^* must maximize H at all times, where the shadow value satisfies the following equation

$$\dot{\lambda} = r\lambda - H_x = \left(r + g'(x)\right)\lambda - q'(x) \tag{6}$$

From Equation (1), we have $g'(x) \leq 0$. In the name of simplicity, in what follows we adopt the assumption of a linear $g(x)$ from Hopland and Kvamsdal [12]. We can then write $g'(x) = \delta_1 + \delta_2(1 - x)$, where both δ_1 and δ_2 are positive constants. Note that although $g(x)$ is linear, the deterioration process accelerates linearly with the difference between the current state, x, and the "as new" state, $x = 1$. The crucial assumption is that of acceleration; linearity is the simplest specification. We have $g(1) = \delta_1 > 0$ and $g'(x) = -\delta_2 < 0$, as above. The latter means that the rate of natural deterioration (without maintenance) increases with δ_2 as x falls.

From the maximum principle, we have

$$H_m = -c'(m) + \lambda \leq 0 \tag{7}$$

For inner solutions with a positive shadow value, the inequality in Eq. (7) holds with equality. That is, $c'(m) = \lambda$. Because $c(m)$ is convex per assumption, this equality provides a one-to-one relationship between the policy variable and the shadow value. Notably, the relationship is increasing such that a higher shadow value implies a higher maintenance level. Let γ be the inverse of $c'(m)$, and we have, for inner solutions, the well-defined equation

$$m = \gamma(\lambda) \tag{8}$$

That is, if we can solve for the shadow value, we implicitly solve for the optimal maintenance schedule. For corner solutions, we simply have

$$m = 0 \quad \text{if} \quad \lambda \leq 0 \tag{9}$$

To solve for the shadow value, we return to Eq. (6) under the assumption of a linear $g(x)$. That is, $\dot{\lambda} = (r - \delta_2)\lambda - q'(x)$. The equation has integrating factor $\mathrm{e}^{-(r-\delta_2)t}$ and can be solved by integrating over the period (t, T). We get

$$\lambda(t) = \mathrm{e}^{-(r-\delta_2)(T-t)}\Lambda + \int_t^T \mathrm{e}^{-(r-\delta_2)(s-t)} q'(x(s))ds \tag{10}$$

In Equation (10), we follow the convention from above and write $\lambda(T) = \Lambda$. At $t = T$, any positive shadow value reflects the salvage value. Thus, if the salvage value is zero, we have a zero shadow value at the terminal time. Polynomial forms of $\gamma(\lambda)$ with no constant term – a typical case in important examples – will then yield a zero maintenance level at the terminal time.

The terminal condition on the shadow value is $\Lambda = \mathrm{e}^{-rT}S_x(X, T)$. The interpretation of the current shadow value is that it reflects the discounted marginal value from investing in the state variable. In our maintenance model, investing in the state of a facility is through maintenance. The terminal condition can thus be interpreted as follows: At the time of decommissioning, the marginal (present) value of continued maintenance is equal to the marginal change to the salvage value. With the terminal condition, Eq. (10) becomes

$$\lambda(t) = \mathrm{e}^{-(r-\delta_2)(T-t)}\mathrm{e}^{-rT}S_X\left(X, T\right) + \int_t^T \mathrm{e}^{-(r-\delta_2)(s-t)} q'(x(s))ds \tag{11}$$

Evaluated at $t = T$, the terminal condition holds. The solution for $\lambda(\tau)$ is positive for all $t \leq T$, thus excluding the degenerate solution with $T = 0$ where the optimal policy nevertheless is of no significance. Equation (11) is well defined and via Eq. (8), we have obtained the optimal maintenance schedule.

If we return to the interpretation of the current shadow value and consider Eq. (11), the discounted stream of utility generated by a marginal investment in the state variable, from the present to the end of the planning horizon, we note that Eq. (11) has two parts. The first reflects the discounted contribution from the terminal state in terms of salvage value; the second reflects the discounted stream of marginally better service quality from the current time to the end of the planning horizon. In more detail, the first part shows that the salvage term itself is a constant ($e^{-rT}S_x(X, T)$) in the shadow price expression that has exponentially increasing influence on the shadow price schedule. Without the salvage term, the shadow price goes to zero at $t = T$.

Based on Eqs. (2) and (6), we can draw a phase diagram illustrating the system dynamics and thus characterize the optimal maintenance schedule. The zero locus $(\dot{x} = 0)$ of Eq. (2) is given by $\lambda = c'(g(x))$. $g(x)$ is linear in x. Thus, if $c'(\cdot)$ is linear, which is the case for quadratic costs, the zero locus of Eq. (2) is linear. Equation (6) yields the following expression for the zero locus $(\dot{\lambda} = 0)$: $\lambda = q'(x)/(r - \delta_2)$. That is, the zero locus of Eq. (6) has the same shape as $q'(x)$. Note that as $q'(x) \geq 0$, the locus is above the λ-axis if $r > \delta_2$ and below if $r < \delta_2$.

To illustrate, let us assume that costs are quadratic such that the locus $(\dot{x} = 0)$ is linear. Further, if the service quality $q(x)$ has an s-shape, $q'(x)$ will be bell-shaped, as will the $\dot{\lambda} = 0$ locus. An s-shaped service quality function seems reasonable. It implies that at high facility state levels, small changes to the state has little impact on the service quality. That is, whether a facility is "as new" or almost "as new" does not matter much for service provision in the facility. Similarly, at low facility state levels, small changes to the state has little impact on service quality. However, there is a mid-range of facility state levels where the service quality responds more to changes. A basic assumption in the following discussion is that there is no salvage value, but we will comment on the case with a non-zero salvage value when appropriate. Finally, let us assume that $r > \delta_2$ such that the $\dot{\lambda} = 0$ locus is above the λ-axis. The resulting phase diagram is shown in Fig. 1.

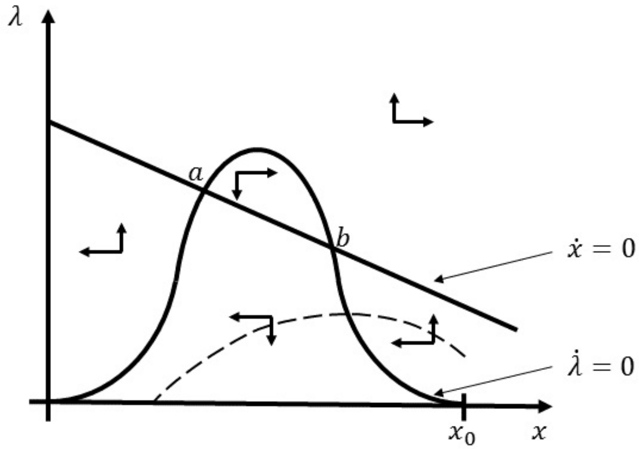

Fig. 1. Phase diagram based on Eqs. (2) and (6) and with a zero salvage value.
Solid curves represent the zero loci of (2) and (6),
the dashed curve illustrates the generic optimal maintenance schedule

As noted above, the optimal maintenance level is increasing in λ. Thus, above the $\dot{x}=0$ locus, the maintenance level is higher than the level required to keep the state fixed, and x will increase. This feature is shown by arrows pointing to the right in Fig. 1. Similarly, below the $\dot{x}=0$ locus, maintenance is insufficient to keep the state fixed, and x will decrease; shown by arrows pointing to the left. Turning to the $\dot{\lambda}=0$ locus, we note that the governing equation for the shadow value, Eq. (6), is linear and increasing in λ. Thus, for a given x-level, λ is increasing above the zero locus and decreasing below. These features are shown by arrows pointing upward above the locus and downward below. In Fig. 1, the loci are drawn such that they intersect (in points a and b), something that not necessarily is the case, but providing for a somewhat more interesting phase diagram.

The dashed curve in Fig. 1 traces a generic optimal maintenance schedule. The curve starts at $x = x_0$, at some level below the $\dot{x} = 0$ locus. x_0 is the highest possible state and points above the locus, implying an increasing state variable, is not relevant. Thus, the optimal level of maintenance in the initial phase allows the condition to slide downwards. As the condition slides, the maintenance level is increasing alongside an increasing deterioration of the facility. If the shadow value increases rapidly such that the maintenance increases to a level where the deterioration is halted and reversed, the path enters the region above the $\dot{x}=0$ locus and both the state and maintenance level will increase until $t = T$. That is, maintenance spending is invested to increase the physical capital towards the time horizon. Such maintenance schedules are only optimal with a nonzero and significant salvage value. However, with no salvage value, as we have currently assumed, the path will at some point ($x > x_b$) meet the $\dot{\lambda}=0$ locus. Note that with no salvage value, the shadow price goes to zero at the time horizon, that is, the optimal path meets the horizontal axis in the diagram. At the $\dot{\lambda}=0$ locus, maintenance remain insufficient to keep the facility condition from sliding and the path continues into the region below the locus. In this region, both maintenance and the condition decreases until one reaches the time horizon, $t = T$.

The phase diagram can also describe optimal paths if the facility condition is below the “as new” level initially. Say that the initial condition is somewhere between where the zero loci meets (between x_a and x_b). If the shadow value is high enough such that the path starts above the $\dot{x}=0$ locus, but still below the $\dot{\lambda}=0$ locus, the initial phase has increasing x. With incentives to build up physical capital toward the horizon, to receive a high salvage value, the path may cut the $\dot{\lambda}=0$ locus above the $\dot{x}=0$ locus, and both the state and maintenance level will increase until $t = T$. This phase is analogous to the potential phase outlined above, with a significant salvage value. With no or a lower salvage value, the path will cut the $\dot{x}=0$ locus and enter a phase where both the facility condition and maintenance decreases towards the horizon. Further below, we illustrate some of the generic optimal paths described above in numerical examples.

5. An exact solution

In what follows, we make some simplifying assumptions to enable an exact, formal solution to the facility maintenance problem stated above. The main, simplifying assumption is that the salvage value is zero. As discussed above, for a range of cases, this has no effect on the principal analysis. We also assume that both $q(x)$ and $c(m)$ passes through the origin (that is, $c_F = 0$).

For the terminal time T, which is freely chosen, we have that the Hamiltionian (5) should equal zero because the Hamiltonian is the shadow price of time

$$H\left(X, M, \Lambda\right) = q(X) - c(M) + \Lambda\left(M - g(X)\right) = 0 \tag{12}$$

As above, capital variables denote variables evaluated at T, for example, $\lambda(T) = \Lambda$. Further, $X\Lambda = 0$ must hold, with both $X \geq 0$ and $\Lambda \geq 0$. That is, either $X = 0$ or $\Lambda = 0$. If $X = 0$, and with $q(0) = 0$ and $g(0) = 0$, Eq. (12) yields $\Lambda M = c(M)$. If $\Lambda > 0$, $\Lambda M = c(M)$ has two possible solutions, one with $M = 0$ and one with $M > 0$ ($c(m)$ is convex and passes through the origin). If $\Lambda = 0$, we have $M = 0$; see Eq. (9). Equation (12) reduces to $q(X) = 0$. We have assumed that $q(x)$ passes through the origin and is non-decreasing. Thus, from $X\Lambda = 0$, $X = 0$ must hold.

For inner solutions, we have $\lambda = c'(m)$, which is monotone in m. If we take the derivative with respect to time and use Eq. (6), we obtain

$$c''(m)\dot{m} = \left(r - \delta_2\right) c'(m) - q'(x) \tag{13}$$

Using Eq. (2), we can rewrite (13) as

$$\left(m - g(x)\right) c''(m) \frac{dm}{dx} = \left(r - \delta_2\right) c'(m) - q'(x) \tag{14}$$

With known $x(T) = X$ and $m(T) = M$ (see above), we can solve Eq. (14) by integration. When x =1 (the original initial condition), we stop the integration. This solution to Eq. (14) sets up a correspondence between the state variable x and the maintenance level along the optimal path. Let $m = \mu(x)$ define maintenance as a function of x. The state equation (2) can then be written $\dot{x} = \mu(x) - g(x) = h(x)$.

Let

$$T_1 = \int_X^1 \frac{ds}{h(s)}$$

that is, the time it takes for the state variable to pass from $x = X$ to $x = 1$ along the optimal path. We illustrate our solution for the state variable in Fig. 2.

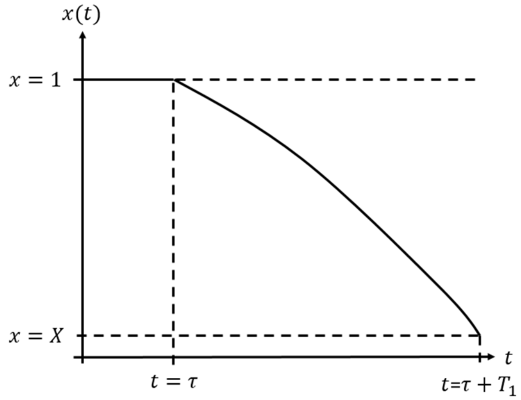

Fig. 2. Illustration of the optimal path for the state variable.
The solid line represents $x(t)$

As indicated in the figure, there may be an initial period, $t \in (0, \tau)$, where x is optimally kept at $x = 1$ with $\mu(1) = m = g(1)$. That is, there may be an initial plateau in $x(t)$. We have continuity of λ that ensures continuity of m, see Eq. (8), which again ensures continuity of x. Thus, there is no jump at $t = \tau$. If $\mu(1) < g(1)$, that is, the integrated solution from Eq. (14), at $x = 1$, falls short of the level required to keep x constant at $x = 1$, then there is no initial period where x is kept constant at $x = 1$ because there can be no jump in x.

We have that $H_x = rc'(m) \geq 0$. If $\mu(1) = g(1)$, that is, there may be an initial plateau, x has to decrease immediately after $t = \tau$. As we want to maximize the Hamiltonian, we want to stay on the plateau as long as possible. That is, with no constraint on the terminal time, we have $T \to \infty$. Whenever there is an upper constraint on T, which in all practical matters there will be, the final period of length T_1 is described by the solution from integrating Eq. (14), as discussed above.

6. Numerical examples

In our examples, we use the following s-shaped curve for the service quality function

$$q(x) = \frac{1}{1 + \mathrm{e}^{Q_0 - \theta x}} \tag{15}$$

Costs are quadratic in the level of maintenance $c(m) = c_0 m^2 + c_F$. The parameter c_F is the fixed running cost. We consider a case where the phase diagram in Fig. 1 applies, thus, $r > \delta_2$, and where the salvage function is as follows

$$S(x,t) = \mathrm{e}^{\alpha t}\left(S_0 + \psi\sqrt{\theta x}\right) \tag{16}$$

That is, the only time dependence in the salvage value is the appreciation of real estate. Further, the salvage value has two parts; one constant, S_0, pertaining to land value, and one that varies with the facility state $\psi\sqrt{\theta x}$; decreasing returns to scale. Parameter values are given in the Appendix.

The solution is governed by the Hamilton–Jacobi–Bellman equation:

$$V_t(x,t) + \max_{T,\, m\geq 0}\left(Q(x,m) + V_x(x,t)\, f(x,m)\right) = 0 \tag{17}$$

In Equation (17), $V(x, t)$ is the optimal value function, which is subject to the end-point constraint $V(X, T) = \mathrm{e}^{-rT} S(X, T)$. In what follows, we consider numerical solutions of (17).

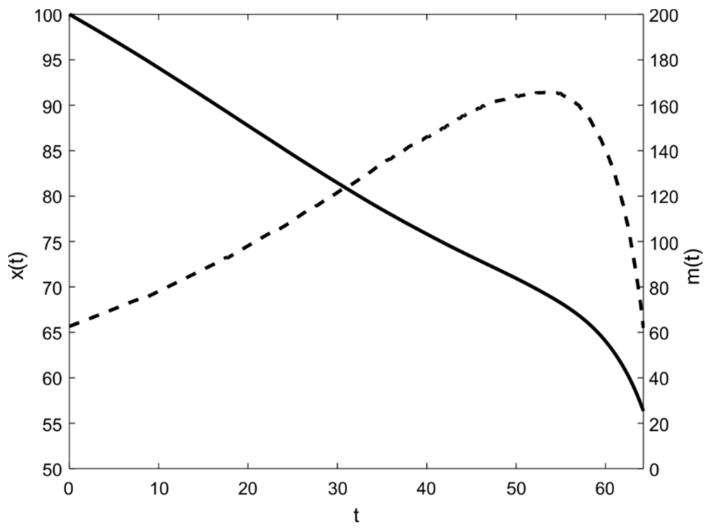

Fig. 3. Optimal time paths for $x(t)$ – solid curve given in percent relative to “as new state”, and $m(t)$ – dashed curve, given in percent relative to “as new steady state level”. Basic case with $x(t)$ =1 (parameters in the Appendix)

Figure 3 displays the optimal time paths for $x(t)$ and $m(t)$ in the basic case. Both are given in percent; $x(t)$ in percent of the “as new” state x_0, $m(t)$ in percent of the level needed to maintain the "as new" state indefinitely, which is simply given by δ_1. We observe that the optimal paths agree with the generic optimal path indicated in the phase diagram in Fig. 1. In particular, the state declines slowly throughout the period, while the maintenance level first has a phase with increasing effort before dropping off toward the end. The optimal decommissioning time is at $T = 64.25$. Further numerical details from this and subsequent examples are provided in the Appendix.

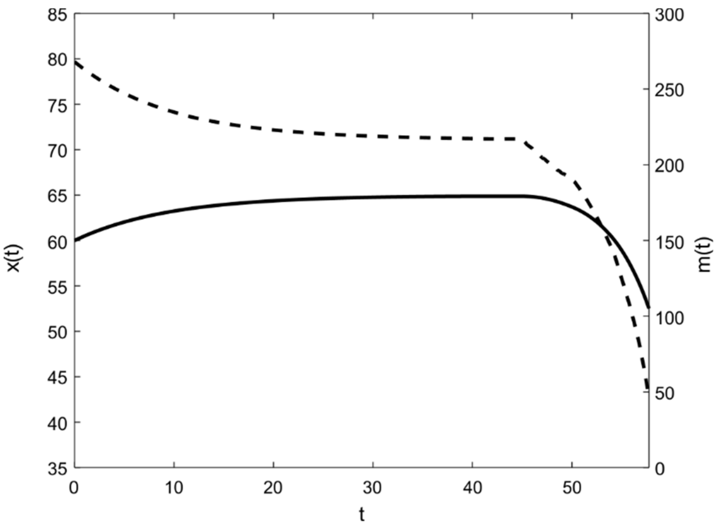

Fig. 4. Optimal time paths for $x(t)$ – solid curve given in percent relative to "as new state" and $m(t)$ – dashed curve given in percent relative to "as new steady state level". Basic case with $x(t) = 0.6$ (parameters in the Appendix)

Figure 4 illustrates optimal time paths in the basic case for a facility that is not "as new" at $t = 0$. Rather, the state at the beginning of the planning horizon is at 60%. The example corresponds to situations where the maintenance regime is changed the optimal regime at some point after the facility was new. The behavior of the optimal paths are different than what is observed in Fig. 3, and corresponds to a path in the phase diagram that has its starting point between x_a and x_b and above the $\dot{x} = 0$ locus (Fig. 1). The state is increasing initially, before leveling off and dropping towards the end. The maintenance level starts out at a high level and decreases throughout at a varying rate. Notably, maintenance is most of the time much higher than in Fig. 3. Both the reduced initial state and the increased maintenance level reduces the utility of the facility and its service production, and the initial net present value is reduced by more than 60% when compared to the initial net present value of a brand new facility. The optimal decommissioning time is at $T = 57.75$, which is just a few years short of the case with a brand new facility. That it is optimal to retain a facility for so long when its initial state is close to the decommissioning state of the brand new facility exposes the strong time dependence (non-autonomy) of the maintenance problem.

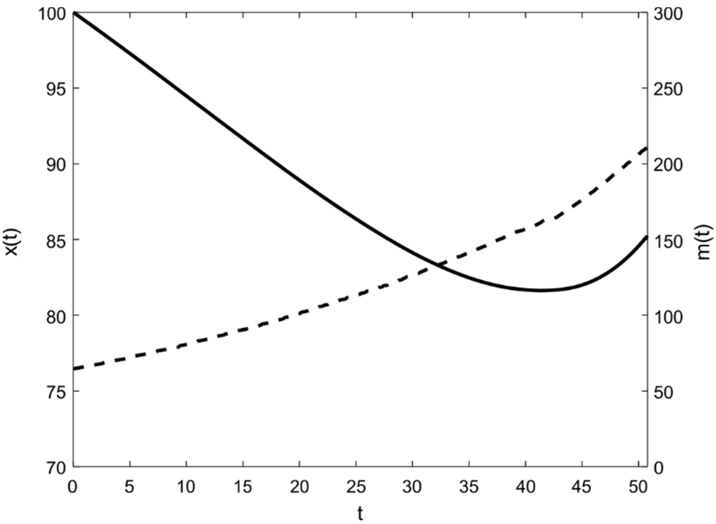

Fig. 5. Optimal time paths for $x(t)$ – solid curve given in percent relative to "as new state" and $m(t)$ – dashed curve given in percent relative to "as new steady state level". Case with linear salvage value (parameters in the Appendix)

Figure 5 illustrates optimal solutions for our example where the logarithm in the salvage value function is replaced with a linear relationship such that $S(x, t) = e^{\alpha t}(S_0 + \psi\theta x)$. This makes the salvage value larger and more sensitive to the facility state, and returns to scale are constant. The date for optimal decommissioning is earlier than in the base case ($T = 50.75$) and the facility is generally kept at in better condition throughout the project horizon. In particular, towards the end of the project horizon maintenance spending is increased to levels such that the facility state improves. In contrast, in our first example (Fig. 3), maintenance spending dropped off quickly towards the date for decommissioning. The larger salvage value contributes to a higher net present value and represents more than five times the salvage value from the case with logarithmic salvage value (the increased realized salvage value depends on both the functional form and the shorter time horizon). Nevertheless, most of the added value is spent on higher maintenance expenditures, and the initial net present value is just 2% higher in the case with a linear salvage value.

The final twist to our example is to consider a higher real estate appreciation rate ($\alpha = r$) together with an increase (a tripling) in the running maintenance costs parameter (without the increased running maintenance costs, the example falls into the "infinite horizon" case discussed above). Optimal solutions are shown in Fig. 6. Note that the salvage value is again logarithmic (Eq. (17)). The optimal date for decommissioning is $T = 33.0$, again significantly earlier than in the base case. As in our first example, the facility state falls – notably faster – throughout the project period. Maintenance efforts are increasing slowly for most of the time before dropping off towards the end. The realized salvage value is again higher than in the base case, but contrary to the linear case that had a high facility state at decommissioning, the facility state at decommissioning is relatively low. The higher realized salvage value thus depends on a higher appreciation rate and the shorter time horizon (smaller effect from discounting). The shorter time horizon and the generally lower facility state means that lesser services are delivered, and the initial net present value is reduced by approximately 7% when compared to the base case.

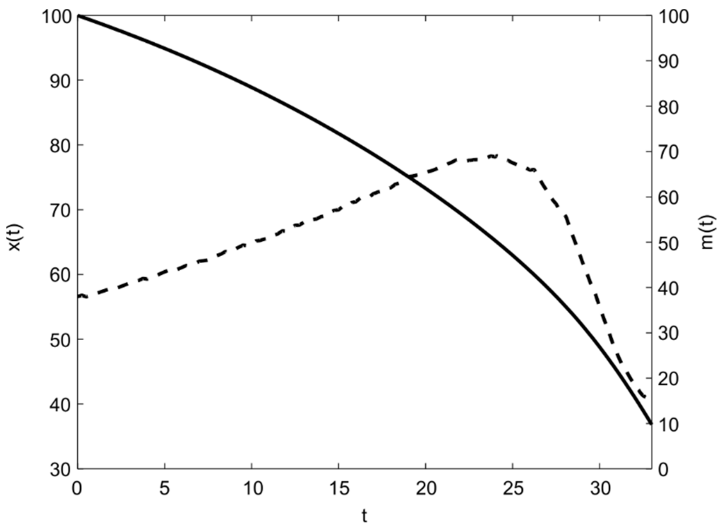

Fig. 6. Optimal time paths for $x(t)$ – solid curve given in percent relative to "as new state" and $m(t)$ – dashed curve given in percent relative to "as new steady state level". Case with $\alpha = r$ and $c_0 = 3$ (further parameters in the Appendix)

Our examples demonstrate that our model facilitates a rich set of scenarios and wide range optimal behaviors, although the model is kept at a conceptual level.

7. Concluding remarks

Facilities represent a significant part of public capital, but investigation into maintenance and its relation to public services and facility deterioration has been limited. We develop a basic theory of maintenance for public facilities. The theory enables us to derive optimal paths for maintenance and the decommissioning time. Our examples demonstrate a wide array of behavior for the optimal solutions, depending on the parametrization and auxiliary assumptions about initial conditions and functional forms. The examples also illustrate some of the trade-offs that are involved in facilities management. For example, when the real estate appreciation rate is higher, the facility is decommissioned much earlier but still with a lower facility state.

Extensions of our theory that would be of interest includes a more comprehensive (multidimensional) measure of the facility state and a service quality measure that reflects datedness. Time dependence in service quality would reflect technical progress on a structural level, that is, innovations that require changed facility structures. Further effects of datedness could also be considered, for example in the salvage function or the maintenance costs.

References

[1] AGÉNOR P.-R., *Infrastructure investment and maintenance expenditure. Optimal allocation rules in a growing economy*, J. Pub. Econ. Th., 2009, 11 (2), 233–250.

[2] BENSOUSSAN A., SETHI S.P., *The machine maintenance and sale age model of Kamien and Schwartz revisited*, Man. Sci., 2007, 53 (12), 1964–1976.

[3] BORGE L.-E., HOPLAND A.O., *Maintenance and building conditions in Norwegian local governments. Economic and political determinants*, Working Paper No. 8/2012, Department of Economics, Norwegian University of Science and Technology, Trondheim 2012.

[4] BORGE L.-E., HOPLAND A.O., *Schools and public buildings in decay. The role of political fragmentation*, Econ. Govern., 2017, 18 (1), 85–105.

[5] COOKE F.L., *Implementing TPM in plant maintenance. Some organisational barriers*, Int. J. Qual. Reliab. Man., 2000, 17 (9), 1003–1016.

[6] COOKE F.L., *Plant maintenance strategy. Evidence from four British manufacturing firms*, J. Qual. Maint. Eng., 2003, 9 (3), 239–249.

[7] DE HAAN J., STURM J.-E., SIKKEN B.J., *Government capital formation. Explaining the decline*, Weltwirtschaft. Arc. Rev. World Econ., 1996, 132 (1), 55–74.

[8] FRASER K., *Facilities management. The strategic selection of a maintenance system*, J. Facil. Man., 2014, 12 (1), 18–37.

[9] FRASER K., HVOLBY H.-H., WATANABE C., *Review of the three most popular maintenance systems. How well is the energy sector represented?*, Int. J. Global Energy Iss., 2011, 35 (2/3/4), 287–309.

[10] GAUTEPLASS A.Å., HOPLAND A.O., *Using game theory to stimulate provision of local public facilities*. Prop. Man., 2017, 35 (4), 368–379.

[11] HOPLAND A.O., *Can game theory explain poor maintenance of regional government facilities?*, Facil., 2015, 33 (3–4), 195–205.

[12] HOPLAND A.O., KVAMSDAL S.F., *Optimal maintenance scheduling of local public purpose buildings*, Prop. Man., 2016, 34 (2), 120–135.

[13] HOPLAND A.O., KVAMSDAL S.F., *Building conditions in Norwegian local governments. Trends and determinants*, Facil., 2019, 37 (3/4), 141–156.

[14] HOPLAND A.O., KVAMSDAL S.F., *Concerns among local government facility managers. A Norwegian survey*, Facil., 2018, 36, (5/6), 230–243.

[15] KALAITZIDAKIS P., KALYVITIS S., *On the macroeconomic implications of maintenance in public capital*, J. Pub. Econ., 2004, 88, 695–712.

[16] KAMIEN M.I., SCHWARTZ N.L., *Optimal maintenance and sale age for a machine subject to failure*, Man. Sci., 1971, 17 (8), B495–B504.

[17] KAMIEN M.I., SCHWARTZ N.L., *Dynamic Optimization*, 2nd Ed., Elsevier, Amsterdam 1991.
[18] OXLEY H., MARTIN J.P., *Controlling government spending and deficit. Trends in the 1980s and prospects for the 1990s*, OECD Econ. Stud., 1991, 17, 145–189.
[19] RIOJA F.K., *Productiveness and welfare implications of public infrastructure. A dynamic two-sector general equilibrium analysis*, J. Dev. Econ., 1999, 58, 387–404.
[20] RIOJA F.K., *Filling potholes: macroeconomic effects of maintenance versus new investments in public infrastructure*, J. Pub. Econ., 2003, 87, 2281–2304.
[21] STURM J.-E., *Public capital expenditures in the OECD countries: The causes and impact of the decline in public capital spending*, Edward Elgar, Cheltenham 1998.

Appendix

Table A1. Parameter values

Parameter	α	c_0	c_F	δ_1	δ_2	ψ	Q_0	r	S_0	θ
Value	0.05	1	0.1	0.3	0.05	0.1	10	0.08	0.2	20

Table A2. Numerical results

Example	Base case $x(0) = 1$	Base case $x(0) = 0.6$	Linear salvage value	High α $\alpha = r$, $c_0 = 3$
T	64.25	57.75	50.75	33.0
$x(T)$	0.56	0.53	0.85	0.37
NPV(0)	10.53	3.90	10.75	9.81
Salvage discounted	0.077	0.092	0.42	0.47
Salvage nominal	13.30	9.41	24.09	6.61

Received 24 July 2018
Accepted 11 March 2019