Supplementary material for “Green’s theorem in seismic imaging across the scales” and “Monitoring induced distributed double-couple sources using Marchenko-based virtual receivers”

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1 Classical homogeneous Green’s function representation

1.1 Definition of the homogeneous Green’s function

Consider an inhomogeneous lossless acoustic medium with mass density $\rho(\mathbf{x})$ and compressibility $\kappa(\mathbf{x})$. Here $\mathbf{x}$ denotes the Cartesian coordinate vector $\mathbf{x} = (x_1, x_2, x_3)$; the x_3 axis is pointing downward. In this medium a space ($\mathbf{x}$) and time (t) dependent source distribution $q(\mathbf{x}, t)$ is present, with q defined as the volume-injection rate density. The acoustic wave field, caused by this source distribution, is described in terms of the acoustic pressure $p(\mathbf{x}, t)$ and the particle velocity $v_i(\mathbf{x}, t)$. These field quantities obey the equation of motion and the stress-strain relation, according to

$$\rho \partial_t v_i + \partial_i p = 0, \tag{1}$$

$$\kappa \partial_t p + \partial_i v_i = q. \tag{2}$$

Here ∂_t and ∂_i stand for the temporal and spatial differential operators $\partial/\partial t$ and $\partial/\partial x_i$, respectively. Latin subscripts (except t) take on the values 1 to 3 and Einstein’s summation convention applies to repeated subscripts. When q is an impulsive source at $\mathbf{x} = \mathbf{x}_A$ and $t = 0$, according to

$$q(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_A)\delta(t), \tag{3}$$

then the causal solution of equations (1) and (2) defines the Green’s function, hence

$$p(\mathbf{x}, t) = G(\mathbf{x}, \mathbf{x}_A, t). \tag{4}$$

By eliminating v_i from equations (1) and (2) and substituting equations (3) and (4), we find that the Green’s function $G(\mathbf{x}, \mathbf{x}_A, t)$ obeys the following wave equation

$$\partial_i(\rho^{-1}\partial_i G) - \kappa \partial_t^2 G = -\delta(\mathbf{x} - \mathbf{x}_A)\partial_t \delta(t). \tag{5}$$

Wave equation (5) is symmetric in time, except for the source on the right-hand side, which is anti-symmetric. Hence, the time-reversed Green’s function $G(\mathbf{x}, \mathbf{x}_A, -t)$ obeys the same wave equation, but with opposite sign for the source. By summing the wave equations for $G(\mathbf{x},\mathbf{x}_A,t)$ and $G(\mathbf{x},\mathbf{x}_A,-t)$, the sources on the right-hand sides cancel each other, hence, the function

$$G_{\mathrm{h}}(\mathbf{x},\mathbf{x}_A,t) = G(\mathbf{x},\mathbf{x}_A,t) + G(\mathbf{x},\mathbf{x}_A,-t) \tag{6}$$

obeys the homogeneous equation

$$\partial_i(\rho^{-1}\partial_i G_{\mathrm{h}}) - \kappa\partial_t^2 G_{\mathrm{h}} = 0. \tag{7}$$

Therefore $G_{\mathrm{h}}(\mathbf{x},\mathbf{x}_A,t)$, as defined in equation (6), is called the homogeneous Green's function.

1.2 Reciprocity theorems

We define the temporal Fourier transform of a space- and time-dependent quantity $u(\mathbf{x},t)$ as

$$u(\mathbf{x},\omega) = \int_{-\infty}^{\infty} u(\mathbf{x},t)\exp(i\omega t)\mathrm{d}t, \tag{8}$$

where ω is the angular frequency and i the imaginary unit. To keep the notation simple, we denote quantities in the time and frequency domain by the same symbol. In the frequency domain, equations (1) and (2) transform to

$$-i\omega\rho v_i + \partial_i p = 0, \tag{9}$$

$$-i\omega\kappa p + \partial_i v_i = q. \tag{10}$$

We introduce two independent acoustic states, which will be distinguished by subscripts A and B. Rayleigh's reciprocity theorem is obtained by considering the quantity $\partial_i\{p_A v_{i,B} - v_{i,A} p_B\}$, applying the product rule for differentiation, substituting equations (9) and (10) for both states, integrating the result over a spatial domain $\mathbb{V}$ enclosed by boundary $\mathbb{S}$ with outward pointing normal n_i, and applying the theorem of Gauss (de Hoop, 1988; Fokkema and van den Berg, 1993). Assuming that in $\mathbb{V}$ the medium parameters $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$ in the two states are identical, this yields Rayleigh's reciprocity theorem of the convolution type

$$\int_{\mathbb{V}} \{p_A q_B - q_A p_B\}\mathrm{d}\mathbf{x} = \oint_{\mathbb{S}} \frac{1}{i\omega\rho}\{p_A(\partial_i p_B) - (\partial_i p_A)p_B\}n_i\mathrm{d}\mathbf{x}. \tag{11}$$

We derive a second form of Rayleigh's reciprocity theorem for time-reversed wave fields. In the frequency domain, time-reversal is replaced by complex conjugation. When p is a solution of equations (9) and (10) with source distribution q (and real-valued medium parameters), then p^* obeys the same equations with source distribution $-q^*$ (the superscript * denotes complex conjugation). Making these substitutions for state A in equation (11) we obtain Rayleigh's reciprocity theorem of the correlation type (Bojarski, 1983)

$$\int_{\mathbb{V}} \{p_A^* q_B + q_A^* p_B\}\mathrm{d}\mathbf{x} = \oint_{\mathbb{S}} \frac{1}{i\omega\rho}\{p_A^*(\partial_i p_B) - (\partial_i p_A^*)p_B\}n_i\mathrm{d}\mathbf{x}. \tag{12}$$

1.3 Representation of the homogeneous Green's function

We choose point sources in both states, according to $q_A(\mathbf{x},\omega)=\delta(\mathbf{x}-\mathbf{x}_A)$ and $q_B(\mathbf{x},\omega)=\delta(\mathbf{x}-\mathbf{x}_B)$, with $\mathbf{x}_A$ and $\mathbf{x}_B$ both in $\mathbb{V}$. The fields in states A and B are thus expressed in terms of Green's functions, according to

$$p_A(\mathbf{x},\omega)=G(\mathbf{x},\mathbf{x}_A,\omega), \tag{13}$$

$$p_B(\mathbf{x},\omega)=G(\mathbf{x},\mathbf{x}_B,\omega), \tag{14}$$

with $G(\mathbf{x},\mathbf{x}_A,\omega)$ and $G(\mathbf{x},\mathbf{x}_B,\omega)$ being the Fourier transforms of $G(\mathbf{x},\mathbf{x}_A,t)$ and $G(\mathbf{x},\mathbf{x}_B,t)$, respectively. Making these substitutions in equation (12) and using source-receiver reciprocity of the Green's functions gives (Porter, 1970; Oristaglio, 1989; Wapenaar, 2004; van Manen et al., 2005)

$$G_{\mathrm{h}}(\mathbf{x}_B,\mathbf{x}_A,\omega)=\oint_{\mathbb{S}}\frac{1}{i\omega\rho(\mathbf{x})}\Big(\{\partial_i G(\mathbf{x},\mathbf{x}_B,\omega)\}G^*(\mathbf{x},\mathbf{x}_A,\omega)-G(\mathbf{x},\mathbf{x}_B,\omega)\partial_i G^*(\mathbf{x},\mathbf{x}_A,\omega)\Big)n_i\mathrm{d}\mathbf{x}, \tag{15}$$

where $G_{\mathrm{h}}(\mathbf{x}_B,\mathbf{x}_A,\omega)$ is the homogeneous Green's function in the frequency domain. It is defined as

$$G_{\mathrm{h}}(\mathbf{x},\mathbf{x}_A,\omega)=G(\mathbf{x},\mathbf{x}_A,\omega)+G^*(\mathbf{x},\mathbf{x}_A,\omega)=2\Re\{G(\mathbf{x},\mathbf{x}_A,\omega)\}, \tag{16}$$

where $\Re$ denotes the real part. Equation (15) is an exact representation for the homogeneous Green's function $G_{\mathrm{h}}(\mathbf{x}_B,\mathbf{x}_A,\omega)$.

When $\mathbb{S}$ is sufficiently smooth and the medium outside $\mathbb{S}$ is homogeneous, the two terms under the integral in equation (15) are nearly identical (but with opposite signs), hence

$$G_{\mathrm{h}}(\mathbf{x}_B,\mathbf{x}_A,\omega)=-2\oint_{\mathbb{S}}\frac{1}{i\omega\rho(\mathbf{x})}G(\mathbf{x},\mathbf{x}_B,\omega)\partial_i G^*(\mathbf{x},\mathbf{x}_A,\omega)n_i\mathrm{d}\mathbf{x}. \tag{17}$$

The main approximation is that evanescent waves are neglected at $\mathbb{S}$ (Zheng et al., 2011; Wapenaar et al., 2011).

2 Single-sided homogeneous Green's function representations

2.1 Modification of the configuration

We replace the arbitrary closed boundary $\mathbb{S}$ by a combination of two boundaries $\mathbb{S}_0$ and $\mathbb{S}_A$, as indicated in Figure 1. Here $\mathbb{S}_0$ may be curved, but $\mathbb{S}_A$ is a horizontal boundary, with $\mathbf{n}=(0,0,1)$. The depth level of the focal plane $\mathbb{S}_A$ is defined as $x_{3,A}$ (which is equal to the x_3-coordinate of the point $\mathbf{x}_A$). The domain between boundaries $\mathbb{S}_0$ and $\mathbb{S}_A$ is called $\mathbb{V}_A$. For this configuration, reciprocity theorems (11) and (12) are replaced by

$$\int_{\mathbb{V}_A}\{p_Aq_B-q_Ap_B\}\mathrm{d}\mathbf{x}=\int_{\mathbb{S}_0}\frac{1}{i\omega\rho}\{p_A(\partial_i p_B)-(\partial_i p_A)p_B\}n_i\mathrm{d}\mathbf{x}+\int_{\mathbb{S}_A}\frac{1}{i\omega\rho}\{p_A(\partial_3 p_B)-(\partial_3 p_A)p_B\}\mathrm{d}\mathbf{x} \tag{18}$$

and

$$\int_{\mathbb{V}_A}\{p_A^*q_B+q_A^*p_B\}\mathrm{d}\mathbf{x}=\int_{\mathbb{S}_0}\frac{1}{i\omega\rho}\{p_A^*(\partial_i p_B)-(\partial_i p_A^*)p_B\}n_i\mathrm{d}\mathbf{x}+\int_{\mathbb{S}_A}\frac{1}{i\omega\rho}\{p_A^*(\partial_3 p_B)-(\partial_3 p_A^*)p_B\}\mathrm{d}\mathbf{x}, \tag{19}$$

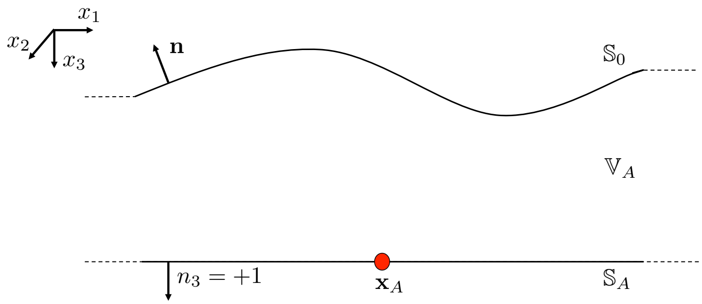

Figure 1. Modified configuration. The boundary $\mathbb{S}$ consists of the combination of boundaries $\mathbb{S}_0$ and $\mathbb{S}_A$.

respectively. In the following we use these reciprocity theorems as the basis for deriving several versions of single-sided homogeneous Green's function representations, each time by applying decomposition to one or more of the integrals in these theorems.

2.2 Single-sided homogeneous Green's function representation: general formulation

Substituting equations (A37) and (A38) for the boundary integrals at $\mathbb{S}_A$ into equations (18) and (19), we obtain

$$\int_{\mathbb{V}_A} (p_A q_B - q_A p_B)\mathrm{d}\mathbf{x} = \int_{\mathbb{S}_0} \frac{1}{i\omega\rho}\Big(p_A(\partial_i p_B) - (\partial_i p_A)p_B\Big)n_i \mathrm{d}\mathbf{x} - \int_{\mathbb{S}_A} \frac{2}{i\omega\rho}\Big((\partial_3 p_A^+)p_B^- + (\partial_3 p_A^-)p_B^+\Big)\mathrm{d}\mathbf{x} \tag{20}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{V}_A} (p_A^* q_B + q_A^* p_B)\mathrm{d}\mathbf{x} = \int_{\mathbb{S}_0} \frac{1}{i\omega\rho}\Big(p_A^*(\partial_i p_B) - (\partial_i p_A^*)p_B\Big)n_i \mathrm{d}\mathbf{x} - \int_{\mathbb{S}_A} \frac{2}{i\omega\rho}\Big((\partial_3 p_A^+)^* p_B^+ + (\partial_3 p_A^-)^* p_B^-\Big)\mathrm{d}\mathbf{x}. \tag{21}$$

For state A we consider the focusing function $f_1(\mathbf{x},\mathbf{x}_A,\omega) = f_1^+(\mathbf{x},\mathbf{x}_A,\omega) + f_1^-(\mathbf{x},\mathbf{x}_A,\omega)$, introduced in section 3.1 in "Green's theorem in seismic imaging across the scales". This focusing function is defined in a truncated version of the medium, which is identical to the actual medium in $\mathbb{V}_A$, but reflection free above $\mathbb{S}_0$ and below $\mathbb{S}_A$. Hence, the condition for the validity of equations (A36), (A37) and (A38) is fulfilled at $\mathbb{S}_A$. The focusing conditions at the focal plane $\mathbb{S}_A$ are

$$[\partial_3 f_1^+(\mathbf{x},\mathbf{x}_A,\omega)]_{x_3=x_{3,A}} = \tfrac{1}{2} i\omega\rho(\mathbf{x}_A)\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},A}), \tag{22}$$

$$[\partial_3 f_1^-(\mathbf{x},\mathbf{x}_A,\omega)]_{x_3=x_{3,A}} = 0. \tag{23}$$

For state B we consider the Green's function $G(\mathbf{x},\mathbf{x}_B,\omega) = G^+(\mathbf{x},\mathbf{x}_B,\omega) + G^-(\mathbf{x},\mathbf{x}_B,\omega)$, with its source at $\mathbf{x}_B$ anywhere in the half-space below $\mathbb{S}_0$. Note that the superscripts $+$ and $-$ in $f_1^\pm(\mathbf{x},\mathbf{x}_A,\omega)$ and $G^\pm(\mathbf{x},\mathbf{x}_B,\omega)$ refer to the propagation direction (downward or upward) at the observation point $\mathbf{x}$. The source of the Green's function at $\mathbf{x}_B$ is omnidirectional. Substituting $q_A(\mathbf{x},\omega) = 0$, $p_A^\pm(\mathbf{x},\omega) = f_1^\pm(\mathbf{x},\mathbf{x}_A,\omega)$, $q_B(\mathbf{x},\omega) = \delta(\mathbf{x}-\mathbf{x}_B)$ and $p_B^\pm(\mathbf{x},\omega) = G^\pm(\mathbf{x},\mathbf{x}_B,\omega)$ into equations (20) and (21), using equations (22) and (23), gives

$$G^{-}(\mathbf{x}_A,\mathbf{x}_B,\omega)+\chi(\mathbf{x}_B)f_1(\mathbf{x}_B,\mathbf{x}_A,\omega)$$
$$=\int_{\mathbb{S}_0}\frac{1}{i\omega\rho(\mathbf{x})}\Big(\{\partial_i G(\mathbf{x},\mathbf{x}_B,\omega)\}f_1(\mathbf{x},\mathbf{x}_A,\omega)-G(\mathbf{x},\mathbf{x}_B,\omega)\partial_i f_1(\mathbf{x},\mathbf{x}_A,\omega)\Big)n_i\mathrm{d}\mathbf{x} \tag{24}$$

and

$$G^{+}(\mathbf{x}_A,\mathbf{x}_B,\omega)-\chi(\mathbf{x}_B)f_1^{*}(\mathbf{x}_B,\mathbf{x}_A,\omega)$$
$$=-\int_{\mathbb{S}_0}\frac{1}{i\omega\rho(\mathbf{x})}\Big(\{\partial_i G(\mathbf{x},\mathbf{x}_B,\omega)\}f_1^{*}(\mathbf{x},\mathbf{x}_A,\omega)-G(\mathbf{x},\mathbf{x}_B,\omega)\partial_i f_1^{*}(\mathbf{x},\mathbf{x}_A,\omega)\Big)n_i\mathrm{d}\mathbf{x}, \tag{25}$$

respectively, where χ is the characteristic function of the domain $\mathbb{V}_A$. It is defined as

$$\chi(\mathbf{x}_B)=\begin{cases}1, & \text{for } \mathbf{x}_B \text{ between } \mathbb{S}_0 \text{ and } \mathbb{S}_A,\\ \frac{1}{2}, & \text{for } \mathbf{x}_B \text{ on } \mathbb{S}=\mathbb{S}_0\cup\mathbb{S}_A,\\ 0, & \text{for } \mathbf{x}_B \text{ outside } \mathbb{S}.\end{cases} \tag{26}$$

Summing equations (24) and (25) and using source-receiver reciprocity for the Green's function on the left-hand side yields

$$G(\mathbf{x}_B,\mathbf{x}_A,\omega)+\chi(\mathbf{x}_B)2i\Im\{f_1(\mathbf{x}_B,\mathbf{x}_A,\omega)\}$$
$$=\int_{\mathbb{S}_0}\frac{2}{\omega\rho(\mathbf{x})}\Big(\{\partial_i G(\mathbf{x},\mathbf{x}_B,\omega)\}\Im\{f_1(\mathbf{x},\mathbf{x}_A,\omega)\}-G(\mathbf{x},\mathbf{x}_B,\omega)\Im\{\partial_i f_1(\mathbf{x},\mathbf{x}_A,\omega)\}\Big)n_i\mathrm{d}\mathbf{x}, \tag{27}$$

where $\Im$ denotes the imaginary part. Taking the real part of both sides of this equation, using equation (16), gives the single-sided representation of the homogeneous Green's function

$$G_{\mathrm{h}}(\mathbf{x}_B,\mathbf{x}_A,\omega)=\int_{\mathbb{S}_0}\frac{2}{\omega\rho(\mathbf{x})}\Big(\{\partial_i G_{\mathrm{h}}(\mathbf{x},\mathbf{x}_B,\omega)\}\Im\{f_1(\mathbf{x},\mathbf{x}_A,\omega)\}-G_{\mathrm{h}}(\mathbf{x},\mathbf{x}_B,\omega)\Im\{\partial_i f_1(\mathbf{x},\mathbf{x}_A,\omega)\}\Big)n_i\mathrm{d}\mathbf{x}. \tag{28}$$

2.3 Single-sided homogeneous Green's function representation: assuming a homogeneous upper half-space

From here onward we assume that $\mathbb{S}_0$ is a horizontal boundary, with $\mathbf{n}=(0,0,-1)$. Substituting equations (A39) and (A40) for the boundary integrals at $\mathbb{S}_0$ and equations (A47) and (A48) for the volume integrals into equations (20) and (21), we obtain

$$\int_{\mathbb{V}_A}(p_A^{+}q_B^{-}+p_A^{-}q_B^{+}-q_A^{+}p_B^{-}-q_A^{-}p_B^{+})\mathrm{d}\mathbf{x}=$$
$$\int_{\mathbb{S}_0}\frac{2}{i\omega\rho}\Big((\partial_3 p_A^{+})p_B^{-}+(\partial_3 p_A^{-})p_B^{+}\Big)\mathrm{d}\mathbf{x}-\int_{\mathbb{S}_A}\frac{2}{i\omega\rho}\Big((\partial_3 p_A^{+})p_B^{-}+(\partial_3 p_A^{-})p_B^{+}\Big)\mathrm{d}\mathbf{x} \tag{29}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{V}_A}(p_A^{+*}q_B^{+}+p_A^{-*}q_B^{-}+q_A^{+*}p_B^{+}+q_A^{-*}p_B^{-})\mathrm{d}\mathbf{x}=$$
$$\int_{\mathbb{S}_0}\frac{2}{i\omega\rho}\Big((\partial_3 p_A^{+})^{*}p_B^{+}+(\partial_3 p_A^{-})^{*}p_B^{-}\Big)\mathrm{d}\mathbf{x}-\int_{\mathbb{S}_A}\frac{2}{i\omega\rho}\Big((\partial_3 p_A^{+})^{*}p_B^{+}+(\partial_3 p_A^{-})^{*}p_B^{-}\Big)\mathrm{d}\mathbf{x}. \tag{30}$$

We apply these theorems to the situation in which the upper half-space above $\mathbb{S}_0$ is homogeneous (for the Green's function as well as for the focusing function). For state A we consider again the focusing function $f_1(\mathbf{x},\mathbf{x}_A,\omega) = f_1^+(\mathbf{x},\mathbf{x}_A,\omega) + f_1^-(\mathbf{x},\mathbf{x}_A,\omega)$, defined in a truncated version of the medium. For state B we consider the Green's function $G(\mathbf{x},\mathbf{x}_B,\omega) = G^{+,+}(\mathbf{x},\mathbf{x}_B,\omega)+G^{-,+}(\mathbf{x},\mathbf{x}_B,\omega)+G^{+,-}(\mathbf{x},\mathbf{x}_B,\omega)+G^{-,-}(\mathbf{x},\mathbf{x}_B,\omega)$, with its source at $\mathbf{x}_B$ anywhere in the half-space below $\mathbb{S}_0$. Note that we introduced two superscripts. The first superscript refers again to the propagation direction at the observation point $\mathbf{x}$. The second superscript refers to the radiation direction of the source at $\mathbf{x}_B$. Substituting $q_A^+(\mathbf{x},\omega) = q_A^-(\mathbf{x},\omega) = 0$, $p_A^\pm(\mathbf{x},\omega) = f_1^\pm(\mathbf{x},\mathbf{x}_A,\omega)$, $q_B^+(\mathbf{x},\omega) = \delta(\mathbf{x}-\mathbf{x}_B)$, $q_B^-(\mathbf{x},\omega) = 0$ and $p_B^\pm(\mathbf{x},\omega) = G^{\pm,+}(\mathbf{x},\mathbf{x}_B,\omega)$ into equations (29) and (30), using equations (22) and (23) and $G^{+,+}(\mathbf{x},\mathbf{x}_B,\omega) = 0$ for $\mathbf{x}$ at $\mathbb{S}_0$ (because the upper half-space is homogeneous), gives

$$G^{-,+}(\mathbf{x}_A,\mathbf{x}_B,\omega) + \chi(\mathbf{x}_B) f_1^-(\mathbf{x}_B,\mathbf{x}_A,\omega) = \int_{\mathbb{S}_0} \frac{2}{i\omega\rho(\mathbf{x})} G^{-,+}(\mathbf{x},\mathbf{x}_B,\omega)\partial_3 f_1^+(\mathbf{x},\mathbf{x}_A,\omega)\mathrm{d}\mathbf{x} \tag{31}$$

and

$$G^{+,+}(\mathbf{x}_A,\mathbf{x}_B,\omega) - \chi(\mathbf{x}_B)\{f_1^+(\mathbf{x}_B,\mathbf{x}_A,\omega)\}^* = -\int_{\mathbb{S}_0} \frac{2}{i\omega\rho(\mathbf{x})} G^{-,+}(\mathbf{x},\mathbf{x}_B,\omega)\{\partial_3 f_1^-(\mathbf{x},\mathbf{x}_A,\omega)\}^*\mathrm{d}\mathbf{x}. \tag{32}$$

Next, substituting $q_A^+(\mathbf{x},\omega) = q_A^-(\mathbf{x},\omega) = 0$, $p_A^\pm(\mathbf{x},\omega) = f_1^\pm(\mathbf{x},\mathbf{x}_A,\omega)$, $q_B^+(\mathbf{x},\omega) = 0$, $q_B^-(\mathbf{x},\omega) = \delta(\mathbf{x}-\mathbf{x}_B)$ and $p_B^\pm(\mathbf{x},\omega) = G^{\pm,-}(\mathbf{x},\mathbf{x}_B,\omega)$ into equations (29) and (30), using equations (22) and (23) and $G^{+,-}(\mathbf{x},\mathbf{x}_B,\omega) = 0$ for $\mathbf{x}$ at $\mathbb{S}_0$, gives

$$G^{-,-}(\mathbf{x}_A,\mathbf{x}_B,\omega) + \chi(\mathbf{x}_B) f_1^+(\mathbf{x}_B,\mathbf{x}_A,\omega) = \int_{\mathbb{S}_0} \frac{2}{i\omega\rho(\mathbf{x})} G^{-,-}(\mathbf{x},\mathbf{x}_B,\omega)\partial_3 f_1^+(\mathbf{x},\mathbf{x}_A,\omega)\mathrm{d}\mathbf{x} \tag{33}$$

and

$$G^{+,-}(\mathbf{x}_A,\mathbf{x}_B,\omega) - \chi(\mathbf{x}_B)\{f_1^-(\mathbf{x}_B,\mathbf{x}_A,\omega)\}^* = -\int_{\mathbb{S}_0} \frac{2}{i\omega\rho(\mathbf{x})} G^{-,-}(\mathbf{x},\mathbf{x}_B,\omega)\{\partial_3 f_1^-(\mathbf{x},\mathbf{x}_A,\omega)\}^*\mathrm{d}\mathbf{x}. \tag{34}$$

Summing equations (31) – (34), using source-receiver reciprocity for the Green's function on the left-hand side and $G^{+,+}(\mathbf{x},\mathbf{x}_B,\omega) = G^{+,-}(\mathbf{x},\mathbf{x}_B,\omega) = 0$ for $\mathbf{x}$ at $\mathbb{S}_0$, we obtain

$$\begin{aligned} &G(\mathbf{x}_B,\mathbf{x}_A,\omega) + \chi(\mathbf{x}_B) 2i\Im\{f_1(\mathbf{x}_B,\mathbf{x}_A,\omega)\} \\ &\quad = \int_{\mathbb{S}_0} \frac{2}{i\omega\rho(\mathbf{x})} G(\mathbf{x},\mathbf{x}_B,\omega)\partial_3\big(f_1^+(\mathbf{x},\mathbf{x}_A,\omega) - \{f_1^-(\mathbf{x},\mathbf{x}_A,\omega)\}^*\big)\mathrm{d}\mathbf{x}. \end{aligned} \tag{35}$$

Taking the real part of both sides gives the single-sided representation of the homogeneous Green's function for the situation that the upper half-space is homogeneous

$$G_\mathrm{h}(\mathbf{x}_B,\mathbf{x}_A,\omega) = 4\Re\int_{\mathbb{S}_0} \frac{1}{i\omega\rho(\mathbf{x})} G(\mathbf{x},\mathbf{x}_B,\omega)\partial_3\big(f_1^+(\mathbf{x},\mathbf{x}_A,\omega) - \{f_1^-(\mathbf{x},\mathbf{x}_A,\omega)\}^*\big)\mathrm{d}\mathbf{x}. \tag{36}$$

We conclude by deriving source-receiver reciprocity relations for the decomposed Green's functions $G^{\pm,\pm}(\mathbf{x},\mathbf{x}_B,\omega)$. We consider equation (29), but replace $\mathbb{V}_A$ by the entire space $\mathbb{R}^3$. In this situation there are only outgoing waves at $\mathbb{S}$. Hence, equation (29) simplifies to

$$\int_{\mathbb{R}^3} \big(p_A^+ q_B^- + p_A^- q_B^+ - q_A^+ p_B^- - q_A^- p_B^+\big)\mathrm{d}\mathbf{x} = 0. \tag{37}$$

First we substitute $q_A^+ = \delta(\mathbf{x}-\mathbf{x}_A)$, $q_A^- = 0$, $p_A^\pm = G^{\pm,+}(\mathbf{x},\mathbf{x}_A,\omega)$, $q_B^+ = \delta(\mathbf{x}-\mathbf{x}_B)$, $q_B^- = 0$ and $p_B^\pm = G^{\pm,+}(\mathbf{x},\mathbf{x}_B,\omega)$. This gives

$$G^{-,+}(\mathbf{x}_B,\mathbf{x}_A,\omega) = G^{-,+}(\mathbf{x}_A,\mathbf{x}_B,\omega). \tag{38}$$

Next, we substitute $q_A^+ = \delta(\mathbf{x}-\mathbf{x}_A)$, $q_A^- = 0$, $p_A^\pm = G^{\pm,+}(\mathbf{x},\mathbf{x}_A,\omega)$, $q_B^+ = 0$, $q_B^- = \delta(\mathbf{x}-\mathbf{x}_B)$ and $p_B^\pm = G^{\pm,-}(\mathbf{x},\mathbf{x}_B,\omega)$. This gives

$$G^{+,+}(\mathbf{x}_B,\mathbf{x}_A,\omega) = G^{-,-}(\mathbf{x}_A,\mathbf{x}_B,\omega). \tag{39}$$

Finally, we substitute $q_A^+ = 0$, $q_A^- = \delta(\mathbf{x}-\mathbf{x}_A)$, $p_A^\pm = G^{\pm,-}(\mathbf{x},\mathbf{x}_A,\omega)$, $q_B^+ = 0$, $q_B^- = \delta(\mathbf{x}-\mathbf{x}_B)$ and $p_B^\pm = G^{\pm,-}(\mathbf{x},\mathbf{x}_B,\omega)$. This gives

$$G^{+,-}(\mathbf{x}_B,\mathbf{x}_A,\omega) = G^{+,-}(\mathbf{x}_A,\mathbf{x}_B,\omega). \tag{40}$$

Note that equation (39) does not include a minus sign, unlike the corresponding relation for the flux-normalised decomposed Green's functions (Wapenaar, 1996a). This is due to the definition of $q^\pm$ in equation (A46). As a result of this definition, we have the following simple expression for the full Green's function

$$G(\mathbf{x},\mathbf{x}_A,\omega) = G^{+,+}(\mathbf{x},\mathbf{x}_A,\omega) + G^{-,+}(\mathbf{x},\mathbf{x}_A,\omega) + G^{+,-}(\mathbf{x},\mathbf{x}_A,\omega) + G^{-,-}(\mathbf{x},\mathbf{x}_A,\omega). \tag{41}$$

Appendix A: Decomposition of the integrals in the reciprocity theorems

A1 Matrix-vector wave equation

By eliminating v_1 and v_2 from equations (9) and (10), we obtain the following matrix-vector wave equation in the space-frequency domain

$$\partial_3 \mathbf{q} = \boldsymbol{\mathcal{A}}\mathbf{q} + \mathbf{d}, \tag{A1}$$

where

$$\mathbf{q} = \begin{pmatrix} p \\ v_3 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ q \end{pmatrix}, \quad \boldsymbol{\mathcal{A}} = \begin{pmatrix} 0 & \mathcal{A}_{12} \\ \mathcal{A}_{21} & 0 \end{pmatrix}, \tag{A2}$$

with

$$\mathcal{A}_{12} = i\omega\rho, \tag{A3}$$

$$\mathcal{A}_{21} = i\omega\kappa - \frac{1}{i\omega}\partial_\alpha \tfrac{1}{\rho}\partial_\alpha \tag{A4}$$

(Corones, 1975; Ursin, 1983; Fishman and McCoy, 1984; Wapenaar and Berkhout, 1989; de Hoop, 1996). Here ∂_α stands for the spatial differential operator $\partial/\partial x_\alpha$. Greek subscripts take on the values 1 and 2 and Einstein's summation convention applies to repeated subscripts. The notation in the right-hand side of equation (A4) should be understood in the sense that differential operators act on all factors to the right of it. Hence, operator $\partial_\alpha \frac{1}{\rho}\partial_\alpha$, applied via equation (A1) to p, stands for $\partial_\alpha(\frac{1}{\rho}\partial_\alpha p)$.

A2 Decomposition of the matrix-vector wave equation

For the decomposition of the matrix-vector wave equation, we first recast the operator matrix $\boldsymbol{\mathcal{A}}$ into a more symmetric form. To this end we define an operator $\mathcal{H}_2$, according to

$$\mathcal{H}_2 = -i\omega\sqrt{\rho}\mathcal{A}_{21}\sqrt{\rho} = k^2 + \sqrt{\rho}\partial_\alpha\tfrac{1}{\rho}\partial_\alpha\sqrt{\rho}, \tag{A5}$$

with

$$k^2 = \frac{\omega^2}{c^2}, \quad c = \frac{1}{\sqrt{\rho\kappa}}. \tag{A6}$$

After some bookkeeping it follows that $\mathcal{H}_2$ can be written as a 2D Helmholtz operator

$$\mathcal{H}_2 = k_s^2 + \partial_\alpha\partial_\alpha \tag{A7}$$

(Wapenaar and Berkhout, 1989; de Hoop, 1992), with the scaled wavenumber k_s obeying

$$k_s^2 = \frac{\omega^2}{c^2} - \frac{3(\partial_\alpha\rho)(\partial_\alpha\rho)}{4\rho^2} + \frac{(\partial_\alpha\partial_\alpha\rho)}{2\rho} \tag{A8}$$

(Brekhovskikh, 1960). We now rewrite operator matrix $\boldsymbol{\mathcal{A}}$ as

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} 0 & i\omega\rho \\ -\frac{1}{i\omega\sqrt{\rho}}\mathcal{H}_2\frac{1}{\sqrt{\rho}} & 0 \end{pmatrix}. \tag{A9}$$

The decomposition of this matrix is not unique. Flux-normalized decomposition is discussed by de Hoop (1996) and Wapenaar (1996b). Here we discuss a symmetric form of pressure-normalized decomposition, modified after Wapenaar and Berkhout (1989). We decompose the matrix as follows

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{L}}^{-1}, \tag{A10}$$

with

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\omega\rho}\mathcal{H}_1^s & -\frac{1}{\omega\rho}\mathcal{H}_1^s \end{pmatrix}, \quad \boldsymbol{\mathcal{H}} = \begin{pmatrix} i\mathcal{H}_1^s & 0 \\ 0 & -i\mathcal{H}_1^s \end{pmatrix}, \quad \boldsymbol{\mathcal{L}}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \left(\frac{1}{\omega\rho}\mathcal{H}_1^s\right)^{-1} \\ 1 & -\left(\frac{1}{\omega\rho}\mathcal{H}_1^s\right)^{-1} \end{pmatrix}. \tag{A11}$$

Here

$$\mathcal{H}_1^s = \sqrt{\rho}\mathcal{H}_1\tfrac{1}{\sqrt{\rho}}, \tag{A12}$$

where $\mathcal{H}_1$ is the square-root of the Helmholtz operator, according to

$$\mathcal{H}_1\mathcal{H}_1 = \mathcal{H}_2. \tag{A13}$$

We decompose the wave vector $\mathbf{q}$ and the source vector $\mathbf{s}$ as follows

$$\mathbf{q} = \boldsymbol{\mathcal{L}}\mathbf{p}, \quad \mathbf{p} = \boldsymbol{\mathcal{L}}^{-1}\mathbf{q}, \tag{A14}$$

$$\mathbf{d} = \boldsymbol{\mathcal{L}}\mathbf{s}, \quad \mathbf{s} = \boldsymbol{\mathcal{L}}^{-1}\mathbf{d}, \tag{A15}$$

with

$$\mathbf{p}=\begin{pmatrix} p^+ \\ p^- \end{pmatrix}, \quad \mathbf{s}=\begin{pmatrix} s^+ \\ s^- \end{pmatrix}. \tag{A16}$$

Substitution of equations (A14) and (A15) into the matrix-vector wave equation (A1), using equation (A10), yields

$$\partial_3 \mathbf{p}=\boldsymbol{\mathcal{B}}\mathbf{p}+\mathbf{s}, \tag{A17}$$

with

$$\boldsymbol{\mathcal{B}}=\boldsymbol{\mathcal{H}}-\boldsymbol{\mathcal{L}}^{-1}\partial_3\boldsymbol{\mathcal{L}}, \tag{A18}$$

or

$$\partial_3\begin{pmatrix} p^+ \\ p^- \end{pmatrix}=\begin{pmatrix} i\mathcal{H}_1^s & 0 \\ 0 & -i\mathcal{H}_1^s \end{pmatrix}\begin{pmatrix} p^+ \\ p^- \end{pmatrix}-\frac{1}{2}\begin{pmatrix} \left(\frac{1}{\rho}\mathcal{H}_1^s\right)^{-1}\partial_3\left(\frac{1}{\rho}\mathcal{H}_1^s\right) & -\left(\frac{1}{\rho}\mathcal{H}_1^s\right)^{-1}\partial_3\left(\frac{1}{\rho}\mathcal{H}_1^s\right) \\ -\left(\frac{1}{\rho}\mathcal{H}_1^s\right)^{-1}\partial_3\left(\frac{1}{\rho}\mathcal{H}_1^s\right) & \left(\frac{1}{\rho}\mathcal{H}_1^s\right)^{-1}\partial_3\left(\frac{1}{\rho}\mathcal{H}_1^s\right) \end{pmatrix}\begin{pmatrix} p^+ \\ p^- \end{pmatrix}+\begin{pmatrix} s^+ \\ s^- \end{pmatrix}. \tag{A19}$$

This is a system of coupled one-way wave equations for downgoing and upgoing waves, p^+ and p^-, respectively. With the definitions of $\mathbf{q}$ and $\mathbf{p}$ in equations (A2) and (A16), respectively, and $\boldsymbol{\mathcal{L}}$ in equation (A11), it follows from equation (A14) that

$$p=p^+ + p^-. \tag{A20}$$

Hence, the decomposed fields p^+ and p^- are indeed pressure-normalised downgoing and upgoing waves.

A3 Symmetry properties of the operators

For an operator $\mathcal{U}$, of which the entries are operators containing space-dependent medium parameters and differential operators ∂_1 and ∂_2, we introduce its transpose $\mathcal{U}^t$ and its adjoint (i.e., complex conjugate transpose) $\mathcal{U}^\dagger$ via

$$\int_{\mathbb{A}}(\mathcal{U}f)^t g\,\mathrm{d}\mathbf{x}=\int_{\mathbb{A}} f(\mathcal{U}^t g)\,\mathrm{d}\mathbf{x} \tag{A21}$$

and

$$\int_{\mathbb{A}}(\mathcal{U}f)^* g\,\mathrm{d}\mathbf{x}=\int_{\mathbb{A}} f^*(\mathcal{U}^\dagger g)\,\mathrm{d}\mathbf{x}, \tag{A22}$$

where $\mathbb{A}$ is an infinite horizontal integration boundary at arbitrary depth x_3, and $f=f(\mathbf{x})$ and $g=g(\mathbf{x})$ are space-dependent functions with sufficient decay along $\mathbb{A}$ towards infinity. For the Helmholtz operator $\mathcal{H}_2$, defined in equation (A7), we have

$$\mathcal{H}_2^t=\mathcal{H}_2, \tag{A23}$$

meaning $\mathcal{H}_2$ is a symmetric operator. Since we consider a lossless medium, we also have

$$\mathcal{H}_2^\dagger=\mathcal{H}_2^*=\mathcal{H}_2, \tag{A24}$$

meaning $\mathcal{H}_2$ is also a self-adjoint operator.

The square-root operator $\mathcal{H}_1$, defined in equation (A13), is a pseudo-differential operator. It obeys the following symmetry property

$$\mathcal{H}_1^t = \mathcal{H}_1, \tag{A25}$$

meaning $\mathcal{H}_1$ is a symmetric operator (Wapenaar and Grimbergen, 1996). Ignoring evanescent waves, we have

$$\mathcal{H}_1^\dagger = \mathcal{H}_1^* \approx \mathcal{H}_1. \tag{A26}$$

Hence, this operator is not self-adjoint. In the following we replace approximation signs by equal signs whenever the only approximation is the negligence of evanescent waves. Operator $\mathcal{H}_1^s$, defined in equation (A12), obeys the following symmetry properties

$$\left(\tfrac{1}{\rho}\mathcal{H}_1^s\right)^t = \tfrac{1}{\rho}\mathcal{H}_1^s, \tag{A27}$$

$$\left(\tfrac{1}{\rho}\mathcal{H}_1^s\right)^\dagger = \tfrac{1}{\rho}\mathcal{H}_1^s. \tag{A28}$$

From these symmetry relations, we find that $\boldsymbol{\mathcal{L}}$, defined in equation (A11), obeys the following properties

$$\boldsymbol{\mathcal{L}}^t\mathbf{N}\boldsymbol{\mathcal{L}} = \begin{pmatrix} 0 & -\frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right) \\ \frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right)^t \\ \frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right)^t & 0 \end{pmatrix} \tag{A29}$$

and, ignoring evanescent waves,

$$\boldsymbol{\mathcal{L}}^\dagger\mathbf{K}\boldsymbol{\mathcal{L}} = \begin{pmatrix} \frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right) & 0 \\ 0 & -\frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right) \end{pmatrix} = \begin{pmatrix} \frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right)^\dagger & 0 \\ 0 & -\frac{2}{\omega}\left(\frac{1}{\rho}\mathcal{H}_1^s\right)^\dagger \end{pmatrix}, \tag{A30}$$

with

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A31}$$

A4 Decomposition of the boundary integrals

For the boundary integrals along $\mathbb{S}_A$ appearing in equations (18) and (19) we introduce the following compact notation (using $\frac{1}{i\omega\rho}\partial_3 p = v_3$)

$$\int_{\mathbb{S}_A} \frac{1}{i\omega\rho}\{p_A(\partial_3 p_B) - (\partial_3 p_A)p_B\}\mathrm{d}\mathbf{x} = \int_{\mathbb{S}_A} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\mathrm{d}\mathbf{x} \tag{A32}$$

and

$$\int_{\mathbb{S}_A} \frac{1}{i\omega\rho}\{p_A^*(\partial_3 p_B) - (\partial_3 p_A^*)p_B\}\mathrm{d}\mathbf{x} = \int_{\mathbb{S}_A} \mathbf{q}_A^\dagger\mathbf{K}\mathbf{q}_B\mathrm{d}\mathbf{x}, \tag{A33}$$

respectively. With the decomposition of $\mathbf{q}$ defined in equation (A14), the properties of $\boldsymbol{\mathcal{L}}$ formulated in equations (A29) and (A30), and the definition of $\mathbf{p}$ in equation (A16) we obtain

$$\int_{\mathbb{S}_A} \mathbf{q}_A^t\mathbf{N}\mathbf{q}_B\mathrm{d}\mathbf{x} = \int_{\mathbb{S}_A} \mathbf{p}_A^t\boldsymbol{\mathcal{L}}^t\mathbf{N}\boldsymbol{\mathcal{L}}\mathbf{p}_B\mathrm{d}\mathbf{x} = -\int_{\mathbb{S}_A} \frac{2}{\omega}\left(p_A^+(\tfrac{1}{\rho}\mathcal{H}_1^s)^t p_B^- - p_A^-(\tfrac{1}{\rho}\mathcal{H}_1^s)^t p_B^+\right)\mathrm{d}\mathbf{x} \tag{A34}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{S}_A} \mathbf{q}_A^\dagger \mathbf{K} \mathbf{q}_B \mathrm{d}\mathbf{x} = \int_{\mathbb{S}_A} \mathbf{p}_A^\dagger \boldsymbol{\mathcal{L}}^\dagger \mathbf{K} \boldsymbol{\mathcal{L}} \mathbf{p}_B \mathrm{d}\mathbf{x} = \int_{\mathbb{S}_A} \frac{2}{\omega} \big(p_A^{+*} (\tfrac{1}{\rho}\mathcal{H}_1^s)^\dagger p_B^+ - p_A^{-*} (\tfrac{1}{\rho}\mathcal{H}_1^s)^\dagger p_B^- \big) \mathrm{d}\mathbf{x}. \tag{A35}$$

Assuming that in state A there are no vertical derivatives of the medium parameters at $\mathbb{S}_A$, we find from equation (A19)

$$\partial_3 p_A^\pm = \pm i \mathcal{H}_1^s p_A^\pm \quad \text{at } \mathbb{S}_A. \tag{A36}$$

Using this in equations (A34) and (A35) and substituting the results in equations (A32) and (A33), we obtain

$$\int_{\mathbb{S}_A} \frac{1}{i\omega\rho} \{ p_A (\partial_3 p_B) - (\partial_3 p_A) p_B \} \mathrm{d}\mathbf{x} = -\int_{\mathbb{S}_A} \frac{2}{i\omega\rho} \Big((\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \Big) \mathrm{d}\mathbf{x} \tag{A37}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{S}_A} \frac{1}{i\omega\rho} \{ p_A^* (\partial_3 p_B) - (\partial_3 p_A^*) p_B \} \mathrm{d}\mathbf{x} = -\int_{\mathbb{S}_A} \frac{2}{i\omega\rho} \Big((\partial_3 p_A^+)^* p_B^+ + (\partial_3 p_A^-)^* p_B^- \Big) \mathrm{d}\mathbf{x}. \tag{A38}$$

When $\mathbb{S}_0$ in equations (18) and (19) is also a horizontal boundary, with $\mathbf{n} = (0,0,-1)$, we obtain (assuming that in state A there are no vertical derivatives of the medium parameters at $\mathbb{S}_0$)

$$\int_{\mathbb{S}_0} \frac{-1}{i\omega\rho} \{ p_A (\partial_3 p_B) - (\partial_3 p_A) p_B \} \mathrm{d}\mathbf{x} = \int_{\mathbb{S}_0} \frac{2}{i\omega\rho} \Big((\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \Big) \mathrm{d}\mathbf{x} \tag{A39}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{S}_0} \frac{-1}{i\omega\rho} \{ p_A^* (\partial_3 p_B) - (\partial_3 p_A^*) p_B \} \mathrm{d}\mathbf{x} = \int_{\mathbb{S}_0} \frac{2}{i\omega\rho} \Big((\partial_3 p_A^+)^* p_B^+ + (\partial_3 p_A^-)^* p_B^- \Big) \mathrm{d}\mathbf{x}. \tag{A40}$$

A5 Decomposition of the volume integrals

Assuming both $\mathbb{S}_0$ and $\mathbb{S}_A$ are horizontal boundaries, we introduce the following compact notation for the volume integrals in equations (18) and (19)

$$\int_{\mathbb{V}_A} \{ p_A q_B - q_A p_B \} \mathrm{d}\mathbf{x} = \int_{\mathbb{V}_A} \big(\mathbf{d}_A^t \mathbf{N} \mathbf{q}_B + \mathbf{q}_A^t \mathbf{N} \mathbf{d}_B \big) \mathrm{d}\mathbf{x} \tag{A41}$$

and

$$\int_{\mathbb{V}_A} \{ p_A^* q_B + q_A^* p_B \} \mathrm{d}\mathbf{x} = \int_{\mathbb{V}_A} \big(\mathbf{d}_A^\dagger \mathbf{K} \mathbf{q}_B + \mathbf{q}_A^\dagger \mathbf{K} \mathbf{d}_B \big) \mathrm{d}\mathbf{x}, \tag{A42}$$

respectively. With the decomposition of $\mathbf{q}$ and $\mathbf{d}$ defined in equations (A14) and (A15), the properties of $\boldsymbol{\mathcal{L}}$ formulated in equations (A29) and (A30), and the definition of $\mathbf{p}$ and $\mathbf{s}$ in equation (A16), we obtain

$$\begin{aligned} \int_{\mathbb{V}_A} \big(\mathbf{d}_A^t \mathbf{N} \mathbf{q}_B + \mathbf{q}_A^t \mathbf{N} \mathbf{d}_B \big) \mathrm{d}\mathbf{x} &= \int_{\mathbb{V}_A} \big(\mathbf{s}_A^t \boldsymbol{\mathcal{L}}^t \mathbf{N} \boldsymbol{\mathcal{L}} \mathbf{p}_B + \mathbf{p}_A^t \boldsymbol{\mathcal{L}}^t \mathbf{N} \boldsymbol{\mathcal{L}} \mathbf{s}_B \big) \mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{V}_A} \frac{2}{\omega} \big(s_A^+ (\tfrac{1}{\rho}\mathcal{H}_1^s)^t p_B^- - s_A^- (\tfrac{1}{\rho}\mathcal{H}_1^s)^t p_B^+ + p_A^+ (\tfrac{1}{\rho}\mathcal{H}_1^s) s_B^- - p_A^- (\tfrac{1}{\rho}\mathcal{H}_1^s) s_B^+ \big) \mathrm{d}\mathbf{x} \end{aligned} \tag{A43}$$

and, ignoring evanescent waves,

$$\int\limits_{\mathbb{V}_A} \left(\mathbf{d}_A^\dagger \mathbf{K} \mathbf{q}_B + \mathbf{q}_A^\dagger \mathbf{K} \mathbf{d}_B\right) \mathrm{d}\mathbf{x} = \int\limits_{\mathbb{V}_A} \left(\mathbf{s}_A^\dagger \boldsymbol{\mathcal{L}}^\dagger \mathbf{K} \boldsymbol{\mathcal{L}} \mathbf{p}_B + \mathbf{p}_A^\dagger \boldsymbol{\mathcal{L}}^\dagger \mathbf{K} \boldsymbol{\mathcal{L}} \mathbf{s}_B\right) \mathrm{d}\mathbf{x}$$

$$= \int\limits_{\mathbb{V}_A} \frac{2}{\omega} \left(s_A^{+*} (\tfrac{1}{\rho} \mathcal{H}_1^s)^\dagger p_B^+ - s_A^{-*} (\tfrac{1}{\rho} \mathcal{H}_1^s)^\dagger p_B^- + p_A^{+*} (\tfrac{1}{\rho} \mathcal{H}_1^s) s_B^+ - p_A^{-*} (\tfrac{1}{\rho} \mathcal{H}_1^s) s_B^-\right) \mathrm{d}\mathbf{x}. \tag{A44}$$

From $\mathbf{s} = \boldsymbol{\mathcal{L}}^{-1} \mathbf{d}$, and the definitions of $\mathbf{d}$, $\boldsymbol{\mathcal{L}}^{-1}$ and $\mathbf{s}$ in equations (A2), (A11) and (A16), we find

$$s^\pm = \pm \left(\frac{2}{\omega\rho} \mathcal{H}_1^s\right)^{-1} q. \tag{A45}$$

We define new decomposed sources q^+ and q^-, according to

$$q^\pm = \pm \frac{2}{\omega\rho} \mathcal{H}_1^s s^\pm. \tag{A46}$$

Using this definition in equations (A43) and (A44) and substituting the results in equations (A41) and (A42), we obtain

$$\int\limits_{\mathbb{V}_A} \{p_A q_B - q_A p_B\} \mathrm{d}\mathbf{x} = \int\limits_{\mathbb{V}_A} \left(p_A^+ q_B^- + p_A^- q_B^+ - q_A^+ p_B^- - q_A^- p_B^+\right) \mathrm{d}\mathbf{x} \tag{A47}$$

and, ignoring evanescent waves,

$$\int\limits_{\mathbb{V}_A} \{p_A^* q_B + q_A^* p_B\} \mathrm{d}\mathbf{x} = \int\limits_{\mathbb{V}_A} \left(p_A^{+*} q_B^+ + p_A^{-*} q_B^- + q_A^{+*} p_B^+ + q_A^{-*} p_B^-\right) \mathrm{d}\mathbf{x}. \tag{A48}$$

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