

Stability of elliptic vortex solitons in anisotropic nonlocal media

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Elliptic vortex solitons are investigated in anisotropic nonlocal media with more general formulations. We address the existence and dynamics of such solitons analytically and numerically. The solution of elliptic vortex solitons depends on the eccentricity of both the input beam and nonlocal response function. With different degrees of nonlocality, we numerically investigate the evolution of the elliptic vortex solitons, and find that, typically, the elliptic vortex solitons with single and double charges collapse into spiraling dipole- and tripole-like soliton clusters, respectively.

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Recently, nonlocal solitons have prompted extensive research^[1] because of their unique physical features, in many nonlinear materials, such as nematic liquid crystals, thermal materials, atomic vapors^[2], Bose–Einstein condensates, and photorefractive materials. In nonlinear optics, nonlocality means that the refractive index of a material at a particular point is not determined solely by the wave intensity at that point (as in local media) but also depends on the wave intensity in its neighborhood^[1]. Nonlocality can promote modulation instability in self-defocusing media, suppress modulation instability in self-focusing media^[3], prevent the catastrophic collapse of high-dimensional self-focusing beams^[4], and provide attractive forces between dark solitons^[5] which always repel in local media. Nonlocality also sustains a series of novel soliton states, such as stable multipole solitons^[6], gap solitons^[7], incoherent solitons^[8], and vector solitons^[9].

Circular vortices^[10], associated with phase singularities in the center, have drawn considerable attention in nonlocal media^[11–15]. Although the nonlocality can stabilize the vortex solitons, their dynamics depend crucially upon the actual forms of the nonlocal response function. In highly nonlocal media with Gaussian-type response function, there are no restrictions on the topological charge of the stable ring vortex solitons^[11,12]. With Helmholtz-type response function, only vortex solitons with single topological charge can be stable^[12]. The stable vortex solitons with single charge has been experimentally observed in nonlocal media with thermal optical nonlinearity^[13].

In contrast to the degenerated circular symmetry, elliptic symmetry is more generalized, which offers more degrees of freedom. Elliptic bright solitons can be achieved by introducing anisotropic nonlinearity of

photorefractive crystals^[16]. Experimental and theoretical works also have shown that elliptic bright solitons and twin-vortex solitons can be stable in thermal media with anisotropic nonlinearity^[13]. It has been addressed that elliptically shaped vortex solitons can exist in defocusing nonlinear media imprinted with a composite Mathieu lattice^[17], and can even exist in isotropic nonlocal nonlinear media^[18]. In anisotropic nonlocal media, elliptic dipole-mode solitons^[19] and spiraling elliptic solitons^[20] have been discussed.

In this work, we investigate analytically and numerically elliptic vortex quasi-solitons in anisotropic nonlocal nonlinear media. The evolution of the elliptic vortex solitons is also investigated. We find that the typical elliptic vortex solitons with single and double charges collapse into spiraling dipole- and tripole-like solitons, respectively.

Considering optical beams propagating in nonlocal media, the propagation of optical beams is governed by the following nonlocal nonlinear Schrödinger equations:

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \nabla_{\perp}^2 u + u \delta n(I) = 0, \quad (1)$$

$$\delta n(I) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(x-x', y-y') |u(x', y', z)|^2 dx' dy', \quad (2)$$

where $\nabla_{\perp}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian and $R(x, y)$ is the normalized nonlocal response function. In particular, the kernel R of the nonlocal response is determined by the nonlocal process of actual physical system. For example,

$R(x, y) = 1/(2\pi)^2 \iint [1 - \sqrt{\pi k} e^{k^2} \operatorname{erfc}(k)] e^{i(k_x x + k_y y)} dk_x dk_y$ describes the nonlocal response of a dipolar Bose–Einstein condensate where the nonlocal character of the interatomic potential is due to a long-range interaction

of dipoles^[14]. To make the problem solvable analytically, we assume the anisotropic nonlocal response function in the elliptic Gaussian form throughout this work.

$$R(x, y) = \frac{\beta}{\pi\sigma^2} \exp\left(-\frac{x^2 + \beta^2 y^2}{\sigma^2}\right), \quad (3)$$

where σ is the width of the response function, which determines the degree of nonlocality. When $\sigma \rightarrow 0$, we recover the local Kerr model, whereas we have the limit of strongly nonlocal nonlinear when $\sigma \rightarrow \infty$. β is the elliptic parameter, and when $\beta = 1$ the response function returns to be isotropic, in which elliptically modulated self-trapped singular beams have been discussed in detail^[18]. The ellipticity of the response function is $\varepsilon_\beta = \sqrt{1 - \beta^2}$ when $0 < \beta < 1$. The circular response function corresponds to $\varepsilon_\beta \rightarrow 0$, whereas a strongly stretched elliptic one corresponds to $\varepsilon_\beta \rightarrow 1$.

Let us restate the problem of solving Eq. (1) as an Euler–Lagrange equation corresponding to the variational principle^[21] of the Lagrangian density

$$L_r = \frac{i}{2} \left(u^* \frac{\partial u}{\partial z} - u \frac{\partial u^*}{\partial z} \right) - \frac{1}{2} \left(\left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial y} \right|^2 \right) + \frac{1}{2} |u|^2 \delta n(I). \quad (4)$$

As we know, Hermite–Gaussian solitons, Laguerre–Gaussian solitons^[22], and Ince–Gaussian solitons^[23] are exact solutions of the Snyder–Mitchell model (the linear model of the nonlocal nonlinear Schrödinger equation in the limit of strong nonlocality). Basically, one can assume the trial function in one form of them. For convenience in the variational calculations described below, we introduce the trial function of the elliptic vortex beam^[24]

$$u(x, y, z) = A \left(x^2 + \alpha^2 y^2 \right)^{\frac{m}{2}} \exp \left[-\frac{x^2 + \alpha^2 y^2}{2w^2} \right] \exp \left[i m \tan^{-1} \left(\frac{\alpha y}{x} \right) + i \mu z \right], \quad (5)$$

where m denotes topological charge, A is the amplitude of the solution, μ is the propagation constant, and α is related to ellipticity of the beam. When $0 < \alpha < 1$, major axis is in y direction (Fig. 1(a)) and the ellipticity is $\varepsilon_\alpha = \sqrt{1 - \alpha^2}$; while major axis is in x direction when $\alpha > 1$, and in this case, the behaviors of the beams agree with the case of $0 < \alpha < 1$, so we consider only the case of $0 < \alpha < 1$ below. The elliptic hollow beam turns into ring vortex beam with $\alpha = 1$. The total power of the elliptic vortex beam with the topological charge m is

$$P_0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x, y, z)|^2 dx dy = \frac{m! \pi A^2 w^{2m+2}}{\alpha}. \quad (6)$$

So the total powers of the elliptic vortex beam with single charge ($m = 1$) and double charge ($m = 2$) are

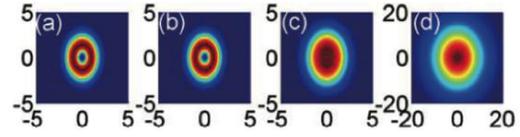


Fig. 1 (a) Intensity distribution and (b–d) nonlinear refractive index of the elliptic vortex beam when the beam width parameter is $w = 1$. The degree of the nonlocality is $\sigma = 0.1$ for (b), $\sigma = 1$ for (c), and $\sigma = 10$ for (d). Here the parameters $\alpha = 0.7$, $\beta = 0.8$. The topological charge is $m = 1$.

$P_{10} = \pi A^2 w^4 / \alpha$ and $P_{20} = 2\pi A^2 w^6 / \alpha$, respectively.

The integral widths of an optical beam along x and y directions are defined as

$$W_X = 2 \left(\langle X^2 \rangle - \langle X \rangle^2 \right)^{\frac{1}{2}}, \quad X \equiv \{x, y\}, \quad (7)$$

$$\langle X \rangle = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X |u|^2 dx dy}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u|^2 dx dy}. \quad (8)$$

It is worth mentioning that the value of $\langle X \rangle$ is zero for the present choice of u in Eq. (5). In this case, the integral widths of the elliptical vortex beams in two directions are

$$w_x^2 = 2(1+m)w^2, \quad w_y^2 = \frac{2(1+m)}{\alpha^2} w^2, \quad (9)$$

according to the above definition. It is clear that $w_x < w_y$ when $\alpha^2 < 1$. The solitons will always feature the elliptic shape in the case of $\alpha \neq 1$.

According to Eqs. (2), (3), and (5), we can get the nonlinear refractive index in nonlocal media induced by the elliptical vortex beams with single and double charges. Based on the definition of the integral widths in Eq. (7), the widths of nonlinear refractive index along x and y directions are represented as

$$W_{1x}^2 = 2(2w^2 + \sigma^2), \quad W_{1y}^2 = 2 \left(\frac{2w^2}{\alpha^2} + \frac{\sigma^2}{\beta^2} \right), \quad (10)$$

for single charge ($m = 1$) and

$$W_{2x}^2 = 2(3w^2 + \sigma^2), \quad W_{2y}^2 = 2 \left(\frac{3w^2}{\alpha^2} + \frac{\sigma^2}{\beta^2} \right), \quad (11)$$

for double charges ($m = 2$), respectively. The corresponding widths with double charges are always larger than the widths with single charge for the same σ , α , and β . We know the circular symmetry of the nonlinear refractive index when $\alpha = \beta = 1$. The nonlinear refractive index will also become circularly symmetric in the limit of high nonlocality ($w \ll \sigma$) with isotropic response function ($\beta = 1$). In the limit $\sigma \rightarrow 0$, the corresponding widths of nonlinear refractive index along x and y directions are equal to the widths of the optical beams. As σ increases, the integral widths of nonlinear refractive index will also increase (Fig. 1).

In Fig. 1, we show the nonlinear refractive index in the case of weak $\sigma = 0.1$, general $\sigma = 1$, and strong nonlocality $\sigma = 10$, with the width parameter $w = 1$. As shown in Fig. 1, the strong nonlocality can average out all spatial variations of the beam intensity distribution, leading to the peak of the nonlinear refractive index in the center even though there is a singularity in the center of the elliptic beam. Thus the strong nonlocality induces a smooth and attractive potential, resulting in the stabilization of the elliptic beam.

Substituting Eqs. (3) and (5) into (4), an effective Lagrangian is obtained by integrating the average Lagrangian density in the whole two-dimensional spatial coordinates

$$L = \langle L_r \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} L_r dx dy. \quad (12)$$

Based on the Euler–Lagrange equations, we obtain the following ordinary differential equation to describe the evolution dynamics of the beam widths in nonlocal media when the topological charge $m = 1, 2$:

$$\frac{d^2 w}{dz^2} = \frac{(1 + \alpha^2)^2}{4w^3} - P_{m0} Q_m, \quad (13)$$

where Q_1 and Q_2 are related to the nonlinear terms in the evolution equation for single and double charges, and are complicated functions of the parameters α, β, w , and σ , which we are not going to address here.

Solutions of elliptic vortex solitons can be obtained by setting $d^2 w/dz^2 = 0$ and $w(z) = w(0) = w_0$, which represents that the beam keeps its initial widths. Then Eq. (13) leads to the critical powers of such elliptic vortex soliton with single and double charges

$$P_{1c} = \frac{(1 + \alpha^2)^2}{4w_0^3 Q_{10}}, \quad P_{2c} = \frac{(1 + \alpha^2)^2}{4w_0^3 Q_{20}}, \quad (14)$$

where $Q_{10} = Q_1|_{w=w_0}$ and $Q_{20} = Q_2|_{w=w_0}$. We define the relative difference of the critical powers between the solitons with single and double charges as $\eta = (P_{2c} - P_{1c})/P_{2c}$. In Fig. 2, we plot the critical power of elliptic vortex solitons P_{1c} with single charge (the case of $m = 2$ can be obtained in the same way) versus the parameters α and β (Fig. 2(a)). As α increases, the critical power will increase, whereas P_{1c} will decrease with increasing β . These results indicate that the critical power will increase with the increase in the ellipticity of the nonlocal response function but decrease with the increase in the ellipticity of the elliptic vortex beams. It can be seen from Fig. 2(b) that the critical power of elliptic vortex solitons with single charge is always smaller than that of double charges, which shows that the vortex with larger topological charge requires larger power to maintain its initial profile. As the degree of nonlocality increases, the relative difference η between two critical powers gets smaller until it vanishes in highly nonlocal nonlinearity (Fig. 2(c)).

Next, we integrate Eq. (1) to investigate the stability and dynamics of the elliptic vortex solitons by

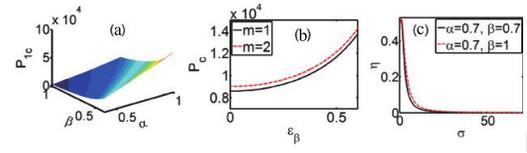


Fig. 2 (a) Critical power of the stationary elliptic vortex solitons as a function of the eccentricity parameters α and β with single charge $m = 1$. (b) Critical powers with different topological charges $m = 1$ and $m = 2$. (c) the relative difference between the critical powers of $m = 1$ and $m = 2$. Here the parameters are (a, b) $\sigma = 10$, (b) $\alpha = 0.5$. The width parameter is fixed at $w = 1$.

the split step beam propagation method. The variational results are used as the initial input of our two-dimensional numerical code. In Fig. 3, we show the dynamics of the elliptic vortex solitons with single charge propagating in nonlinear media with different degrees of nonlocality. In all our simulations, we make the initial beam width $w = 1$. At typical set of the parameters $\alpha = 0.7$ and $\beta = 0.8$, this elliptic vortex soliton is unstable in the anisotropic nonlocal media. The vortex beam splits into a dipole-like soliton with a repulsive force between its two lobes. The nonlocality cannot prevent the repulsion when the degree of nonlocality is weak ($\sigma = 0.2$) (Fig. 3(a)) and general ($\sigma = 1.6$) (Fig. 3(b)), and the two lobes of the dipole-like soliton will fly off. Increasing the degree of nonlocality can effectively improve the stability of the elliptic vortex solitons (compare Figs. 3(a) and (b)). In the case of $\sigma = 10$, the strong nonlocality can induce an attractive force (Fig. 1(d)) to stabilize the vortex beam although it is not completely stable. As shown in Fig. 3(c), the elliptic vortex beam will split into a dipole soliton with anti-clockwise rotation firstly and then evolve into elliptic vortex beam. Subsequently, it splits into another dipole-like solitons with clockwise rotation and evolve into elliptic vortex beam again. Although the vortex beam may split into the dipole soliton during the propagation, the strong nonlocality can eliminate the repulsive tendency, leading to a quasi-stable elliptic vortex soliton.

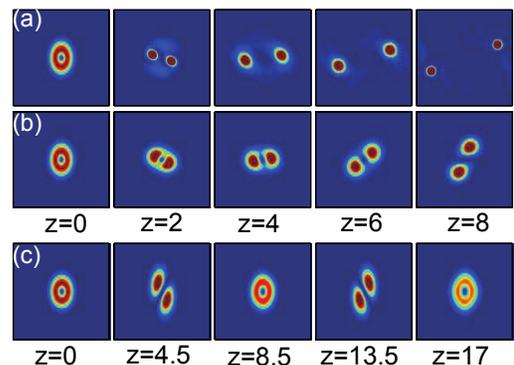


Fig. 3. Dynamics and evolution of the elliptic vortex beams with single charge for the degree of nonlocality of: (a) $\sigma = 0.2$, (b) $\sigma = 1.6$, and (c) $\sigma = 10$ when $\alpha = 0.7$ and $\beta = 0.8$.

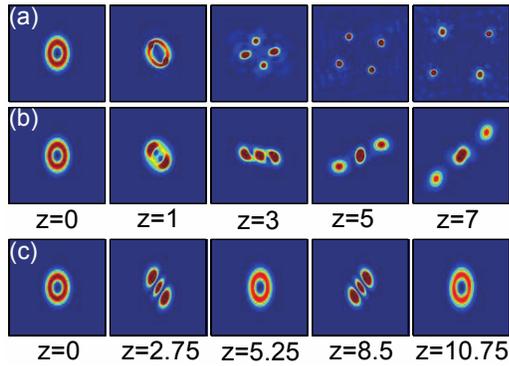


Fig. 4. Dynamics and evolution of the elliptic vortex beams with double charges for the degree of nonlocality of: (a) $\sigma = 0.5$, (b) $\sigma = 1.5$, and (c) $\sigma = 10$ when $\alpha = 0.7$ and $\beta = 0.5$.

We also simulate the propagation of the elliptic vortex beams with double charges at $\alpha = 0.7$ and $\beta = 0.7$. We find that the elliptic vortex beam with higher charge ($m = 2$) is more unstable than the beam with single charge. For instance, in the weak nonlocality $\sigma = 0.5$, the elliptic vortex beam with double charges will split into four fundamental solitons (Fig. 4(a)). On the other hand, the elliptic vortex beam with double charges will split into three fundamental solitons (Fig. 4(b)) when the degree of nonlocality is general ($\sigma = 1.5$). Similar to the case of single charge, the stability of the elliptic vortex beam can also be improved by increasing the degree of nonlocality. As shown in Fig. 4(c), the elliptic vortex beam will split into a tripole-like soliton with anti-clockwise rotation and evolve into vortex. Then it breakups into another tripole-like soliton with clockwise rotation and evolve into elliptic vortex beam again.

Now in order to clarify the stability of elliptic vortex solitons in nonlocal nonlinear media with anisotropic Kerr nonlinearity, we still need to consider the following two cases. One is for elliptic vortex solitons ($0 < \alpha \leq 1$) in nonlocally isotropic media ($\beta = 1$) and the other is for circular vortex solitons ($\alpha = 1$) with anisotropic Kerr nonlinearity ($0 < \beta \leq 1$). First we set $\beta = 1$ and $\sigma = 10$, which means it is the case of strongly nonlocally isotropic media, and we vary α and simulate the solitons. We find that when $\alpha > 0.7$, the elliptic vortex

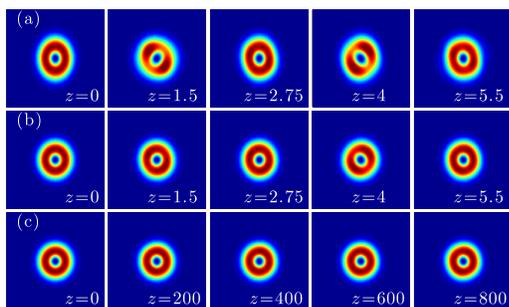


Fig. 5. Dynamics and evolution of the elliptic vortex beams with single charge in the isotropic media with strong nonlocality $\sigma = 10$ and $\beta = 1$ and (a) $\alpha = 0.8$, (b) $\alpha = 0.9$, and (c) $\alpha = 0.95$.

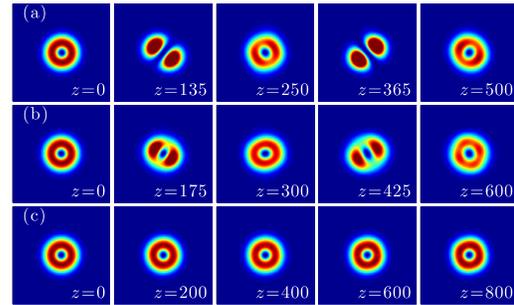


Fig. 6. Dynamics and evolution of ring vortex beams ($\alpha = 1$) with single charge in the anisotropic media with high nonlocality $\sigma = 10$ and (a) $\beta = 0.992$ (b) $\beta = 0.996$, and (c) $\beta = 1$.

solitons will not decay into dipoles during the evolution. As α goes up to 1, the elliptic vortex solitons are getting more and more stable (Fig. 5). We can see that when $\alpha = 0.95$, the elliptic vortex solitons are nearly stable. For ring vortex solitons in anisotropic Kerr nonlinear media, it is found that when β is bigger than 0.995, the circular vortex will not decay into dipoles (Figs. 6(a) and (b)). As an example, we display evolution of circular vortex solitons in anisotropic media in Fig. 6, for three β values, 0.992, 0.996, and 1. It is very clear that as β increases, the stability of circular vortex solitons gets much better.

In conclusion, we study analytically and numerically the self-trapping of elliptic vortex quasi-solitons in nonlocal nonlinear media with anisotropic Kerr nonlinearity. Using the variational approach, the existence of the elliptic vortex quasi-solitons is addressed. It is found that the elliptic vortex quasi-solitons are determined by the eccentricity of both the input beam and nonlocal response function. The evolution of the elliptic vortex solitons is also investigated numerically based on the split step beam propagation method. We find that the typically elliptic vortex solitons with single and double charges collapse into spiraling dipole and tripole-like solitons, respectively.

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