Improvement on controlled-controlled-NOT operation-based entanglement purification protocol

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We show that controlled-controlled-NOT (CCN) operation-based entanglement purification protocol can be further improved. CCN protocol requires Bell state measurements after performing the CCN operations. In the original CCN protocol, the measured states are assumed to be destroyed. However, if controlled-NOT gates are used to perform such Bell state measurements, in some unsuccessful situations of the CCN protocol, one can further purify the two mixed entangled states which are to be measured. In this way, the total efficiency of the CCN protocol is further increased.

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Quantum entanglement is of great significance in quantum information processing and fundamental quantum mechanics. Many schemes have been proposed to generate quantum entangled states[1,2], especially the maximally entangled states, which are often required in quantum information processing. However, it is almost impossible to directly generate and transmit maximally entangled states without degrading their entanglement due to decoherence. Quantum entanglement purification or distillation, which purifies mixed entangled states to maximally entangled state, is thus very important in quantum communication and computation.

The first entanglement purification protocol based on local controlled-NOT (CNOT) operations and classical communication was proposed by Bennett *et al.*[3,4]. This protocol was then developed to purify generalized Werner states by Deutsch *et al.*[5]. Both protocols were regarded as standard entanglement purification protocol afterwards. Since then many theoretical schemes have been proposed[6−10] and verified in the experiment[11]. A basic rule for purifying mixed entangled states by local operations and classical communication (LOCC) is to perform collective operations on two or more mixed entangled states since any LOCC operations on a single mixed entangled state cannot let it purified with nonzero probability[12,13]. Feng *et al.* showed that collectively performing controlled-controlled-NOT (CCN) operations on three Werner states, followed by Bell state measurements, can achieve more efficient purification in comparison with the standard protocol[6]. For simplicity we name this protocol the CCN protocol. The CCN protocol was also applied to the purification of generalized Werner states[10]. Although there are several errors in the original protocol of Ref. [6] as pointed out recently[14], its main results are still valid[15].

In this letter, we show that CCN protocol can be further improved by reconsidering the Bell state measurements using the CNOT gates. The standard and CCN protocols are reviewed briefly.

We assume Alice and Bob each share an ensemble of Werner states of the following form

$$\begin{aligned}\rho =&F|\varphi_{ij}^{+}><\varphi_{ij}^{+}|+\frac{1-F}{3}(|\varphi_{ij}^{-}><\varphi_{ij}^{-}|\\&+|\psi_{ij}^{+}><\psi_{ij}^{+}|+|\psi_{ij}^{-}><\psi_{ij}^{-}|),\end{aligned}\tag{1}$$

with fidelity $F>1/2$, where $|\varphi_{ij}^{\pm}>$ and $|\psi_{ij}^{\pm}>$ are Bell states of particles i and j with the particle i belonging to Alice and the particle j to Bob,

$$\begin{aligned}\varphi_{ij}^{\pm}&=\frac{1}{\sqrt{2}}(|00>_{ij}\pm|11>_{ij}),\\\psi_{ij}^{\pm}&=\frac{1}{\sqrt{2}}(|01>_{ij}\pm|10>_{ij}).\end{aligned}\tag{2}$$

To apply the standard purification protocol[3], Alice and Bob pick up two Werner states (1) and perform local CNOT operations on their own particles. Afterwards they measure the target qubits in the computational basis. If their measurement results are different they have to discard both pairs; otherwise, if their measurement results coincide, the unmeasured pair becomes an entangled state with larger fidelity of the form

$$F'=\frac{F^2+\frac{1}{9}(1-F)^2}{F^2+\frac{2}{3}F(1-F)+\frac{5}{9}(1-F)^2}.\tag{3}$$

The probability for one-round successful purification is

$$P'=\frac{1}{9}(5-4F+8F^2).\tag{4}$$

To apply the CCN protocol[5,13,14], Alice and Bob pick up three Werner states and perform local CCN operations on their own particles. We assume that the first and the second qubits act as control and the third one as target. Then Alice and Bob perform Bell state measurement on their two control qubits. In the case that their measurement results are both $|\varphi^+>$ (or both $|\varphi^->$), the target pair is purified with probability:

$$P'' = \frac{1}{2}F^3 + F^2\frac{1-F}{6} + \frac{7}{2}F\left(\frac{1-F}{3}\right)^2 + \frac{7}{2}\left(\frac{1-F}{3}\right)^3. \quad (5)$$

The fidelity of the target pair becomes

$$F'' = \frac{F^3 + F(\frac{1-F}{3})^2 + 2(\frac{1-F}{3})^3}{F^3 + F^2\frac{1-F}{3} + 7F(\frac{1-F}{3})^2 + 7(\frac{1-F}{3})^3}. \quad (6)$$

In any other cases, the purification of the target pair fails. But if the measurement results of both Alice and Bob are $|\psi^\pm>$, the target pair is kept untouched and can be used for further purification. To understand the last situation we consider that the CCN gate perform a NOT operation only when both control qubits are in state 1, nothing will happen to all three qubits if one or two of the control qubits are in state 0. If the measurement results of both Alice and Bob are $|\psi^\pm>$, one of the control qubits is in state 0, hence nothing will happen to all three Werner states after the CCN gate operations. In the original CCN protocol only the third Werner state is kept for further purification, and the two control pairs were assumed being destroyed by measurement[6,14,15]. In this letter, we show that in this situation, the two control pairs can be further purified by using the CNOT operations as Bell state analyzer.

In Fig. 1 we sketch CCN protocol with CNOT gate as the Bell state analyzer for both observers Alice and Bob. For CCN operation, the first and the second qubits act as control, and the third as target; for CNOT operation, the first qubit acts as control and the second as target. In Fig. 1, the Hadamard gates do not always work and hence are plotted in dotted lines.

In contrast to the original CCN protocol[6], the Bell states are measured step by step using the CNOT gates. After the operation of the CNOT gate, the Bell states (2) become

$$\begin{aligned} |\varphi^\pm_{\mu\upsilon}> &\Rightarrow \frac{1}{\sqrt{2}}(|0>_\mu \pm |1>_\mu)|0>_\upsilon, \\ |\psi^\pm_{\mu\upsilon}> &\Rightarrow \frac{1}{\sqrt{2}}(|0>_\mu \pm |1>_\mu)|1>_\upsilon . \end{aligned} \quad (7)$$

Note here the two particles μ and ν belong to Alice or to Bob. The transformations (7) indicate that after performing the CNOT operation, the measurement of the second qubit (i.e., the target of CNOT gate) can sufficiently identify $|\varphi^\pm_{\mu\upsilon}>$ and $|\psi^\pm_{\mu\upsilon}>$ with the output 0 and 1, respectively.

Fig. 1. CCN protocol with CNOT gate as the Bell state analyzer for both observers Alice and Bob. The dotted line in Hadamard gates means that Alice and Bob take the Hadamard gates operation as required.

On one hand, if both Alice and Bob's second qubits are measured in 0; the input state of the CNOT gates is in $|\varphi^+_{\mu\upsilon}>$ or $|\varphi^-_{\mu\upsilon}>$. In order to complete the CCN protocol, Alice and Bob perform the Hadamard operation on the first qubits (i.e., the control of the CNOT gate), and then measure them in the computational basis. The coincidence of Alice and Bob's measurement results is just corresponding to the successful purification situation of the CCN protocol; otherwise, if their measurement results are different, the purification procedure by CCN protocol fails.

On the other hand, if both Alice and Bob's second qubits are measured in 1; we are sure that the input state of CNOT gate for both Alice and Bob sides is $\psi^+_{\mu\upsilon}$ or $\psi^-_{\mu\upsilon}$. As mentioned above, the CCN gates in this situation do nothing to all three initial Werner states and the third Werner state can be used again as pointed out in Ref. [6]. However, the first and second Werner states can also be purified, rather than discarded as in Ref. [6]. In fact, the CNOT gate operations on the first and second qubits for both Alice and Bob sides play a role of bilateral CNOT operation in the standard purification protocol[3]. After bilateral CNOT operation, both Alice and Bob measure their second qubits (target qubits). If their measurement results coincide, they keep the first qubits (the control qubits), which is in an entangled state with larger entanglement. Otherwise, they discard both pairs. In this way, we obtain an entangled state (that is, the control pair of the CNOT gates) with the fidelity F′ expressed in Eq. (3). In this situation, the Hadamard operations plotted in Fig. 1 are not necessary to be performed any more. The probability for this situation to occur corresponds to the input states of CNOT gates for both the sides of Alice and Bob $|\psi^\pm_{\mu\upsilon}>$, that is, $|\psi^+_{\mu\upsilon}>|\psi^+_{\mu\upsilon}>$, $|\psi^-_{\mu\upsilon}>|\psi^-_{\mu\upsilon}>$, $|\psi^+_{\mu\upsilon}>|\psi^-_{\mu\upsilon}>$, or $|\psi^-_{\mu\upsilon}>|\psi^+_{\mu\upsilon}>$. In addition, the probabilities for such four cases[14] are

$$\begin{aligned} P_1 = P_2 &= \frac{1}{12}(1 - 2F + 4F^2), \\ P_3 = P_4 &= \frac{1}{18}(1 + F - 2F^2). \end{aligned} \quad (8)$$

Hence,

$$\begin{aligned} P' &= P_1 + P_2 + P_3 + P_4 \\ &= \frac{1}{18}(5 - 4F + 8F^2). \end{aligned} \quad (9)$$

In comparison with the successful probability P' of the standard protocol in Eq. (4), the above probability P''' is obviously a half of that one. Considering the probability of line-optics-based entanglement purification protocol[7] which was experimentally realized[11], is also a half of that of the standard protocol, our improvement on the CCN protocol in this letter is of practical importance.

We show that CCN protocol originally proposed in Ref. [6] can be further improved by using CNOT gates as Bell state analyzer. The measurement results of the target qubit of the CNOT gate, which can identify the input state is $|\varphi^\pm_{\mu\upsilon}>$ or $|\psi^\pm_{\mu\upsilon}>$. If both Alice and Bob observe the two control qubits of their CCN gates in $|\psi^+_{\mu\upsilon}>$ or $|\psi^-_{\mu\upsilon}>$, the CCN gates do nothing to each entangled state, while the two control pairs of the CCN gates are purified by CNOT gates according to the standard protocol. In this way, the total efficiency of the CCN protocol is further increased.

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