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A TOPOLOGICAL APPROACH IN THE EXTENDED FRAENKEL-MOSTOWSKI MODEL OF SET THEORY

BY

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Abstract. Lattices of subgroups are presented as algebraic domains. Given an arbitrary group, we define the Scott topology over the subgroups lattice of that group. A basis for this topology is expressed in terms of finitely generated subgroups. Several properties of the continuous functions with respect the Scott topology are obtained; they provide new order properties of groups. Finally there are expressed several properties of the group of permutations of atoms in a permutative model of set theory. We provide new properties of the extended interchange function by presenting some topological properties of its domain. Several order and topological properties of the sets in the Fraenkel-Mostowski model remains also valid in the Extended Fraenkel-Mostowski model, even one axiom in the axiomatic description of the Extended Fraenkel-Mostowski model is weaker than its homologue in the axiomatic description of the Fraenkel-Mostowski model.

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1. Subgroups lattices as domains

Let $(P, \sqsubseteq)$ be a poset. A subset A of P is called lower set if for each $x \in A$, $y \sqsubseteq x$ implies $y \in A$. Upper subsets are defined dually. If $(Q, \sqsubseteq)$ is another poset, then a function $f : P \to Q$ is called monotone if whenever $x \sqsubseteq y$ in P we have $f(x) \sqsubseteq f(y)$ in Q. A subset U of P is directed if it is nonempty and each pair of elements in U has an upper bound in U. A poset $(D, \sqsubseteq)$ in which every directed subset has a supremum is called a directed-complete partial order, or dcpo for short. Let x and y be elements of a dcpo $(D, \sqsubseteq)$. We say that x approximates y if for all directed subsets U of $(D, \sqsubseteq)$ we have that $y \sqsubseteq \sup(U)$ implies $x \sqsubseteq u$ for some $u \in U$. We say that x is compact if it approximates itself. The usual notations are $x \ll y$ iff x approximates y, and $K(D)$ for the set of compact elements of D. We note that $x \sqsubseteq y$ whenever $x \ll y$, and $x' \ll y'$ whenever $x' \sqsubseteq x \ll y \sqsubseteq y'$. We say that a subset B of a dcpo $(D, \sqsubseteq)$ is a basis for $(D, \sqsubseteq)$, if for every element x of $(D, \sqsubseteq)$ there exists a directed subset U of elements in B approximating x, with $\sup(U) = x$. The directness of U shows that whenever B is a basis for $(D, \sqsubseteq)$, for each element x in D we can say that the set of elements in B approximating x is directed, and x is the supremum of the directed set of elements in B approximating it. Using the definition of approximation and the previous result we conclude that for each dcpo $(D, \sqsubseteq)$ with a basis B we have that $K(D) \subseteq B$. A dcpo is called a continuous domain if it has a basis. It is called algebraic domain if it has a basis of compact elements. More details were given in [7].

Let $(G, \cdot)$ be a group. If H is a subgroup of G we denote this by $H \leq G$. If $S \subseteq G$, we denote by $[S]$ the subgroup of G generated by S i.e. the smallest subgroup of G which contains S. Every element of $[S]$ can be expressed as a finite product of elements of S and inverses of elements of S. If $S \subseteq G$ is finite and $H = [S]$ we call H a finitely generated subgroup of G. The set $\mathcal{L}(G)$ of all subgroups of G ordered by inclusion form a complete lattice. If $(H_i)_{i \in I}$ is a family of subgroups of G, the infimum of this family is $\bigcap_{i \in I} H_i$ and the supremum is $[\bigcup_{i \in I} H_i]$.

Proposition 1. *If $(H_i)_{i \in I}$ is a directed family of subgroups of G, then $\bigcup_{i \in I} H_i$ is a subgroup of G, and hence $[\bigcup_{i \in I} H_i] = \bigcup_{i \in I} H_i$.*

Proof. Let $x, y \in \bigcup_{i \in I} H_i$. There are $i, j \in I$ such that $x \in H_i$ and $y \in H_j$. Because of the directness of $(H_i)_{i \in I}$ we can find $k \in I$ such that H_k is an upper bound both of H_i and H_j. This means that $x, y \in H_k$ and hence $xy^{-1} \in H_k \subseteq \bigcup_{i \in I} H_i$. We obtained that $\bigcup_{i \in I} H_i$ is a subgroup of G. $\square$

Remark 1. Before proving the following result we make the trivial remark that if a finite set A is covered by a directed collection $(A_i)_{i \in I}$ of sets, then A will always be contained in some A_i.

Proposition 2. *Each finitely generated subgroup of a group G is compact in $(\mathcal{L}(G), \subseteq)$.*

Proof. Let $H \leq G$ be a finitely generated subgroup of G. Then $H = [S]$ where S is a finite subset of G. Let $(H_i)_{i \in I}$ be a directed family of subgroups of G with $H \subseteq [\bigcup_{i \in I} H_i]$. By Proposition 1 we have $H \subseteq \bigcup_{i \in I} H_i$ and $S \subseteq$

$H \subseteq \bigcup_{i\in I} H_i$. By Remark 1, there exists $j \in I$ such that $S \subseteq H_j$. However $[S] = \bigcap_{\substack{H'\leq G\\ S\subseteq H'}} H'$ and hence $[S] \subseteq H_j$. This means exactly $H \ll H$. $\square$

Proposition 3. *Let H be a subgroup of G. Then H is the directed union of the subgroups generated by the finite subsets of H.*

Proof. Let $H \leq G$ and $A_H = \{[F] \,|\, F \subseteq H$ and F is finite$\}$. We have to prove that A_H is directed and $H = \bigcup_{H'\in A_H} H'$.

Let $[F_1] \in A_H$ and $[F_2] \in A_H$. Then $[F_1 \cup F_2] \in A_H$. Also $[F_1] = \bigcap_{\substack{H'\leq G\\ F_1\subseteq H'}} H'$. We know that $F_1 \subseteq [F_1 \cup F_2]$ and so $[F_1] \subseteq [F_1 \cup F_2]$. Analogue $[F_2] \subseteq [F_1 \cup F_2]$. We obtained that A_H is directed.

Now let $h \in H$. Then $h \in [\{h\}]$ and $[\{h\}] \in A_H$. Hence $H \subseteq \bigcup_{H'\in A_H} H'$. For the reverse inclusion let F be a finite subset of H. Since $[F] = \bigcap_{\substack{H'\leq G\\ F\subseteq H'}} H'$ we get $[F] \subseteq H$ and $\bigcup_{H'\in A_H} H' \subseteq H$. $\square$

Let $F(\mathcal{L}(G))$ be the the set of all finitely generated subgroups of a group G.

Proposition 4. *Let G be a group. Then $(\mathcal{L}(G), \subseteq)$ is a continuous domain and a basis in $(\mathcal{L}(G), \subseteq)$ is precisely $F(\mathcal{L}(G))$.*

Proof. For each subgroup H of G, we define $A_H = \{[F] \,|\, F \subseteq H$ and F is finite$\}$. Clearly $A_H \subseteq F(\mathcal{L}(G))$. By Proposition 3 we know that A_H is directed and $H = \bigcup_{H'\in A_H} H'$. By Proposition 2 we know that whenever $[F] \in A_H$ we have $[F] \ll [F] \subseteq H$ and hence $[F] \ll H$. Using the definition of a basis in a dcpo we get that $F(\mathcal{L}(G))$ is a basis in $(\mathcal{L}(G), \subseteq)$. $\square$

Remark 2. From the general theory of basis we know that $K(\mathcal{L}(G)) \subseteq F(\mathcal{L}(G))$. Now, by Proposition 2 we also have $F(\mathcal{L}(G)) \subseteq K(\mathcal{L}(G))$. We can say that $(\mathcal{L}(G), \subseteq)$ is an algebraic domain and the compact elements in $(\mathcal{L}(G), \subseteq)$ are precisely those in $F(\mathcal{L}(G))$.

In the general theory of domains the following interpolation property is valid (see [7]):

Let $x \ll y$ in a continuous domain $(D, \sqsubseteq)$ with basis B. Then there exists an element $b \in B$ such that $x \ll b \ll y$.

So, if $(D, \sqsubseteq)$ is an algebraic domain and $x \ll y$, then we can find a compact element c such that $x \ll c \ll y$. For the particular case of algebraic domains this result can be easily proved. Indeed, let $x \ll y$. Because D is algebraic we know that y is the directed supremum of the compact elements approximating it. By the definition of the approximation relation there exists $c \in K(D)$ such that $x \sqsubseteq c \ll y$. Because $c \ll c$ we get $x \ll c \ll y$.

The following result can be easily proved using Proposition 2.

Theorem 1. *Let G be a group and H, K be two elements in $(\mathcal{L}(G), \subseteq)$ such that $H \ll K$. There exist a finitely generated subgroup of G denoted by L with $H \ll L \ll K$.*

2. Scott topology over the subgroups lattice of a group

A topology on a space X is a system of subsets of X (called *open sets*), which is closed under finite intersections and infinite unions. The complementary of open sets are called *closed sets* (see [4]). For a dcpo $(D, \sqsubseteq)$ there are many possible choices of topology which transforms D into a topological space. There are well known the Scott topology and the Lawson topology. These were described in [7] for the general case of dcpos. Here we particularize to groups some topological results in the general theory of dcpos and domains. Our purpose is to obtain some new properties of groups.

Let $(D, \sqsubseteq)$ be a dcpo. A subset U is called Scott-closed if it is a lower set and is closed under suprema of directed subsets. A principal ideal (i.e. an ideal of form $\downarrow x$ where $\downarrow x = \{y \in D \mid y \sqsubseteq x\}$) is always a Scott-closed set. Complements of closed sets are called Scott-open. A Scott-open set is an upper set O with the property that every directed set whose supremum belongs to O has a non-empty intersection with O.

Definition 1. Let $(\mathcal{L}(G), \subseteq)$ be the subgroups lattice of G. A subset $(H_i)_{i \in I}$ *of elements in $\mathcal{L}(G)$ is called Scott-closed if the following conditions are satisfied:*

1. For each $i \in I$ and $K \leq G$, $K \subseteq H_i$ implies there exists $j \in I$ such that $K = H_j$.

2. If $(K_j)_{j \in J} \subseteq (H_i)_{i \in I}$ is a directed family of subgroups of G, then $\bigcup_{j \in J} K_j \in (H_i)_{i \in I}$.

Definition 2. Let $(\mathcal{L}(G), \subseteq)$ be the subgroups lattice of G. A subset $(H_i)_{i \in I}$ *of elements in $\mathcal{L}(G)$ is called Scott-open if the following conditions are satisfied:*

1. For each $i \in I$ and $K \leq G$, $K \supseteq H_i$ implies there exists $j \in I$ such that $K = H_j$

2. If $(K_j)_{j\in J} \subseteq (H_i)_{i\in I}$ is a directed family of subgroups of G with $\bigcup_{j\in J} K_j \in (H_i)_{i\in I}$, then there are $j \in J$ and $i \in I$ such that $K_j = H_i$.

We already know that $(\mathcal{L}(G), \subseteq)$ is an algebraic domain. From Corollary II-1.15 in [7] we can give the following result:

Theorem 2. *The Scott topology on $\mathcal{L}(G)$ has a basis formed from the sets $\uparrow H = \{K \leq G \,|\, K \supseteq H\}$ where $H \in F(\mathcal{L}(G))$*

Moreover, if $(D, \sqsubseteq)$ is a continuous domain with basis B, we can prove that each open set O in D can be written as the union $O = \bigcup_{x\in O\cap B}\{y \in D \,|\, y \gg x\}$ (see [1]).

Corollary 1. *Let $(H_i)_{i\in I}$ be an open set in $\mathcal{L}(G)$. Then for each $j \in I$ there exists $k \in I$ such that H_k is finitely generated and $H_j \supseteq H_k$.*

A set in $(\mathcal{L}(G), \subseteq)$ is Scott-compact if it is topologically compact with respect the Scott topology . A set in $(\mathcal{L}(G), \subseteq)$ is saturated if it is the intersection of some open sets in $(\mathcal{L}(G), \subseteq)$. Using properties of sober spaces we can prove that in an algebraic domain a set is Scott-open iff it is Scott-compacted and saturated (see [7]).

Theorem 3. *Let $(H_i)_{i\in I}$ be a set of elements in $\mathcal{L}(G)$. Then $(H_i)_{i\in I}$ is Scott-open if and only if $(H_i)_{i\in I}$ is Scott-compacted and saturated.*

Definition 3. A function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is called Scott-continuous if and only if it is topologically continuous with respect the Scott topology over $\mathcal{L}(G)$.

Theorem 4. *A function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is Scott-continuous if and only if f is monotone and for each directed family $(H_i)_{i\in I}$ of subgroups of G we have $f(\bigcup_{i\in I} H_i) = \bigcup_{i\in I} f(H_i)$.*

Proof. Let $f : \mathcal{L}(G) \to \mathcal{L}(G)$ be a Scott-continuous function. We show the monotonicity of f. Let $H, K \leq G$ with $H \subseteq K$. The set $\downarrow f(K) = \{U \leq G \,|\, U \subseteq f(K)\}$ is a principal ideal, and hence it is a Scott-closed set. Since f is topologically continuous we have that $f^{-1}(\downarrow f(K))$ is Scott-closed and so, it is a lower set. Since $K \in f^{-1}(\downarrow f(K))$ and $H \subseteq K$, we have $H \in f^{-1}(\downarrow f(K))$. This means $f(H) \subseteq f(K)$ and f is monotone. Let $(H_i)_{i\in I}$ be a directed family of subgroups of G. The set $A = \downarrow (\bigcup_{i\in I} f(H_i))$ is Scott closed and hence $f^{-1}(A)$ is Scott closed. Since $(H_i)_{i\in I} \subseteq f^{-1}(A)$ it follows that $\bigcup_{i\in I} H_i \in f^{-1}(A)$. This means $f(\bigcup_{i\in I} H_i) \subseteq \bigcup_{i\in I} f(H_i)$. The converse inclusion follows from the monotonicity of f.

Let us suppose now that f is monotone and for each directed family $(H_i)_{i\in I}$ of subgroups of G we have $f(\bigcup_{i\in I} H_i) = \bigcup_{i\in I} f(H_i)$. We show that f is topologically continuous. Let O be an open subset in $\mathcal{L}(G)$. Clearly, O is an upper set and then $f^{-1}(O)$ is an upper set because f is monotone. Let $(H_i)_{i\in I}$ be a directed family of subgroups of G whose supremum $H = \bigcup_{i\in I} H_i$ belongs to $f^{-1}(O)$. This means $f(\bigcup_{i\in I} H_i) = \bigcup_{i\in I} f(H_i) \in O$. Because O is Scott open we can find $j \in I$ such that $f(H_j) \in O$ that is $H_j \in f^{-1}(O)$. We proved that $f^{-1}(O)$ is Scott open and f is Scott continuous. $\square$

Theorem 5. *A function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is Scott-continuous if and only if for each $H \leq G$ and each $K \leq G$ finitely generated with $K \ll f(H)$ there exists $L \leq G$ finitely generated with $L \ll H$ such that ($L \subseteq U$ implies $K \subseteq f(U)$) for each subgroup U of G.*

Proof. We denote by $F(\mathcal{L}(G))_H$ the set of all finitely generated subgroups of G approximating H. From Proposition 4 we know that $F(\mathcal{L}(G))$ is a basis in $\mathcal{L}(G)$. From the general theory of basis we have that $F(\mathcal{L}(G))_H$ is directed and each subgroup H of G can be written as $H = \bigcup_{H'\in F(\mathcal{L}(G))_H} H'$.

Let us suppose that the function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is Scott-continuous and $H \leq G$. By Theorem 4 we have $f(H) = \bigcup_{H'\in F(\mathcal{L}(G))_H} f(H')$. If $K \leq G$ is finitely generated with $K \ll f(H)$ there exists $L \in F(\mathcal{L}(G))_H$ such that $K \subseteq f(L)$. If U is another subgroup of G such that $L \subseteq U$, we have also $K \subseteq f(L) \subseteq f(U)$ because f is monotone by Theorem 4.

It remains to prove the converse implication. We start by making the remark that $F(\mathcal{L}(G))_H \subseteq \{L \,|\, L \subseteq K\}$ implies, by taking the supremum, that $H \subseteq K$. This implication is true for all subgroups H and K of G.

Let us prove first that f is monotone. Let $H_1, H_2 \leq G$ with $H_1 \subseteq H_2$. We suppose that $f(H_1) \nsubseteq f(H_2)$. This means that there exists a finitely generated subgroup K of G with $K \ll f(H_1)$ but $K \nsubseteq f(H_2)$. For K, there exists a finitely generated subgroup L of G with $L \ll H_1$ and for which we have the implication: $L \subseteq U$ implies $K \subseteq f(U)$ for each subgroup U of G. However $L \subseteq H_2$ and we must have $K \subseteq f(H_2)$. We get a contradiction with the choice of K, and hence f is monotone.

Let $(H_i)_{i\in I}$ be a directed family of subsets of G. We denote by H the supremum of this family, i.e. $H = \bigcup_{i\in I} H_i$. Because f is monotone, we have $\bigcup_{i\in I} f(H_i) \subseteq f(H)$. We suppose that $f(H) \nsubseteq \bigcup_{i\in I} f(H_i)$. This means there exists a finitely generated subgroup $K \ll f(H)$ with $K \nsubseteq \bigcup_{i\in I} f(H_i)$. For this K we can find a finitely generated subgroup L of G with $L \ll H$ and for which we have the implication: $L \subseteq U$ implies $K \subseteq f(U)$ for each subgroup U of G. However $H = \bigcup_{i\in I} H_i$. From the definition of the approximation relation we have that there exists $j \in I$ such that $L \subseteq H_j$. We obtain now $\bigcup_{i\in I} f(H_i) \supseteq f(H_j) \supseteq f(L) \supseteq K$. We get a contradiction since we chose $K \nsubseteq \bigcup_{i\in I} f(H_i)$. Now we have $f(\bigcup_{i\in I} H_i) = \bigcup_{i\in I} f(H_i)$ and by Theorem 4 we obtain that f is Scott continuous. □

From the general theory of domains we know that each continuous domain $(D, \sqsubseteq)$ with basis B is canonically isomorphic with $Id(B, \ll)$, where $Id(B, \ll)$ is the ideal completion of B i.e the set of all ideals (directed lower sets) of $(B, \ll)$ ordered by inclusion. The requested isomorphism is given by $f : Id(B, \ll) \to D$ defined by $f(A) = \sup(A)$ for each ideal A in $(B, \ll)$, and $g : D \to Id(B, \ll)$ which maps each x in D into the directed set of elements in B approximating x.

Also, for each dcpo E and for each monotone map $h : B \to E$ we can find a continuous function $\widetilde{h} : Id(B, \ll) \to E$ defined by $\widetilde{h}(A) = \sup(h(A))$. The continuity of h follows by applying the associativity of suprema and Theorem 4. Each continuous function α from D onto another dcpo E can be written as $\alpha = \widetilde{\alpha|_B} \circ g$ because of the definition of the basis B. However $g : D \to Id(B, \ll)$ is completely determined only by D and B. Also, $\widetilde{\alpha|_B} : Id(B, \ll) \to E$ is completely determined by $\alpha|_B : B \to E$ which is the restriction of α to B. Hence each continuous function from the domain D onto the dcpo E is completely determined by its restriction to the basis B of D. A complete calculation is given in [7] or [1]. We can give the following result:

Theorem 6. *A continuous function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is completely determined by its values on the finitely generated subgroups of G.*

Several fixed-point theorems in the general order theory remains valid for the domain $(\mathcal{L}(G), \subseteq)$. Tarski fixed-point theorem states that, whenever L is a complete lattice and $f : L \to L$ is a monotone map, we have that $fix(f) = \{x \in L \mid x = f(x)\}$ is non-empty. Moreover, $fix(f)$ form another complete lattice (Theorem 0-2.3 in [7]). We particularize this theorem to $(\mathcal{L}(G), \subseteq)$ which is a complete lattice.

Theorem 7. *Each monotone function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ has a fixpoint. The least of them is given by $\cap\{H \leq G \mid f(H) \subseteq H\}$, and the largest by $[\cup\{H \leq G \mid H \subseteq f(H)\}]$.*

In the case of continuous functions the things are more clear. We have a uniform canonical method for constructing the fixpoints of a continuous function on a pointed dcpo (i.e. a dcpo with a least element). From Proposition II-2.4 in [7] we know that, whenever $(D, \sqsubseteq, \perp)$ is a pointed dcpo (the least element of D is denoted by $\perp$) and $f : D \to D$ is a continuous function, there is a least fixpoint of f which is given by $fix(f) = \sup\{f^n(\perp) \mid n \in \mathbb{N}\}$. Moreover, from Theorem 2.1.19 in [1], the map $fix : [D \longrightarrow D] \to D$, $f \mapsto fix(f)$ is also continuous.

Theorem 8. *Each continuous function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ has a least fixpoint. It is given by $\bigcup_{n\in\mathbb{N}} f^n(\{e\})$ where e is the identity element in G.*

3. Extended Fraenkel-Mostowski model of set theory

The notion of choosing a fresh name often arises when manipulating syntactic expressions; therefore it is necessary to indicate some constraints whenever describing such a syntactic manipulation. Often it is just said that a name is fresh without specifying any restrictions. In such a case, we mean that the fresh name must be different from any name occurring anywhere else in the expression or program. Some programming systems have mechanisms for renaming, for binding a name with a value and for managing sets of such bindings. Modern programming languages are designed to manage bindings and fresh names by using the notions of scopes, workspaces, or environments. Since renaming, binding and fresh names appear in several approaches, it became evident that they deserve to be studied in their own terms.

The nominal logic and semantics was presented by Gabbay and Pitts in [6]; it uses the Fraenkel-Mostowski (FM) model of set theory. We recall that the FM permutation model of set theory was devised in 1930s to prove the independence of the Axiom of Choice (AC) from the other axioms of Zermelo-Fraenkel (ZF) model of set theory (see [8]). The FM model is built using all the axioms of the Zermelo-Fraenkel with atoms (ZFA) model, except the axiom of choice. It has the special property of finite support which claims that for each element x in an arbitrary FM-set we can find a finite set supporting x (in the sense of Definition 6), and the property that the set A of atoms is infinite. In fact, the finite support property says that for each element x in an arbitrary FM-set, we can always find a fresh element for x i.e. an element which is not in the support of x (see Theorem 9 for the definition of support).

Since for α-equivalence classes of λ-terms x, the support of the class of x is represented by the free variables relatively to x (see [6]), this model could be useful in computer science. We think that Fraenkel-Mostowski model can be considered as a more suitable framework for computer science than the classical Zermelo-Fraenkel model.

Let A be an infinite set of atoms. A is characterized by the axiom "$y \in x \Rightarrow x \notin A$" which means that only non-atoms can have elements.

Definition 4. i) A *transposition* is a function $(a\,b) : A \to A$ with the property $(a\,b)(a) = b$, $(a\,b)(b) = a$ and $(a\,b)(n) = n$ for $n \neq a, b$.

ii) A *permutation* of A is a bijection π from A to A.

iii) A *substitution* is a function $\{b|a\} : A \to A$ with the property $\{b|a\}(n) = n$ if $n \neq a$ and $\{b|a\}(a) = b$.

Let S_A be the set of all permutations over A; S_A is a group with the usual composition of permutations. Let $\overline{S_A}$ be the group of finitary permutations (i.e the group of permutations which leave unchanged all but finitely many atoms).

Definition 5. Let X be a set defined by the axioms of ZFA model without axiom of choice. An *interchange function* over X is a function $\cdot : S_A \times X \to X$ defined inductively by $\pi \cdot a = \pi(a)$ for all atoms $a \in A$, and by $\pi \cdot x = \{\pi \cdot y \,|\, y \in x\}$ otherwise. Moreover, it satisfies the following axiom: for each $x \in X$, there is a finite nonempty set $S \subset A$ such that for each $\pi \in Fix(S) \cap \overline{S_A}$ we have $\pi \cdot x = x$.

An *FM set* is a pair $(X, \cdot)$, where X is a set defined by ZFA model without axiom of choice, and $\cdot : S_A \times X \to X$ is an interchange function over X. We simply use X whenever no confusion arises.

Remark 3. Since S_A is a group, the interchange function $\cdot : S_A \times X \to X$ is an action of the group S_A on the set X because we have $Id \cdot x = x$ and $\pi \cdot \pi^{'} \cdot x = (\pi \circ \pi^{'}) \cdot x$ for all $\pi, \pi' \in S_A$. Therefore we can see an FM set $(X, \cdot)$ as a set provided by an action of S_A on X.

Definition 6. Let X be an FM set. We say that $S \subset A$ *supports* x whenever for each $\pi \in Fix(S) \cap \overline{S_A}$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$.

From [6] we know the following result:

Theorem 9. *Let X be an FM set, and for each $x \in X$ let us define $\mathcal{F}_x = \{S \subset A \,|\, S\, finite,\, S\, supports\, x\}$. Then $\mathcal{F}_x$ has a least element which also supports x. We call this element* the support of x, *and we denote it by $S(x)$ or $supp(x)$.*

The axiomatic description of the FM model is related to the nominal logic (see [6]).

Definition 7. The following axioms gives a complete characterization of the Fraenkel-Mostowski model:

1. $\forall x.(\exists y.y \in x) \Rightarrow x \notin A$ (only non-atoms can have elements)
2. $\forall x, y.(x \notin A\, and\; y \notin A\; and\; \forall z.(z \in x \Leftrightarrow z \in y)) \Rightarrow x = y$ (axiom of extensionality)
3. $\forall x, y.\exists z.z = \{x, y\}$ (axiom of pairing)
4. $\forall x.\exists y.y = \{z \,|\, z \subset x\}$ (axiom of powerset)
5. $\forall x.\exists y.y \notin A\, and\, y = \{z \,|\, \exists w.(z \in w\, and\, w \in x)\}$ (axiom of union)
6. $\forall x.\exists y.(y \notin A\, and\, y = \{f(z) \,|\, z \in x\})$, for each functional formula $f(z)$ (axiom of replacement)
7. $\forall x.\exists y.(y \notin A\, and\, y = \{z \,|\, z \in x\, and\, p(z)\})$, for each formula $p(z)$ (axiom of separation)
8. $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$ (induction principle)
9. $\exists x.(\emptyset \in x\, and\, (\forall y.y \in x \Rightarrow y \cup \{y\} \in x))$ (axiom of infinite)
10. $A\, is\, not\, finite$
11. $\forall x.\exists S \subset A.S\, is\, finite\, and\, S\, supports\, x$ (the finite support property)

Therefore, we can see an FM model like a ZFA model with an infinite set A of atoms infinite and with an additional property of finite support.

A new axiomatic model of set theory was described in [2]; it is called the Extended Fraenkel-Mostowski (EFM) model of set theory. The axioms of the EFM model are precisely those of the Zermelo-Fraenkel with atoms (ZFA) without choice model of set theory; additionally we assume that the set A of atoms is infinite and each subset of A is either finite or cofinite. The description of the EFM model was motivated in [3] by the fact that we can work with a weaker set of axioms which are the axioms 1-11' in the description of EFM model to obtain similar properties for *permutative renamings* as in the FM model (that assumed stronger axioms).

The following definition generalizes the notion defined in Definition 5.

Definition 8. Let X be a set defined by the axioms of ZFA model without axiom of choice. An *extended interchange function* over X is a function $\cdot : S_A \times X \to X$ defined inductively by $\pi \cdot a = \pi(a)$ for all atoms $a \in A$, and by $\pi \cdot x = \{\pi \cdot y \,|\, y \in x\}$ otherwise. Moreover, each subset of A is either finite or cofinite.

An *EFM set* is a pair $(X, \cdot)$, where X is a set defined by ZFA model without choice, and $\cdot : S_A \times X \to X$ is an extended interchange function on X. We simply use X whenever no confusion arises.

Remark 4. Since S_A is a group, the extended interchange function $\cdot : S_A \times X \to X$ is an action of the group S_A on the set X; we have $Id \cdot x = x$ and $\pi \cdot \pi^{'} \cdot x = (\pi \circ \pi^{'}) \cdot x$ for all $\pi, \pi' \in S_A$. Therefore we can see an EFM set $(X, \cdot)$ like a set provided by an action of S_A on X.

Definition 9. The following axioms gives a complete characterization of the Extended Fraenkel-Mostowski model:

1. $\forall x.(\exists y.y \in x) \Rightarrow x \notin A$ (only non-atoms can have elements)
2. $\forall x, y.(x \notin A \, and \; y \notin A \; and \; \forall z.(z \in x \Leftrightarrow z \in y)) \Rightarrow x = y$ (axiom of extensionality)
3. $\forall x, y.\exists z.z = \{x, y\}$ (axiom of pairing)
4. $\forall x.\exists y.y = \{z \,|\, z \subset x\}$ (axiom of powerset)
5. $\forall x.\exists y.y \notin A \, and \, y = \{z \,|\, \exists w.(z \in w \, and \, w \in x)\}$ (axiom of union)
6. $\forall x.\exists y.(y \notin A \, and \, y = \{f(z) \,|\, z \in x\})$, for each functional formula $f(z)$ (axiom of replacement)

7. $\forall x.\exists y.(y \notin A\, and\, y = \{z \,|\, z \in x\, and\, p(z)\})$, for each formula $p(z)$ (axiom of separation)

8. $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$ (induction principle)

9. $\exists x.(\emptyset \in x\, and\, (\forall y.y \in x \Rightarrow y \cup \{y\} \in x))$ (axiom of infinite)

10. $A\, is\, not\, finite$

11'. $Each\, subset\, of\; A\, is\, either\; finite\, or\, cofinite$(axiom on structure of A)

Therefore, an EFM model is similar to ZFA model with an infinite set A of atoms infinite, having the additional property that each subset of A is either finite or cofinite.

Remark 5. Axiom 11' of EFM model is a direct consequence of Axiom 11 of FM model as it is proved in [2]; thus, the EFM model is a natural extension of the FM model.

Remark 6. If we assume the set of axioms 1-11 (in Definition 7) to be valid we say that *we work in the axiomatic Fraenkel-Mostowski model of set theory* and if we assume the set of axioms 1-11' (in Definition 9) to be valid we say that *we work in the axiomatic Extended Fraenkel-Mostowski model of set theory.*

In [2] we proved the following important result:

Theorem 10. *If we work in the Extended Fraenkel-Mostowski axiomatic model of set theory, then each subgroup of S_A which is finitely generated is also finite.*

From Theorem 4 in [2] we know that the property of S_A described in Theorem 10 remains valid also if we work in the Fraenkel-Mostowski model of set theory. Hence we have:

Theorem 11. *If we work in the Fraenkel-Mostowski axiomatic model of set theory, then each subgroup of S_A which is finitely generated is also finite.*

4. Several properties of the group of permutations of atoms in the Extended Fraenkel-Mostowski model

The results in Sections 1 and 2 were given in the general case when G is an arbitrary group. A particular class of groups is represented by the locally finite groups which are groups whose all finitely generated subgroups are also finite. Hence, if G is a locally finite group, then a subgroup H of G is finitely generated if and only if H is finite. The results presented in Sections 1 and 2 can be particularized to locally finite groups by replacing "finitely generated" with "finite".

Let now G be locally finite group. We have that $\mathcal{L}(G)$ is an algebraic domain and the smallest basis in $\mathcal{L}(G)$ (i.e. the set of all compact elements in $\mathcal{L}(G)$) is formed precisely from the finite subgroups of G. The Scott topology on $\mathcal{L}(G)$ has a basis formed from the sets $\uparrow H = \{K \leq G \,|\, K \supseteq H\}$ where the subgroups H are precisely the finite subgroups of G. A continuous function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is completely determined by its values on the finite subgroups of G. A function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is Scott-continuous if and only if for each $H \leq G$ and each $K \leq G$ finite with $K \ll f(H)$ there exists $L \leq G$ finite with $L \ll H$ such that ($L \subseteq U$ implies $K \subseteq f(U)$) for each subgroup U of G. Whenever $H \ll K$ in $\mathcal{L}(G)$ we can interpolate a finite subgroup of G between H and K. All these results are valid whenever G is a locally finite group. There are some important classes of locally finite groups for which the results presented before look quite interesting. Some examples of locally finite groups (for which the previous results are valid) are: infinite direct sums of finite groups (see [9]) hamiltonian groups i.e. the non-abelian groups whose all subgroups are normal (see [9]), torsion solvable groups (see [5]). For these particular class of groups we obtain interesting results using the general order theory for groups and the locally finiteness property. For example: If G is a torsion solvable group, then the Scott topology on $\mathcal{L}(G)$ has a basis formed from the sets $\uparrow H = \{K \leq G \,|\, K \supseteq H\}$ where the subgroups H are precisely the finite subgroups of G; if G is a hamiltonian group, then a continuous function $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is completely determined by its values on the finite subgroups of G.

An important locally finite group is presented in [2]. First we make the remark that none of the results presented in this paper, except Theorem 3, does not use the Axiom of Choice in its proof. Clearly the structure of A assumed axiomatically in the description of the EFM model contradicts the Axiom of Choice. The definitions and results presented in Sections 1 and 2, except Theorem 3, are also valid if we work with the axioms of the EFM model of set theory described in Definition 9 instead of the axioms of the usual Zermelo-Fraenkel model of set theory. The indexing of several families of subgroups in this paper was done *only* for an easy writing; it does not play a special role. So, to avoid any logical contradiction in the EFM framework, we can present these families without using indexes; some examples of how we can express the results in this paper without using indexes are Theorem 14 and Theorem 16. From Theorem 10 we know that S_A is locally finite which means that the set of finitely generated subgroups of S_A is identical with the set of finite subgroups of S_A. Since the results presented in this paper, except Theorem 3, keep the same proofs (*which does not use* AC) if we work with the EFM axioms, we can present now several properties of S_A in the EFM framework.

From Proposition 4, Remark 2 and Theorem 10 we obtain:

Theorem 12. *$\mathcal{L}(S_A)$ is an algebraic domain and the smallest basis in $\mathcal{L}(S_A)$ (i.e. the set of all compact elements in $\mathcal{L}(S_A)$) is formed precisely from the finite subgroups of S_A.*

Theorem 2 and Theorem 10 give us a description of a topological basis for the Scott topology over $\mathcal{L}(S_A)$:

Theorem 13. *The Scott topology on $\mathcal{L}(S_A)$ has a basis formed from the sets $\uparrow H = \{K \leq S_A \,|\, K \supseteq H\}$ where the subgroups H are precisely the finite subgroups of S_A.*

From Corollary 1 and Theorem 10 we also get the following result:

Theorem 14. *Let $\mathcal{F}$ be an open set in $\mathcal{L}(S_A)$ (i.e. an open family of subgroups with respect the Scott topology). Then for each subgroup H of S_A which belongs to $\mathcal{F}$, there exists a finite subgroup K of S_A with the property that $K \in \mathcal{F}$ and $H \supseteq K$.*

From Theorem 6 we know that a continuous function on a subgroups lattice is completely determined by its behavior on a basis. In this way, because of Theorem 10, we obtain:

Theorem 15. *A continuous function $f : \mathcal{L}(S_A) \to \mathcal{L}(S_A)$ is completely determined by its values on the finite subgroups of S_A.*

Theorem 4 can be easily adapted to the EFM approach. We obtain the following theorem:

Theorem 16. *A function $f : \mathcal{L}(S_A) \to \mathcal{L}(S_A)$ is Scott-continuous if and only if f is monotone and for each directed family $\mathcal{F}$ of subgroups of G we have $f(\bigcup_{H\in\mathcal{F}} H) = \bigcup_{H\in\mathcal{F}} f(H)$.*

From Theorem 5 and Theorem 10 we get a continuity criterion:

Theorem 17. *A function $f : \mathcal{L}(S_A) \to \mathcal{L}(S_A)$ is Scott-continuous if and only if for each $H \leq S_A$ and each $K \leq S_A$ finite with $K \ll f(H)$ there exists $L \leq S_A$ finite with $L \ll H$ such that ($L \subseteq U$ implies $K \subseteq f(U)$) for each subgroup U of S_A.*

An interpolation property in $\mathcal{L}(S_A)$ follows from Proposition 1 and Theorem 10:

Theorem 18. *Whenever $H \ll K$ in $\mathcal{L}(S_A)$ we can interpolate a finite subgroup of S_A between H and K.*

A fixed-point theorem in the domain $\mathcal{L}(S_A)$ is obtained from Theorem 8 and Theorem 10:

Theorem 19. *Each continuous function $f : \mathcal{L}(S_A) \to \mathcal{L}(S_A)$ has a least fixpoint. It is given by $\bigcup_{n\in\mathbb{N}} f^n(\{id_A\})$ where id_A is the identity map on A.*

From Theorem 11 we can conclude that the results expressed in Theorems 12,13,14,15,16,17,18 and 19 remain valid also if we work in the classical Fraenkel-Mostowski model of set theory.

In this section we have presented several order and topological properties of the group S_A which is a part both of the domain of the interchange function and of the domain of the extended interchange function. The results presented in this section are valid both in the EFM and in the FM settings, even for constructing the EFM model we replaced a strong axiom (the finite support property) of the FM model with a consequence of it which assumes only a certain structure of A. For proving the results of Section 4 we do not need to assume that any element, in each arbitrary set, have finite support; it is enough to assume only that each subset of A is either finite or cofinite.

The properties of the *domain* of the extended interchange function (interchange function) are properties of the extended interchange function (interchange function). Since the sets in the EFM approach (respectively in the FM approach) are pairs $(X, \cdot)$ where X is a ZFA-set and $\cdot$ is an extended interchange function over X (respectively $\cdot$ is an interchange function over

X), we can also say that the properties of extended interchange function (interchange function) provide properties of *EFM-sets (FM-sets)*. From the results expressed in this section we conclude that the sets have some similar order and topological properties both in the EFM and in the FM settings.

5. Conclusion

This paper makes a connection between the classical group theory and the domain theory. For each group, its subgroups lattice is an algebraic domain with respect the standard inclusion of subgroups. A basis for this domain is represented precisely by the finitely generated subgroups (Proposition 4 and Remark 2). Some results in the classical order theory are particularized to this domain and several properties of various classes of groups are obtained. We also make a connection between the topology and the group theory by defining the Scott topology on the subgroups lattice of a group. A basis for this topology is expressed in terms of finitely generated subgroups (Theorem 2). Continuous functions defined with respect the Scott topology have also interesting properties (Theorem 5 and Theorem 6).

The main purpose of this paper was to obtain some order properties for S_A, where S_A is the set of all bijections of the set A of atoms in the EFM model of set theory constructed in [2]. These new properties of S_A are presented in Section 4 and are valid both in the FM and in the EFM framework. The main aim of constructing the EFM model was to prove that, even we relax one of the axioms of the FM model, we get similar algebraic properties of interchange function (see [2]) and similar properties of permutative renamings (see [3]) as in the FM model. In this paper, in Section 4, we prove that the extended interchange function (which is the natural generalization of the interchange function for the EFM approach) and the interchange function also have similar order and topological properties. In the same way we did in [3] where the properties of S_A obtained in [2] were extended to nominal logic, we could be able, in a future work, to extend the results obtained in the fourth section of this paper also to nominal logic or to other parts of computer science which deals with permutative models of set theory.

REFERENCES

1. Abramsky, S.; Jung, A. – *Domain Theory*, Handbook of logic in computer science, Vol. 3, 1–168, Handb. Log. Comput. Sci., 3, Oxford Univ. Press, New York, 1994.
2. Alexandru, A.; Ciobanu, G. – *An extension of a permutative model of set theory*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), 58 (2012), 1–18.
3. Alexandru, A.; Ciobanu, G. – *Permutative renamings in Extended Fraenkel-Mostowski model*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), accepted for publication.
4. Bourbaki, N. – *Elements de Mathematiques*, Livre III: Topologie generale, Hermann, 1971.
5. Dixon, M.R. – *Sylow theory, formations and Fitting classes in locally finite groups*, Series in Algebra, 2, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
6. Gabbay, M.J.; Pitts, A.M. – *A new approach to abstract syntax with variable binding*, Formal Aspects of Computing, 13 (2001), 341–363.
7. Gierz, G.; Hofmann, K.H.; Keimel, K.; Lawson, J.D.; Mislove, M.; Scott, D.S. – *Continuous Lattices and Domains*, Encyclopedia of Mathematics and its Applications, 93, Cambridge University Press, Cambridge, 2003.
8. Jech, T.J. – *The Axiom of Choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75, North-Holland Publishing Co., Amsterdam-London; Amercan Elsevier Publishing Co., Inc., New York, 1973.
9. Robinson, Derek J.S. – *A Course in the Theory of Groups*, Second edition, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.

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