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STABILITY AND OSCILLATION OF THE SOLUTIONS FOR A GENERALIZED COUPLED OSCILLATORS MODEL

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Abstract

In this paper, the stability and oscillation of the solutions for a class of generalized Hamiltonian parity-time non-symmetry model is investigated. By means of the mathematical analysis method, some sufficient conditions to guarantee the stability and oscillation of the solutions are obtained. Computer simulations are provided to demonstrate our results.

1. Introduction

It is known that a nonlinear system is a system in which the small change in input may produce an incommensurably large change in response. A coupled system of simple oscillators may often produce many new phenomena than isolated oscillator model. It gains much attention for engineers, physicists and mathematicians in last decade [1-15]. For

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example, Beregov and Melkikh have considered a system of autonomous inductively coupled Van der Pol generators, the mathematical model is the following:

$$\begin{cases} x_1''(t) - (\mu_1 - x_1^2(t))x_1'(t) + x_1(t) = -M_{12}x_2''(t), \\ x_2''(t) - (\mu_2 - x_2^2(t))x_2'(t) + x_2(t) = -M_{12}x_1''(t), \end{cases} \tag{1}$$

where M_{12} is the coefficient of mutual induction which is either positive or negative number. The authors established the presence of a strange non-chaotic attractor and several stable limiting cycles [1]. Guin et al. have investigated a bilaterally coupled Rayleigh-Duffing oscillators model as follows:

$$\begin{cases} x_1''(t) = a_1x_1(t) - b_1x_1^3(t)) + c_1x_1'(t) - d_1x_1'^3(t) + k_1f(x_1'(t), x_2'(t)), \\ x_2''(t) = a_2x_2(t) - b_2x_1^3(t)) + c_2x_2'(t) - d_2x_2'^3(t) + k_2g(x_1'(t), x_2'(t)). \end{cases} \tag{2}$$

With the increase of coupling factor between Rayleigh-Duffing oscillators, birth of periodic oscillations was observed. Dynamics becomes chaotic through a quasi-periodic route but for even higher coupling factor, synchronized stable periodic oscillations in Rayleigh-Duffing oscillators were found [2]. Tsoy has presented several models with parity-time symmetry. Hamiltonian functions for two and three linear oscillators coupled via coordinates and accelerations are derived. The mathematical model of two nonlinear oscillators is the following ([3], model (5), page 464):

$$\begin{cases} x_1''(t) + 2\gamma x_1'(t) + w_0^2x_1(t) + k_2x_2(t) + \mu_2x_2''(t) = 0, \\ x_2''(t) - 2\gamma x_2'(t) + w_0^2x_2(t) + k_1x_1(t) + \mu_1x_1''(t) = 0. \end{cases} \tag{3}$$

Regions of stable dynamics for two coupled oscillators are obtained. Numerical solutions for model (3) are provided. The author pointed out that there is an infinite growth of coordinates and velocities in system (3) above the threshold. It means that the model under consideration is incomplete, so that more terms are necessary for an adequate description of the process. Then the author provided the following generalized Hamiltonian parity-time symmetry model:

$$\begin{cases} x_1''(t) + 2\gamma x_1'(t) + w_0^2 x_1(t) + kx_2(t) + \mu x_2''(t) + (x_1^2(t) + 3x_2^2(t))x_1(t) = 0, \\ x_2''(t) - 2\gamma x_2'(t) + w_0^2 x_2(t) + kx_1(t) + \mu x_1''(t) + (3x_1^2(t) + x_2^2(t))x_2(t) = 0. \end{cases} \tag{4}$$

However, the author did not discuss any dynamical properties for model (4). In other words, the dynamical property for system (4) is still an open problem. In this paper, we discuss the following general coupled oscillators model:

$$\begin{cases} x_1''(t) + \gamma_1 x_1'(t) + w_1^2 x_1(t) + k_2 x_2(t) + \mu_2 x_2''(t) + (x_1^2(t) + 3x_2^2(t))x_1(t) = 0, \\ x_2''(t) - \gamma_2 x_2'(t) + w_2^2 x_2(t) + k_1 x_1(t) + \mu_1 x_1''(t) + (3x_1^2(t) + x_2^2(t))x_2(t) = 0, \end{cases} \tag{5}$$

where $\mu_1\mu_2 \neq 1, \gamma_i, w_i^2, k_i, \mu_i (i = 1, 2)$ may be different numbers. By means of the mathematical analysis method, the boundedness, stability and oscillations of the solutions for model (5) are derived. It was emphasized that model (4) is a special case of system (5). Therefore, some dynamical behaviour of the generalized Hamiltonian parity-time symmetry model has been provided.

2. Preliminaries

System (5) can be written as the following:

$$\begin{cases} x_1''(t) = -\gamma_1 x_1'(t) - w_1^2 x_1(t) - k_2 x_2(t) - \mu_2 x_2''(t) - (x_1^2(t) + 3x_2^2(t))x_1(t), \\ x_2''(t) = \gamma_2 x_2'(t) - w_2^2 x_2(t) - k_1 x_1(t) - \mu_1 x_1''(t) - (3x_1^2(t) + x_2^2(t))x_2(t), \end{cases} \tag{6}$$

or

$$\begin{cases} x_1''(t) = \dfrac{1}{1-\mu_1\mu_2}\left[-\gamma_1 x_1'(t) - w_1^2 x_1(t) - k_2 x_2(t) - \mu_2\gamma_2 x_2'(t) + \mu_2 w_2^2 x_2(t)\right. \\ \qquad\qquad \left. + \mu_2 k_1 x_1(t) + \mu_2(3x_1^2(t) + x_2^2(t))x_2(t) - (x_1^2(t) + 3x_2^2(t))x_1(t)\right], \\ x_2''(t) = \dfrac{1}{1-\mu_1\mu_2}\left[\gamma_2 x_2'(t) - w_2^2 x_2(t) - k_1 x_1(t) + \mu_1\gamma_1 x_1'(t) + \mu_1 w_1^2 x_1(t)\right. \\ \qquad\qquad \left. + \mu_1 k_2 x_2(t) + \mu_1(x_1^2(t) + 3x_2^2(t))x_1(t) - (3x_1^2(t) + x_2^2(t))x_2(t)\right]. \end{cases} \tag{7}$$

For convenience, system (7) can be written as an equivalent four dimensional first order system:

$$\begin{cases} x_1'(t) = \dfrac{1}{1-\mu_1\mu_2}\, x_2(t), \\ x_2'(t) = (\mu_2 k_1 - w_1^2)x_1(t) - \gamma_1 x_2(t) + (\mu_2 w_2^2 - k_2)x_3(t) - \mu_2\gamma_2 x_4(t) \\ \qquad\quad + \mu_2(3x_1^2(t) + x_3^2(t))x_3(t) - (x_1^2(t) + 3x_3^2(t))x_1(t), \\ x_3'(t) = \dfrac{1}{1-\mu_1\mu_2}\, x_4(t), \\ x_4'(t) = (\mu_1 w_1^2 - k_1)x_1(t) + \mu_1\gamma_1 x_2(t) + (\mu_1 k_2 - w_2^2)x_3(t) + \gamma_2 x_4(t) \\ \qquad\quad + \mu_1(x_1^2(t) + 3x_3^2(t))x_1(t) - (3x_1^2(t) + x_3^2(t))x_3(t). \end{cases} \tag{8}$$

The system (8) can be expressed in the following matrix form:

$$x'(t) = Ax(t) + f(x(t)), \tag{9}$$

where $x(t) = (x_1(t),\ x_2(t),\ x_3(t),\ x_4(t))^T$, A is a 4 by 4 matrix, and $f(x)$ is a 4 by 1 vector:

$$A = (a_{ij})_{4\times 4} = \begin{pmatrix} 0 & \dfrac{1}{1-\mu_1\mu_2} & 0 & 0 \\ \mu_2 k_1 - w_1^2 & -\gamma_1 & \mu_2 w_2^2 - k_2 & -\mu_2\gamma_2 \\ 0 & 0 & 0 & \dfrac{1}{1-\mu_1\mu_2} \\ \mu_1 w_1^2 - k_1 & \mu_1\gamma_1 & \mu_1 k_2 - w_2^2 & \gamma_2 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu_2(3x_1^2(t) + x_3^2(t))x_3(t) - (x_1^2(t) + 3x_3^2(t))x_1(t) \\ 0 \\ \mu_1(x_1^2(t) + 3x_3^2(t))x_1(t) - (3x_1^2(t) + x_3^2(t))x_3(t) \end{pmatrix}.$$

The linearized system of (9) is

$$x'(t) = Ax(t). \tag{10}$$

Obviously, system (9) can be seen as a distributed system of system (10). The distributed term is $f(x)$.

Lemma 1. *Assume that the following inequality holds*:

$$(\mu_2 k_1 - w_1^2)(\mu_1 k_2 - w_2^2) \neq (\mu_2 w_2^2 - k_2)(\mu_1 w_1^2 - k_1), \tag{11}$$

then there exists a unique equilibrium point which is exactly the zero point for system (8) (*or* (9)).

Proof. An equilibrium point $x^* = [x_1^*, x_2^*, x_3^*, x_4^*]^T$ of system (8) is a constant solution of the following system:

$$\begin{cases} \dfrac{1}{1-\mu_1\mu_2} x_2^* = 0, \\ (\mu_2 k_1 - w_1^2)x_1^* - \gamma_1 x_2^* + (\mu_2 w_2^2 - k_2)x_3^* - \mu_2\gamma_2 x_4^* \\ \qquad + \mu_2[3(x_1^*)^2 + (x_3^*)^2]x_3^* - [(x_1^*)^2 + 3(x_3^*)^2]x_1^* = 0, \\ \dfrac{1}{1-\mu_1\mu_2} x_4^* = 0, \\ (\mu_1 w_1^2 - k_1)x_1^* + \mu_1\gamma_1 x_2^* + (\mu_1 k_2 - w_2^2)x_3^* + \gamma_2 x_4^* \\ \qquad + \mu_1[(x_1^*)^2 + 3(x_3^*)^2]x_1^* - [3(x_1^*)^2 + (x_3^*)^2]x_3^* = 0. \end{cases} \tag{12}$$

Noting that $x_2^* = 0,\ x_4^* = 0,$ so system (12) changes to the following:

$$\begin{cases} (\mu_2 k_1 - w_1^2)x_1^* + (\mu_2 w_2^2 - k_2)x_3^* \\ \qquad + \mu_2[3(x_1^*)^2 + (x_3^*)^2]x_3^* - [(x_1^*)^2 + 3(x_3^*)^2]x_1^* = 0, \\ (\mu_1 w_1^2 - k_1)x_1^* + (\mu_1 k_2 - w_2^2)x_3^* \\ \qquad + \mu_1[(x_1^*)^2 + 3(x_3^*)^2]x_1^* - [3(x_1^*)^2 + (x_3^*)^2]x_3^* = 0. \end{cases} \tag{13}$$

We shall prove that $x_1^* = 0,\ x_3^* = 0.$ Indeed, system (13) can be written as a matrix form:

$$Bx^* = \mathbf{0}, \tag{14}$$

where $x^* = [x_1^*, x_3^*]^T$, B is a 2 by 2 matrix:

$$B = \begin{pmatrix} \mu_2 k_1 - w_1^2 - [(x_1^*)^2 + 3(x_3^*)^2] & \mu_2 w_2^2 - k_2 + \mu_2[3(x_1^*)^2 + (x_3^*)^2] \\ \mu_1 w_1^2 - k_1 + \mu_1[(x_1^*)^2 + 3(x_3^*)^2] & \mu_1 k_2 - w_2^2 - [3(x_1^*)^2 + (x_3^*)^2] \end{pmatrix}.$$

Firstly, suppose that for any x_1^*, x_3^*, matrix B is a nonsingular matrix. In other words, the following inequality holds:

$$\{\mu_2 k_1 - w_1^2 - [(x_1^*)^2 + 3(x_3^*)^2]\}\{\mu_1 k_2 - w_2^2 - [3(x_1^*)^2 + (x_3^*)^2]\} \neq$$

$$\{\mu_2 w_2^2 - k_2 + \mu_2[3(x_1^*)^2 + (x_3^*)^2]\}\{\mu_1 w_1^2 - k_1 + \mu_1[(x_1^*)^2 + 3(x_3^*)^2]\}, \quad (15)$$

then based on the linear algebraic Cramer's rule, system (14) only has trivial solution, namely, $x_1^* = 0$, $x_3^* = 0$. However, when $x_1^* = 0$, $x_3^* = 0$. Equation (15) changes to (11). The proof is completed.

Lemma 2. *Assume that* $\mu_1 < 0$, $\mu_2 < 0$, *then the solutions of system* (8) (*or* (9)) *are bounded.*

Proof. To prove the boundedness of the solutions in system (8), we construct a Lyapunov function $V(t) = \sum_{i=1}^4 \frac{1}{2} x_i^2(t)$. Calculating the derivative of $V(t)$ through system (8), we have

$$V'(t)|_{(8)} = \sum_{i=1}^{4} x_i(t) x_i'(t)$$

$$= \frac{1}{1-\mu_1\mu_2} x_1(t)x_2(t) + x_2(t)[(\mu_2 k_1 - w_1^2)x_1(t) - \gamma_1 x_2(t) + (\mu_2 w_2^2 - k_2)x_3(t)$$

$$- \mu_2\gamma_2 x_4(t) + \mu_2(3x_1^2(t) + x_3^2(t))x_3(t) - (x_1^2(t) + 3x_3^2(t))x_1(t)]$$

$$+ \frac{1}{1-\mu_1\mu_2} x_3(t)x_4(t) + x_4(t)[(\mu_1 w_1^2 - k_1)x_1(t) + \mu_1\gamma_1 x_2(t)$$

$$+ (\mu_1 k_2 - w_2^2)x_3(t) + \gamma_2 x_4(t) + \mu_1(x_1^2(t)$$

$$+ 3x_3^2(t))x_1(t) - (3x_1^2(t) + x_3^2(t))x_3(t)]$$

$$= [\frac{1}{1-\mu_1\mu_2} + \mu_2 k_1 - w_1^2]x_1(t)x_2(t) - \gamma_1 x_2^2(t) + (\mu_2 w_2^2 - k_2)x_2(t)x_3(t)$$

$$- \mu_2\gamma_2 x_2(t)x_4(t) + [\frac{1}{1-\mu_1\mu_2} + \mu_1 k_2 - w_2^2]x_3(t)x_4(t) + (\mu_1 w_1^2 - k_1)x_1(t)x_4(t)$$

$$+ \gamma_2 x_4^2(t) + \mu_2(3x_1^2(t) + x_3^2(t))x_2(t)x_3(t) - (x_1^2(t) + 3x_3^2(t))x_1(t)x_2(t)$$

$$+ \mu_1(x_1^2(t) + 3x_3^2(t))x_1(t)x_4(t) - (3x_1^2(t) + x_3^2(t))x_3(t)x_4(t). \tag{16}$$

Noting that as $x_i(t) \to +\infty (i = 1, \cdots, 4)$, $(3x_1^2(t) + x_3^2(t))x_2(t)x_3(t)$, $(x_1^2(t) + 3x_3^2(t))x_1(t)x_2(t)$, $(x_1^2(t) + 3x_3^2(t))x_1(t)x_4(t)$, and $(3x_1^2(t) + x_3^2(t))x_3(t)x_4(t)$ are higher order infinity than $x_i(t)x_j(t)(i, j = 1, \cdots, 4)$. Therefore, when $\mu_1 < 0$, $\mu_2 < 0$, there exists $M > 0$ such that $V'(t)|_{(8)} < 0$ as $|x_i| > M$, $(i = 1, \cdots, 4)$. This means that the all solutions of system (8) are bounded.

3. Stability of the Solutions

Theorem 1. *Assume that all solutions of system* (8) (*or* (9)) *are bounded. If zero is the unique equilibrium point of system* (8) (*or* (9)) *for selecting parameter values. Let* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *be characteristic values of matrix A. If* $\alpha_i < 0$, *or* $Re(\alpha_i) < 0$, $(i = 1, \cdots, 4)$, *then the trivial solution is stable.*

Proof. Let $\nu_i = [\nu_{i1}, \cdots, \nu_{i4}]^T$ be the corresponding characteristic vectors of α_i $(i = 1, 2, 3, 4)$. Then the solution to the linearized system (10) is the following:

$$x_i(t) = c_1\nu_{1i}\exp(\alpha_1 t) + c_2\nu_{2i}\exp(\alpha_2 t) + c_3\nu_{3i}\exp(\alpha_3 t) + c_4\nu_{4i}\exp(\alpha_4 t), \tag{17}$$

where $c_i (i = 1, 2, 3, 4)$ are any constants. Since $\alpha_i < 0$ or $\mathrm{Re}(\alpha_i) < 0$, $(i = 1, \cdots, 4)$, this means that the trivial solution of system (10) is stable.

Noting that $f_1(x) = f_3(x) = 0$ in system (9). Both $f_2(x)$ and $f_4(x)$ are higher infinitesimal as $x_1(t) \to 0$ and $x_3(t) \to 0$. Hence, the stability of the trivial solution of system (10) implies the stability of the trivial solution of system (9).

4. Oscillatory Behaviour of the Solutions

Theorem 2. *Assume that all solutions of system* (9) *are bounded. If zero is the unique equilibrium point of system* (9) *for selecting parameter values. Let* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *be characteristic values of matrix A. If there exists some positive* α_k, *or* $Re(\alpha_k) \geq 0$, $(k \in \{1, 2, 3, 4\})$, *then the unique equilibrium point of system* (9) *is unstable. System* (9) (*thus system* (5)) *generates an oscillatory solution.*

Proof. Obviously, the trivial solution of system (9) is unstable if and only if the trivial solution of system (10) is unstable. Therefore, we only need to prove the instability of the trivial solution of system (10). Since α_i $(i = 1, 2, 3, 4)$ are characteristic values of matrix A, then the characteristic equation corresponding to system (10) is the following:

$$\prod_{i=1}^{4}(\lambda - \alpha_i) = 0. \tag{18}$$

Without loss of generality, we assume that $\alpha_1 > 0$, or $\mathrm{Re}(\alpha_1) \geq 0$. Then from (18) we have

$$\lambda - \alpha_1 = 0. \tag{19}$$

Since $\alpha_1 > 0$, or $\mathrm{Re}(\alpha_1) > 0$, this means that there is a positive (or a positive real part) characteristic value of system (10). Therefore, the trivial solution of system (10) is unstable, implies that the trivial solution of system (9) is unstable. If $\mathrm{Re}(\alpha_1) = 0$, this means that system (10) has a pure imaginary root. Since both sin t and cos t can not tend to zero as t tends to infinity. Therefore, the trivial solution is unstable. The boundedness of the solutions of system (9) and the instability of unique equilibrium point will force system (9) to generate an oscillatory solution.

Theorem 3. *Assume that all solutions of system* (9) *are bounded. If zero is the unique equilibrium point of system* (9) *for selecting parameter values. Let* $\mu(A) = \max_{1 \le j \le 4}[a_{jj} + \sum_{i=1, i \ne j}^{4} |a_{ij}|]$ [16]. *If* $\mu(A) > 0$, *then system* (9) *has an oscillatory solution.*

Proof. Let $y(t) = \sum_{i=1}^{4} |x_i(t)|$, from (10), we have

$$y'(t) \le \mu(A) y(t). \tag{20}$$

Consider the scalar differential equation

$$z'(t) = \mu(A) z(t). \tag{21}$$

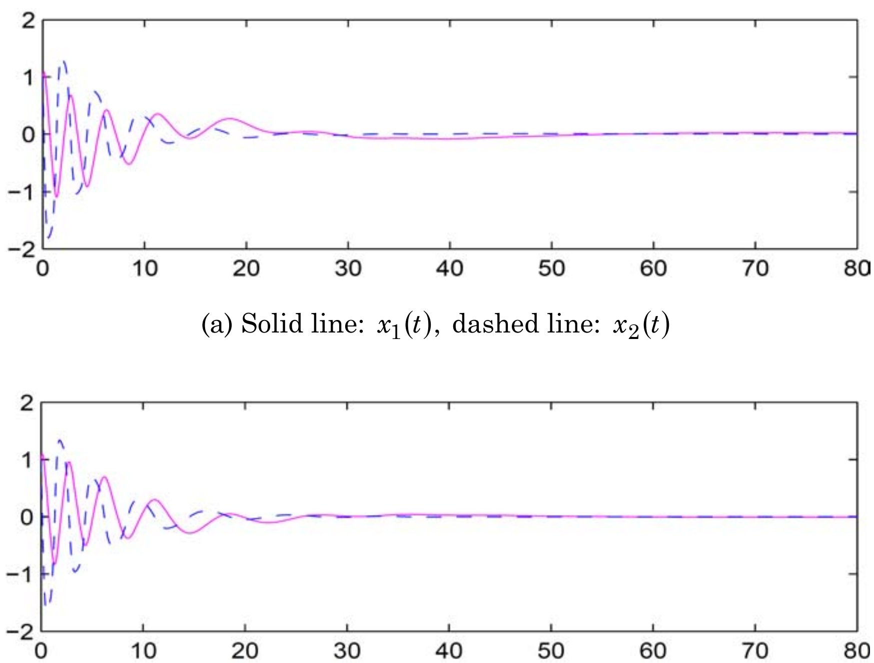
According to the comparison theorem of differential equation, we have $y(t) \le z(t)$. For Equation (21), the characteristic equation associated with (21) is given by

$$\lambda = \mu(A). \tag{22}$$

Since $\mu(A) > 0$, this means that there exists a positive characteristic root of Equation (21). Thus, the trivial solution of Equation (21) is unstable, implying that the trivial solution of Equation (10) is unstable. It suggested that system (9) (thus system (5)) has an oscillatory solution.

5. Simulation Results

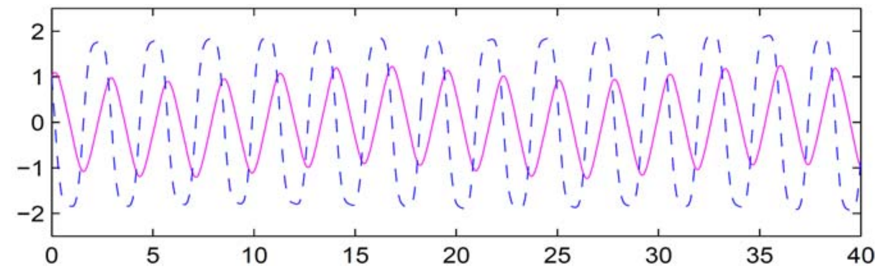
The simulation is based on the equivalent system (8) of (5), first the parameters are selected as follows: $\mu_1 = -0.45$, $\mu_2 = -0.55$, $\gamma_1 = 0.15$, $\gamma_2 = -0.25$, $k_1 = 0.08$, $k_2 = 0.16$, $w_1 = 0.25$, $w_2 = 0.45$, then the characteristic values of A are $-0.1557 \pm 0.6937i$, -0.0001, -0.0887. Based on Theorem 1, the trivial solution is convergent (see Figure 1). When the parameters are selected as $\mu_1 = -0.35$, $\mu_2 = -0.28$, $\gamma_1 = -0.12$, $\gamma_2 = -0.115$, $k_1 = 0.08$, $k_2 = -0.06$, $w_1 = 0.25$, $w_2 = 0.15$, then the characteristic values of A are $0.1514 \pm 0.2681i$, $-0.1489 \pm 0.2381i$. The conditions of Lemma 1 and Lemma 2 are satisfied. Since there is a positive real part of characteristic value $0.1514 + 0.2681i$ of matrix A, based on Theorem 2, there exists an oscillatory solution for system (8) (see Figure 2). When the parameters are selected as $\mu_1 = -0.48$, $\mu_2 = -0.36$, $\gamma_1 = 0.15$, $\gamma_2 = -0.25$, $k_1 = -1.18$, $k_2 = 0.25$, $w_1 = 0.65$, $w_2 = 0.45$, then $\mu(A) = 1.0489$. The conditions of Lemma 1 and Lemma 2 are satisfied. Since $\mu(A) = 1.0489 > 0$, based on Theorem 3, there exists an oscillatory solution for system (8) (see Figure 3).



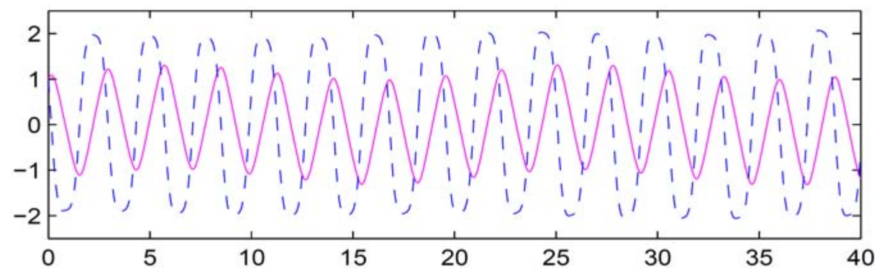
(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$

(b) Solid line: $x_3(t)$, dashed line: $x_4(t)$

Figure 1. The trivial solution is convergent.

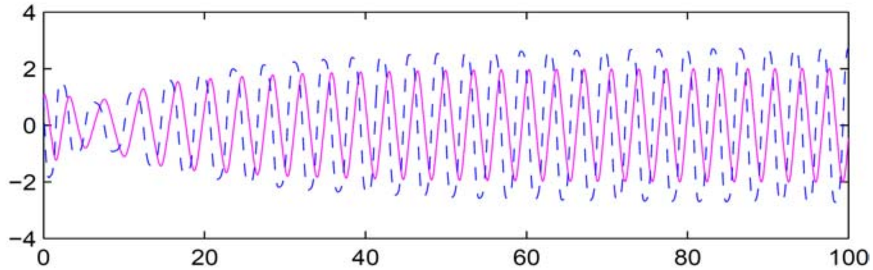


(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$

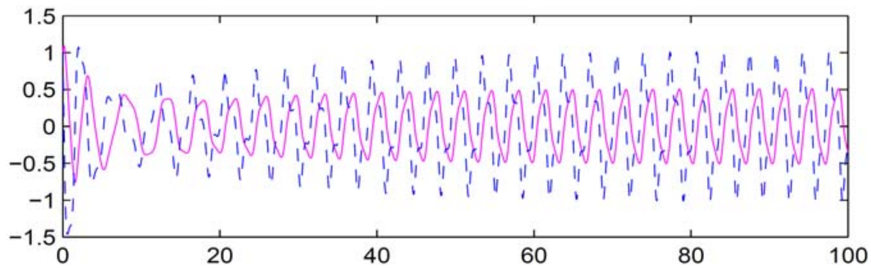


(b) Solid line: $x_3(t)$, dashed line: $x_4(t)$

Figure 2. Oscillatory behaviour of the solutions.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$



(b) Solid line: $x_3(t)$, dashed line: $x_4(t)$

Figure 3. Oscillatory behaviour of the solutions.

6. Conclusion

In this paper, we have discussed the convergence and oscillatory behaviour of the solutions for a generalized coupled oscillators model. Based on mathematical analysis method, we provided some sufficient conditions to guarantee the stability and oscillation of the solutions. Some simulations are provided to indicate the effectness of the criterion.

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