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## AN EXTENSION OF A PERMUTATIVE MODEL OF SET THEORY

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**Abstract.** In this paper we present a new axiomatic model of set theory called the *Extended Fraenkel Mostowski model*. It is defined by replacing an axiom of the Fraenkel-Mostowski model with a consequence of it; the other axioms of the Fraenkel-Mostowski model are left unchanged in the new Extended Fraenkel-Mostowski model.

We use the theory of groups to define the sets both in the Extended Fraenkel-Mostowski and in the Fraenkel-Mostowski approach. Several algebraic properties of the sets in the Fraenkel-Mostowski model remain also valid in the Extended Fraenkel-Mostowski model, even one axiom in the axiomatic description of the Extended Fraenkel-Mostowski model is weaker than its homologue in the axiomatic description of the Fraenkel-Mostowski model.

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**Key words:** Fraenkel-Mostowski model, axiom of choice, interchange function, permutation, group theory.

#### 1. Introduction

The Fraenkel-Mostowski permutation model of set theory (FM model) was introduced in 1930s to prove the independence of the Axiom of Choice (AC) from the other axioms of Zermelo-Fraenkel set theory with atoms (ZFA). The axiom of choice says that, for each indexed family of nonempty sets  $\{A_i \mid i \in I\}$ , there is a function  $f: I \to \bigcup \{A_i \mid i \in I\}$  such that  $f(i) \in A_i$ for each  $i \in I$ . If I is finite, then we can prove by induction that such a function f exists. The statement "For each indexed family of nonempty sets  $\{A_i \mid i \in I \text{ and } I \text{ is finite}\}$  there is a function  $f: I \to \bigcup \{A_i \mid i \in I\}$  such that  $f(i) \in A_i$  for each  $i \in I$  " is only a consequence of other axioms in ZFA, and it is not a form of the axiom of choice (see [7] for details). Informally,

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the axiom of choice says that given any collection of bins, each containing at least one object, it is possible to make a selection of exactly one object from each bin even though there are infinitely many bins and no "rule" of how to choose the objects. According to the work of K.Gödel and P.Cohen, the axiom of choice is logically independent of the other axioms of Zermelo-Fraenkel set theory (ZF). Consequently, if ZF is consistent, then ZFC is consistent, and  $ZF \neg C$  is also consistent. So the decision whether or not it is appropriate to make use of the axiom of choice in a proof cannot be made by appealing to other axioms of set theory. The FM model is built using all the axioms in ZFA model except the axiom of choice; it has also the special property of finite support described in the definition of interchange function  $(ZF\neg C + \text{finite support} - \text{see Definition 2 for the interchange function and})$ Definition 3 for the finite support). Additionally we have the axiom which says that the set of atoms A is infinite. Clearly the finite support property is in contradiction with the axiom of choice because, in our presentation, we obtain that  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$  (where  $\wp(A) = \{X | X \subset A\}$ ,  $\wp_{fin}(A) = \{X | X \subset A, X finite\}, \ \wp_{cofin}(A) = \{X | X \subset A, A \setminus X finite\}\}$ for the set A of atoms; this means we cannot build countable sets of atoms in the sense of Zermelo-Fraenkel theory, and we cannot define the  $\aleph$ -cardinals.

In fact, the finite support property says that we can always find a fresh name (atom) for each element x in an arbitrary set defined by Fraenkel-Mostowski axioms, (i.e. we can always find an atom which is not in the support of x). Since for  $\alpha$ -equivalence classes of  $\lambda$ -terms x, the support of x is represented by the free variables relatively to x, this model could be considered as a more suitable framework for computability theory. As an example how this could change the field, let consider the case of generating new names. New formalisms like  $\pi$ -calculus have fresh operators (e.g., operator new) which allow to generate a fresh name for a channel. The  $\pi$ -calculus embodies the view that in principle most, if not all, distributed computation may usefully be explained in terms of exchanges of names on named communication channels. The axiomatic model which always allows us to find a new name for a communication channel is, as we said before, the Fraenkel-Mostowski model. Also, the Fraenkel-Mostowski model is the axiomatic support for the construction of the nominal logic (see [4]). The Fraenkel-Mostowski model appears naturally in computer science. In this paper we give a mathematical presentation of this model and later, in some future works, we'll use these mathematical results in computer science and give some new results in nominal logic and  $\pi$ -calculus.

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The finite support property of the FM model is very strong. We try to study what happens if we replace this strong axiom with a weaker one. In this article we generalize the FM model by giving a new set of axioms which defines an Extended Fraenkel-Mostowski (EFM) model. We denote the infinite set of atoms (in the ZFA approach) by A, and the group of permutations of atoms (i.e the group of all bijections of A) by  $S_A$ , both in the Fraenkel-Mostowski and in the Extended Fraenkel-Mostowski approach. We prove that some properties of  $S_A$  defined in the EFM model are also properties of  $S_A$  defined in the FM model. We also define an extended interchange function (Definition 4) without using the finite support property of FM model; we use only the new axiom 11' of the EFM model which is a consequence of the axiom 11 of the FM model. We have decided to work with EFM model instead of FM model because many properties of the interchange function (for example many properties of the domain of the interchange function, respectively  $S_A$ ) can be proved by using the new axiomatic model and a more relaxed axiom which is axiom 11' in the description of the EFM model, instead of a very strong axiom which is axiom 11 in the description of the FM model. So, for proving some important properties of the domain of our interchange function (see for example Theorems 1 and 2) it is not necessary to assume that each element of an arbitrary FM set has finite support, as it is done in the axiomatic construction of the FM model (see axiom 11). These properties of  $S_A$  are valid if we assume only the axiom which says that each subset of the set A of atoms is either finite or cofinite (i.e., axiom 11'). This axiom which gives the structure of A is a direct consequence of the finite support property, as it is presented in Example 2.

We describe a new set theory model in which the sets are pairs  $(X, \cdot)$ , where X is defined by ZFA¬C rules, and  $\cdot$  is an extended interchange function which is an action of the group of permutations of atoms on X. Using some group theory results which can be proved without involving the axiom of choice, we obtain some new results about the extended interchange function (when we assume the EFM axiomatic model to be valid) which remain valid also for the interchange function (when we assume the FM axiomatic model to be valid). The properties of the extended interchange function and interchange function obtained here could be successfully applied in nominal logic, since the permutative renamings (described for example by GABBAY and HOFMANN in [2]) are defined as a result of applying the interchange function to a permutation and to an element in an FM set. The notion of renaming can be generalized in the EFM model and some new results can be obtained.

#### 2. The Extended Fraenkel-Mostowski model

We present the Fraenkel-Mostowski model with atoms and the new Extended Fraenkel-Mostowski model by using the notions of transposition, permutation, and substitution. Let A be an infinite set of atoms. A is characterized by the axiom " $y \in x \Rightarrow x \notin A$ " which means that only non-atoms can have elements.

## Definition 1.

- i) A transposition is a function  $(a b) : A \to A$  with the following property: (a b)(a) = b, (a b)(b) = a, and (a b)(n) = n for  $n \neq a, b$ .
- ii) A permutation of A is a bijection  $\pi$  from A to A.
- *iii)* A substitution is a function  $\{b|a\} : A \to A$  with the property  $\{b|a\}(n) = n$  if  $n \neq a$ , and  $\{b|a\}(a) = b$ .

Let  $S_A$  be the set of all permutations over A.  $S_A$  is a group. Let  $\overline{S_A}$  be the group of finitary permutations (i.e the group of permutations which leave unchanged all but finitely many atoms). We prove that  $\overline{S_A}$  is the set of all functions  $\pi$  generated by composing finitely many transpositions. Indeed, let  $\sigma \in \overline{S_A}$  be a function which permutes only a finite number of atoms  $\{a_1, ..., a_n\}$  such that the atoms  $A \setminus \{a_1, ..., a_n\}$  are left unchanged. Formally, we can say that  $\sigma$  is a permutation of the set  $\{a_1, ..., a_n\}$ , and so  $\sigma$  can be expressed as a product of transpositions.

We assume that the set A of atoms is infinite (this is axiom 10 in the characterization both of FM and EFM models). For  $S \subset A$ , we denote by Fix(S) the set  $\{\pi \mid \pi(a) = a \text{ for all } a \in S\}$ .

**Definition 2.** Let X be a set defined by the axioms of ZFA model without axiom of choice. An *interchange function* over X is a function  $\therefore : S_A \times X \to X$  defined inductively by  $\pi \cdot a = \pi(a)$  for all atoms  $a \in A$ , and by  $\pi \cdot x = \{\pi \cdot y \mid y \in x\}$  otherwise. Moreover, it satisfies the following axiom: for each  $x \in X$ , there is a finite nonempty set  $S \subset A$  such that for each  $\pi \in Fix(S) \cap \overline{S_A}$  we have  $\pi \cdot x = x$ .

An *FM* set is a pair  $(X, \cdot)$ , where X is a set defined by ZFA model without axiom of choice, and  $\cdot : S_A \times X \to X$  is an interchange function over X. We simply use X whenever no confusion arises.

The main distinction between ZFA and FM models is given by the fact that the axiom of choice is not used in the FM model. Moreover, the set A of atoms is infinite in FM, and for each element in an arbitrary FM set there is a finite set supporting it (see Definition 3 for the notion of support).

**Remark 1.** Since  $S_A$  is a group, the interchange function  $\cdot : S_A \times X \to X$  is an action of the group  $S_A$  on the set X because we have  $Id \cdot x = x$  and  $\pi \cdot \pi' \cdot x = (\pi \circ \pi') \cdot x$  for all  $\pi, \pi' \in S_A$ . Therefore we can see an FM set  $(X, \cdot)$  as a set provided by an action of  $S_A$  on X.

**Definition 3.** Let X be an FM set. We say that  $S \subset A$  supports x whenever for each  $\pi \in Fix(S) \cap \overline{S_A}$  we have  $\pi \cdot x = x$ , where  $Fix(S) = \{\pi \mid \pi(a) = a, \forall a \in S\}$ .

The interchange function properties always allow us to find a finite set supporting x (in fact this assertion is an axiom, namely the axiom 11 in the description of the FM model). We can generalize the notions defined in Definition 2.

**Definition 4.** Let X be a set defined by the axioms of ZFA model without axiom of choice. An *extended interchange function* over X is a function  $\cdot : S_A \times X \to X$  defined inductively by  $\pi \cdot a = \pi(a)$  for all atoms  $a \in A$ , and by  $\pi \cdot x = \{\pi \cdot y \mid y \in x\}$  otherwise. Moreover, each subset of A is either finite or cofinite.

An *EFM set* is a pair  $(X, \cdot)$ , where X is a set defined by ZFA model without choice, and  $\cdot : S_A \times X \to X$  is an extended interchange function on X. We simply use X whenever no confusion arises.

**Remark 2.** Since  $S_A$  is a group, the extended interchange function  $\cdot : S_A \times X \to X$  is an action of the group  $S_A$  on the set X; we have  $Id \cdot x = x$  and  $\pi \cdot \pi' \cdot x = (\pi \circ \pi') \cdot x$  for all  $\pi, \pi' \in S_A$ . Therefore we can see an EFM set  $(X, \cdot)$  like a set provided by an action of  $S_A$  on X.

It is worth to note that we use the same notation  $\cdot$  for both the interchange function and the extended interchange function. As it will be proved later, an interchange function can be also seen as an extended interchange function.

**Example 1.** The set A of atoms is both an FM set and an EFM set with interchange function, respectively extended interchange function given by  $(a b) \cdot n = (a b)(n)$ , and  $\pi \cdot x$  constructed by induction.

We present now axiomatic descriptions of both the Fraenkel-Mostowski and Extended Fraenkel-Mostowski models. The axiomatic description of the FM model is related to the nominal logic.

**Definition 5.** The following axioms give a complete characterization of the Fraenkel-Mostowski model:

- 1.  $\forall x.(\exists y.y \in x) \Rightarrow x \notin A$  (only non-atoms can have elements)
- 2.  $\forall x, y. (x \notin A \text{ and } y \notin A \text{ and } \forall z. (z \in x \Leftrightarrow z \in y)) \Rightarrow x = y$

(axiom of extensionality)

- 3.  $\forall x, y. \exists z. z = \{x, y\}$  (axiom of pairing)
- 4.  $\forall x. \exists y. y = \{z \mid z \subset x\}$  (axiom of powerset)

5.  $\forall x. \exists y. y \notin A \text{ and } y = \{z \mid \exists w. (z \in w \text{ and } w \in x)\}$  (axiom of union)

6.  $\forall x. \exists y. (y \notin A \text{ and } y = \{f(z) \mid z \in x\})$ , for each functional formula f(z)(axiom of replacement)

7.  $\forall x. \exists y. (y \notin A \text{ and } y = \{z \mid z \in x \text{ and } p(z)\}), \text{ for each formula } p(z)$ 

(axiom of separation)

- 8.  $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$  (induction principle)
- 9.  $\exists x.(\emptyset \in x \text{ and } (\forall y.y \in x \Rightarrow y \cup \{y\} \in x))$  (axiom of infinite)
- $10. \ A \, is \, not \, finite$

11.  $\forall x.\exists S \subset A.S \text{ is finite and } S \text{ supports } x$  (axiom of freshness)

this axiom describes the finite support property, and allows to say that for each x there is  $a \in A$  such that a is fresh for x (i.e. a is not in the support of x).

Thus we can build an FM model like a ZFA model with the set A of atoms infinite and with the additional property of finite support.

**Definition 6.** The following axioms give a complete characterization of the Extended Fraenkel-Mostowski model:

1.  $\forall x.(\exists y.y \in x) \Rightarrow x \notin A$  (only non-atoms can have elements)

2.  $\forall x, y. (x \notin A \text{ and } y \notin A \text{ and } \forall z. (z \in x \Leftrightarrow z \in y)) \Rightarrow x = y$ 

(axiom of extensionality)

- 3.  $\forall x, y. \exists z. z = \{x, y\}$  (axiom of pairing)
- 4.  $\forall x. \exists y. y = \{z \mid z \subset x\}$  (axiom of powerset)

5.  $\forall x. \exists y. y \notin A \text{ and } y = \{z \mid \exists w. (z \in w \text{ and } w \in x)\}$  (axiom of union)

- 6.  $\forall x. \exists y. (y \notin A \text{ and } y = \{f(z) \mid z \in x\})$ , for each functional formula f(z)(axiom of replacement)
- 7.  $\forall x. \exists y. (y \notin A \text{ and } y = \{z \mid z \in x \text{ and } p(z)\})$ , for each formula p(z)(axiom of separation)
- 8.  $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$  (induction principle)
- 9.  $\exists x.(\emptyset \in x \text{ and } (\forall y.y \in x \Rightarrow y \cup \{y\} \in x))$  (axiom of infinite)
- 10. A is not finite

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11'. Each subset of A is either finite or cofinite(axiom of structure for A)

Thus we can build an Extended Fraenkel-Mostowski (EFM) model like a ZFA model with the set A of atoms infinite and an additional property that each subset of A is either finite or cofinite.

**Remark 3.** Axiom 11' of EFM model is a direct consequence of axiom 11 of FM model, and so the EFM model is a natural extension of the FM model. An interchange function can be seen as an extended interchange function, and an FM set can be seen as an EFM set. All these aspects become clearer in Example 2.

**Remark 4.** There is no visible difference between the notions of interchange function and extended interchange function, except there is a restriction on the type. In the FM model the notions of interchange function and extended interchange function are identical. Looking on the set of axioms 1-11 in the description of FM model and axioms 1-11' in the description of EFM model, the interchange function is an extended interchange function if we see the FM model as an EFM model (this is possible because the axiom 11' is a consequence of axiom 11, as we show in Example 2). Also, the extended interchange function is an interchange function if we work in the EFM model and we assume the supplementary axiom of finite support. However we use different expressions for the same notion only to emphasize that, when we use "interchange function" we work in the FM approach (i.e., axioms 1-11), and when we use "extended interchange function" we work in the EFM approach (i.e., axioms 1-11).

Thus we have the following convention:

**Remark 5.** If we assume the set of axioms 1-11 to be valid, we say that we work in the Fraenkel-Mostowski model of set theory; if we assume the set of axioms 1-11' to be valid, we say that we work in the Extended Fraenkel-Mostowski model of set theory.

Let G be a subgroup of  $S_A$ . The function  $\cdot : G \times X \to X$  defined inductively by  $\pi \cdot a = \pi(a)$  for all atoms  $a \in A$ , and by  $\pi \cdot x = \{\pi \cdot y | y \in x\}$ otherwise, for all  $\pi \in G$  is a group action as it is proved in Remark 2. Such a group action is induced by the extended interchange function; however axiom 11' is not restricted to G. When we talk about an action  $\cdot : G \times X \to X$ , we think only of the definition of  $\cdot : G \times X \to X$  by induction as in Definition 4 (we think only to the group theory meaning of  $\cdot : G \times X \to X$ ). There is no restriction for axiom 11' of the EFM model (respectively for the finite support property of the FM model); it is the same for the entire EFM model (respectively for the entire FM model). Formally, we say that an action  $\cdot : G \times X \to X$  is defined as an extended interchange function (respectively as an interchange function) if  $\cdot : G \times X \to X$  is defined by induction as in Definition 4 (respectively as in Definition 2), and axiom 11' (respectively the finite support property) is the same for the entire EFM model (respectively for the entire FM model).

**Definition 7.** A *G*-renaming is an orbit of an (arbitrary) atom under an action  $\cdot : G \times A \to A$  defined as an extended interchange function.

**Remark 6.** The notion of *G*-renaming comes from the nominal logic [4]. Some properties of renamings defined as in [4] or [2] will be presented in future work by using nominal techniques, groups theory and some results of this paper.

We give now some results which finally allow us to say that the domain of the interchange function and the domain of the extended interchange

function have some similar properties. The first two of them provide some algebraic properties of the domain of the extended interchange function. Some general properties of permutation groups used for proving these results can be found in [1].

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**Theorem 1.** If we work in the extended Fraenkel-Mostowski model of set theory, then  $S_A$  is a torsion group.

**Proof.** We prove that  $S_A$  is a torsion group, i.e., every element of  $S_A$ has finite order. Let  $\overline{S_A}$  be the subgroup of  $S_A$  with the property that every permutation in  $\overline{S_A}$  keeps fixed every point of A except a finite number of them (this number could be 0). First we prove that the cycles of an arbitrary  $\sigma \in S_A$  are finite. Moreover, there is  $m \in \mathbb{N}$  such that all but finite many cycles of  $\sigma$  have length m. Let us suppose that  $\sigma$  has an infinite cycle. If we assume that  $\sigma$  has at least two infinite cycles, then the set of points of one of these cycles is infinite and coinfinite. However every subset of A is finite or cofinite, and this means that every cycle of  $\sigma$  is finite. If we suppose now that  $\sigma$  has only one infinite cycle, we obtain that  $\sigma \circ \sigma$  is a permutation with at least two infinite cycles, and so we get a contradiction. Now, for every  $n \in \mathbb{N}$ , the set of points in the cycles of  $\sigma$  which have length n is also finite or cofinite. If there is n such that this set is cofinite, then the proof is finished. If not, then there is an infinite number of different cycle lengths. We can define a partition of the set of cycle lengths into two infinite sets Xand Y. Then the set of points from cycles with length in X is infinite and coinfinite, and so we get a subset of A (the set of points from cycles with length in X) which is neither finite nor cofinite. Again we contradict that  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$ . Therefore for every  $\sigma \in S_A$  except a finite number of them, there is  $m \in \mathbb{N}$  such that  $\sigma^m \in \overline{S_A}$ . Since  $\overline{S_A}$  is a torsion group, we conclude that  $S_A$  is a torsion group.

All the cycles of  $\sigma$  are finite, and we have a finite number of cycle lengths. According to Theorem 5.1.2 in [3], the order of  $\sigma$  is the least common multiple of these cycle lengths.

**Remark 7.** Since the set of cycle lengths is a subset of  $\mathbb{N}$  (built by using the axiom of infinity), we could define a partition of the set of cycle lengths into two infinite sets X and Y. The construction of  $\mathbb{N}$  is not in contradiction with the axioms of the EFM model and of the FM model. However, as we can see in the next section, the set A of atoms cannot be partitioned into two infinite subsets  $A_1$  and  $A_2$ . This is the reason why the axiom of choice fails in both EFM and FM approaches.

**Theorem 2.** If we work in the extended Fraenkel-Mostowski model of set theory, then each subgroup of  $S_A$  which is finitely generated is also finite.

**Proof.** We prove that, if  $G \leq S_A$  is a finitely generated group, then G is finite. Let  $\sigma_1, \sigma_2, \ldots, \sigma_m \in S_A$  and  $G = \langle \sigma_1, \ldots, \sigma_m \rangle$ . A G-renaming is the orbit of an element  $\alpha \in A$  under the canonical action of G on A defined as in Definition 4. We prove that there is an r depending on (m)such that all but finitely many G-renamings (under the canonical action of renaming on atoms defined by the extended interchange function) have size r. Let us assume that there is an infinite G-renaming under the canonical action of G on A defined by  $(\sigma, \alpha) \mapsto \sigma(\alpha)$ . As a G-renaming is the orbit of an element  $\alpha \in A$ , we claim that this orbit contains a countable infinite subset. We define a word with k letters in  $\sigma_1, \sigma_2, \ldots, \sigma_m$  to be a finite composition of k permutations and inverses from the set  $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ . This terminology comes from the theory of free groups; however word in  $\sigma_1, \sigma_2, \ldots, \sigma_m$  in our approach is not an element of the free group on the generators  $\sigma_1, \sigma_2, \ldots, \sigma_m$ . Of course the set of words with k letters is finite for each k. We consider an infinite sequence of words in  $\sigma_1, \sigma_2, \ldots, \sigma_m$ . If there is  $a \in A$  with infinite orbit, we can define the image  $\{a_1, a_2, \ldots, a_m\}$ of a under the words of the sequence by  $a_1 = \sigma_1(a), \ldots, a_m = \sigma_m(a)$ . Let  $a_{m+1} = \sigma_{m+1}(a)$ , where  $\sigma_{m+1}$  is the first word in the sequence such that  $\sigma_{m+1}(a) \notin \{a_1, a_2, ..., a_m\}$ . Such a  $\sigma_{m+1}$  exists because we suppose that the obit of a is infinite. Indeed we can define a method of covering the sequence of words in  $\sigma_1, \sigma_2, \ldots, \sigma_m$  in the following way: first we cover the words with two letters (in an alphabetically ordered way that is in the same way as we "read a dictionary", since the set of letters is finite and hence well ordered; note that each *finite* set can be well-ordered and for proving this we do not need the axiom of choice), secondly we cover the words with three letters and so on. We pick up the *first* word we find with the requested property that the image of a under it is not a member of  $\{a_1, a_2, \ldots, a_m\}$ . We present a constructive method of picking up the first element in the sequence with the requested property. The method of covering the sequence presented before could induce an well-order relation on that sequence if we consider only the distinct words in the sequence (note that if we form all the possible words with the letters  $\sigma_1, \sigma_2, \ldots, \sigma_m$  we could have words in  $\sigma_1, \sigma_2, \ldots, \sigma_m$ with different number of letters but which are equal); this order is called "lexicographic order". With  $a_{m+1}$  already found we repeat the procedure described before and we find  $a_{m+2}$  and so on. Thus, we obtain a countable

subset of A. This contradicts the fact that all subsets of A are finite or cofinite. In fact, if there is a countable subset B of A, then  $A \setminus B$  is finite. It follows that A is countable, i.e. there is a bijection from  $\mathbb{N}$  to A, and so we get a contradiction (see Remark 9). If a finite number (greater than 1) of G-renamings sizes occur infinitely, then we contradict the special property of A that  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$ . Indeed, let us suppose we have an infinite number of G-renamings with size k and an infinite number of Grenamings with size l. Since the *G*-renamings which are different are also disjoint, we conclude that elements in the G-renamings with size k form a set which is both infinite and coinfinite. If the G-renamings are arbitrarily large, then we contradict again the structure of A. The proof of this fact is similar to that of Theorem 1: we use a partition of the G-renamings sizes into two infinite sets X and Y, and see that the set of elements with Grenamings sizes in X is infinite and coinfinite. We can conclude that all but finitely many G-renamings have size r. Let us suppose that G is generated by permutations  $\sigma_1, \sigma_2, \ldots, \sigma_m$ . There are infinitely many *G*-renamings, and all but finitely many of these G-renamings have the size r (otherwise, if we assume that there are only a finite number of G-renamings, because each G-renaming is finite, it follows that A is finite, i.e. a contradiction). Let us suppose that G is infinite. We define an equivalence relation on the set of G-renamings saying that two G-renamings are equivalent iff the actions of G on them are isomorphic. Since G is finitely generated, it follows that there is only a finite number of homomorphisms from G to  $S_r$ , and so there is only a finite number of equivalence classes (we can identify an action with its associated representation by permutations). It follows that one equivalence class (denoted by  $\mathcal{O}$ ) is infinite; otherwise, if all the equivalence classes are finite, because there are only finitely many equivalence classes, then there is only a finite number of G-renamings, i.e. a contradiction.

Let  $X_0$  be a *G*-renaming which is a representative of  $\mathcal{O}$ . The action of *G* on  $X_0$  can be seen as an homomorphism *f* from *G* to  $S_r$ . In fact this action is seen like the associated representation by permutations  $\psi : G \to S(X_0)$ defined by  $\psi(\sigma)(x) = \sigma(x)$ . Since  $|X_0| = r$ , there is an isomorphism  $\varphi$ :  $S(X_0) \to S_r$ , and  $f = \varphi \circ \psi$ . If  $\sigma \in Ker(f)$ , then  $\sigma$  fixes all the elements in  $X_0$ , and so  $\sigma$  fixes all the elements whose *G*-renamings are in  $\mathcal{O}$ . Definition 4.19 of [6] is useful to see how isomorphic actions look like (actually several results used here can be found in [6]). The number of elements of *A* fixed by  $\sigma$  is infinite, and because  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$ , the number of unfixed elements of *A* by  $\sigma$  is finite. Therefore, Ker(f) is formed by permutations

which keeps fixed all but finitely many atoms.

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We also have that  $\overline{S_A}$  is locally finite. Indeed let  $\pi_1, ..., \pi_k \in \overline{S_A}$  such that  $\pi_1$  permutes the atoms from a finite subset of A named  $U_1, ..., \pi_k$  permutes the atoms from a finite subset of A named  $U_k$ . Let  $U = U_1 \cup ... \cup U_k$ . Then each of  $\pi_1, ..., \pi_k$  is a permutation of U. If u is the finite cardinal of U, then we obtain that  $[\{\pi_1, ..., \pi_k\}] \leq S(U) \cong S_u$  and of course  $[\{\pi_1, ..., \pi_k\}]$  is finite.

Now, since  $Ker(f) \leq \overline{S_A}$  and  $\overline{S_A}$  is locally finite, we have that Ker(f) is locally finite (see [6]). There is also an isomorphism from G/Ker(f) to  $S_r$ ; the proof is similar to the proof of the fundamental isomorphism theorem for groups, and it does not use the axiom of choice. Since Ker(f) is locally finite and G/Ker(f) is finite, it follows that G is also locally finite (this result is proved by O.J.Schmidt for the general case: if for a group H there is  $K \triangleleft H$  such that both K and H/K are locally finite, then H is locally finite). Moreover, if H/K is finite, the axiom of choice is not used (see [6]). Since G is finitely generated, we have that G is finite, and this completes the proof.

**Remark 8.** We can also use a result saying that for a finitely generated group G, every subgroup of finite index in G is finitely generated; the proof does not use the axiom of choice (see [6]). Using the notations of the previous theorem, we get that Ker(f) is a subgroup of finite index in G. Then Ker(f) is finitely generated. Since Ker(f) is also a locally finite group (it is a subgroup of the locally finite group  $\overline{S_A}$ ), then Ker(f) is finite. Since Ker(f) and G/Ker(f) are both finite, it follows that G is finite; the proof is similar with the proof of Lagrange theorem. According to Remark 12, we can choose a system of representatives for the set of left cosets modulo Ker(f); moreover, G/Ker(f) is finite. It is easy to proceed in this way for the last part of the proof of Theorem 2.

A general result in ZF group theory says that a torsion group which is in the same time soluble and finitely generated is also a finite group. Another result says that a torsion group which is in the same time nilpotent and finitely generated is also a finite group (see [5]). The result presented in Theorem 2 is stronger; we do not need neither solubility nor nilpotency.

**Remark 9.**  $\aleph$ -cardinals are not defined in the EFM model (respectively in the FM model). When we say a "countable set" in the EFM model (respectively in the FM model), we intuitively think of a countable set

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defined like in Zermelo-Fraenkel model. Such a countable set is defined by axiom of infinity, axiom of separation and axiom of extensionality (i.e., a set of the form  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ ); see [7] for details. We do not define effectively a countable set, and talking about such a countable set we think of the construction of  $\mathbb{N}$  in Zermelo-Fraenkel model. For the existence of a numbering sequence 1, 2, ..., we use the axiom of infinity, and not the axiom of choice. A countable set (like  $\mathbb{N}$ ) can have both infinite and coinfinite subsets; for example, 2 $\mathbb{N}$ . However A cannot be in a bijection with  $\mathbb{N}$ .

# 3. Connections between the Fraenkel-Mostowski model and the Extended Fraenkel-Mostowski model

Since in the axiomatic description of the EFM model we assumed that  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$ , Theorems 1 and 2 are not valid in a general theory of urelements in ZFA model. In fact, we can prove Theorems 1 and 2 only in the EFM and FM settings.

In this section we prove that some algebraic properties of the domain of the extended interchange function are also properties of the domain of the interchange function.

**Definition 8.** Let X be an FM set. We say that  $S \subset A$  supports x whenever for each  $\pi \in \overline{S_A}$  which keeps each  $s \in S$  unchanged, we obtain an invariant renaming of x; i.e., for each  $\pi \in Fix(S) \cap \overline{S_A}$  we have  $\pi \cdot x = x$ , where  $Fix(S) = \{\pi \mid \pi(a) = a, \forall a \in S\}$ .

The interchange function properties always allow to find a finite set supporting x.

**Theorem 3.** Let X be a FM set. For each  $x \in X$  there is a unique minimal set which supports x; this set is called the support of x, and it is denoted by S(x).

**Proof.** We define  $S(x) = \bigcap \{S \subset A \mid S \text{ finite and } S \text{ supports } x\}$ . We have to prove that if  $S_1$  and  $S_2$  supports x, then  $S_1 \cap S_2$  supports x. Indeed, let  $\pi$  be a permutation from  $Fix(S_1 \cap S_2) \cap \overline{S_A}$ . We have to prove that  $\pi \cdot x = x$ . Since each permutation  $\pi$  of this type is generated by composing finitely many transpositions ( $\pi \in \overline{S_A}$ ), we have to prove the finite support property only for transpositions. This means that for each  $a, b \notin S_1 \cap S_2$  we have  $(a \ b) \cdot x = x$ . The cases  $a, b \notin S_1$  and  $a, b \notin S_2$  are obvious because

 $S_1$  and  $S_2$  support x, and by Definition 2 we have  $(ab) \cdot x = x$ . Now let  $a \notin S_1$  and  $b \notin S_2$ . Since  $S_1 \cup S_2$  is finite and A is infinite, we can find  $c \in A \setminus (S_1 \cup S_2)$  and  $a \neq c \neq b$ . Since  $a, c \notin S_1$  and  $S_1$  supports x, we have  $(ca) \cdot x = x$ . Since  $b, c \notin S_2$  and  $S_2$  supports x, we have  $(cb) \cdot x = x$ . It follows that  $(ab) \cdot x = (ab) \cdot (cb) \cdot x = ((ab) \circ (cb)) \cdot x = ((cb) \circ (ac)) \cdot x = (cb) \cdot (ac) \cdot x = x$ . The case when  $a \notin S_2$  and  $b \notin S_1$  is similar. The proof is complete because we know that there is at least one finite set supporting x (by axiom 11), and so the support S(x) is well defined.  $\Box$ 

**Example 2.** We present the supports for various subsets of A:

1. If  $B \subset A$  and B is finite, then S(B) = B.

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- 2. If  $C \subset A$  and C is cofinite, then  $S(C) = A \setminus C$ .
- 3. If  $D \subset A$  is not finite or cofinite, then a finite set supporting A cannot be defined. Indeed, let us suppose that S supports D (this means that for each  $\pi \in Fix(S) \cap \overline{S_A}$  we have  $\pi(D) = D$ ). If D is of the form  $\{a, c, e, ...\}$ , then at least  $\{a, c, e, ...\}$  or  $\{b, d, f, ...\}$  (which is  $C_D = A \setminus D$ ) must be fixed by each  $\pi$ . This means that S cannot be finite. Thus we get  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$ , and so we cannot accept in our model subsets of A which are in the same time infinite and coinfinite because of the "finite support" property. This property of A is characteristic for both the FM and EFM models.

Note 1. In the previous example,  $\{a, c, e, ...\}$  is only a convention of describing D. We do not have any choice of atoms in the construction of D, like a is the first atom of A, c is the third atom of A, etc. We use the form  $\{a, c, e, ...\}$  for D because, by intuition, it is easy to see who is  $C_D$  in this case. However the atoms composing D are arbitrarily presented in the structure of D, with no preliminary choice; the single condition is that both D and  $C_D$  are infinite. For example D can also be of form  $\{a, c, u, e, g, i, gh, ar_1, ...\}$ , where each of  $a, c, u, e, g, i, gh, ar_1, ...$  are arbitrary atoms and both  $\{a, c, u, e, g, i, gh, ar_1, ...\}$  and its complement are infinite.

**Remark 10.** Analyzing the structure of A (in both FM and EFM models we have  $\wp(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$ ), we can prove that the axiom of choice fails in both FM and EFM models. Indeed, if we assume that the axiom of choice is true, then for the family of disjoint sets  $\{a, b\}$ ,  $\{c, d\}$ ,

..., we should be able to find a set M which contains exactly one element from each of these sets  $\{a, b\}, \{c, d\}, \ldots$  However  $M \subset A$  should be infinite and coinfinite at the same time, and this contradicts the structure of A.

We present some algebraic connections between the domain of the extended interchange function the domain of the interchange function.

**Theorem 4.** The properties of  $S_A$  described in Theorem 1 and Theorem 2 remain also valid in the Fraenkel-Mostowski axiomatic model of set theory.

**Proof.** By Example 2 we see that the finite support property implies that each subset of A is either finite or cofinite, and these results presented initially for the EFM model remains also valid for the FM model, since in the proofs of Theorem 1 and Theorem 2 we use only the specific structure of A (i.e axiom 11'), and not the finite support property.

**Remark 11.** It is not trivial to adapt for the FM model the results presented in the previous sections. We could be tempted to say that the support of an arbitrary permutation of A is exactly the set of atoms it moves, because such a definition is given in classical permutation group theory. In this case, because of the finite support property, we should get that  $S_A = \overline{S_A}$  and many of the results presented in Section 2 could become trivial. According to Definition 3, we cannot say that the support of an arbitrary bijection of A is the set of atoms it moves. Indeed, let  $\sigma \in S_A$  and let S be the set of atoms moved by  $\sigma$ . The canonical action of  $S_A$  on  $S_A$  is given by the composition of permutations, which is the internal operation in  $S_A$ . Supposing that  $S_A$  can be organized as a FM set, we prove that S does not support  $\sigma$ , and so S is not the support of  $\sigma$  according to Definition 3. Indeed, let  $\pi \in \overline{S_A}$  such that  $\pi \in Fix(S)$ , i.e.  $\pi \in \overline{S_A}$  such that  $\pi(a) = a$  for all atoms a with the property that  $\sigma(a) \neq a$ . If we suppose that S supports  $\sigma$  then  $\pi \circ \sigma = \sigma$ . Let b be an atom such that  $\sigma(b) = b, \pi(b) = c$ , and  $\pi(a) = a$  for all atoms a with the property that  $\sigma(a) \neq a$ , where a, b, c are distinct atoms. Then  $\pi(\sigma(b)) = \pi(b) = c \neq \sigma(b)$  and  $\pi \circ \sigma \neq \sigma$ . Therefore S does not support  $\sigma$ .

Even if the axiom of choice fails in the FM and EFM approach, a weaker form of the axiom of choice (where the choice is from finite families) is valid. This remark has been used implicitly in this paper; for example, in the proof of Theorem 2.

**Remark 12.** The axiom of choice says that for each family  $\mathcal{F}$  of nonempty disjoint sets, we can find a system of representatives (which is a set that contains exactly one element from each set in  $\mathcal{F}$ ). If  $\mathcal{F}$  is a finite family of disjoint nonempty sets, this statement is a consequence of Axioms 1-9 (and not a form of the axiom of choice). Indeed, if  $\mathcal{F}$  contains only one nonempty set U, then we can find an element  $x_0 \in U$  (because U is nonempty). By the axiom of pairing we obtain the set  $\{x_0\}$  which is a set of representatives for  $\mathcal{F}$ . By the induction principle we can obtain a set of representatives for each finite family  $\mathcal{F}$  of disjoint nonempty sets (see [7] for details).

An important goal of describing the EFM model is to prove that some properties of  $S_A$  which are valid in the FM approach (for example Theorems 1 and 2) remain also valid if we consider a weaker axiom 11' in the description of EFM model instead of the axiom 11 in the description of FM model and work with the set of axioms 1-11'. As we have already explained, for the proof of some important properties of  $S_A$  we do not necessarily need to assume that for each element in an FM set there is a finite nonempty set supporting it (according to Definition 3) as we did in the axiomatic description of the FM model (by axiom 11). Instead of using axiom 11 to prove some properties of  $S_A$  (e.g., Theorems 1 and 2), we can use only a consequence of it (axiom 11') which says that each subset of A is either finite or cofinite. This section shows that the domain of the extended interchange function defined in EFM approach (Definition 4) and the domain of the interchange function defined in FM approach (Definition 2) have similar properties. Thus we can work in a model where, instead of an axiom which forces each element to have finite support (axiom 11 of the FM model), we use an axiom only for the structure of A (axiom 11' of the EFM model), and finally obtain similar properties of  $S_A$ .

We do not say that the EFM model is better than the FM model. The finite support property has important benefits (some of them are presented in [4]). However, we emphasize that we can relax the set of axioms in the description of the FM model and the effect is that, for the group of permutations of atoms, we obtain similar properties in both FM and EFM models. A future work will be devoted to prove that the permutative renamings (defined in [2] and [4]) have similar properties both in FM and EFM approach. In this paper we give the mathematical results that can be used then in a computer science paper where the permutative renamings are described in

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an axiomatic model without the finite support property.

## 4. Conclusion and further work

FM sets are described by the ZFA axioms together with the group action  $\cdot$  which is called *interchange function*. Since the ZFA model has already been investigated for a long time, the important new properties of the FM sets are related to the properties of the interchange function.

In this article we give a new axiomatic model of set theory, respectively the EFM model, which is a natural extension extension of the FM model in the sense that one of the FM axioms was replaced by one of its consequences. In fact we describe the EFM model by replacing the finite support property of the FM model with an axiom which says only that the set of atoms has a special structure: each subset of atoms is either finite or cofinite. We naturally extend the notion of *interchange function*, and get the notion of extended interchange function. Several properties of the extended interchange function are presented, and finally we make a comparison between the FM and EFM models by providing some algebraic connections between the interchange function and the extended interchange function. The main idea is to prove that some algebraic properties of the domain of the interchange function remain valid even we replace a strong axiom (namely axiom 11 of the FM model) with a weaker consequence of it (namely the axiom 11' of the EFM model). Theorem 1, Theorem 2 and Theorem 4 emphasize this idea.

Since the FM model is often used in the computer science (nominal logic and semantics), in a future work we'll try to analyze how some classical results are affected if we use the axiomatic EFM approach instead of the FM approach.

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