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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR GENERALIZED $(r; g, s, m, \varphi)$-PREINVEX FUNCTIONS

Abstract. In the present paper, a new class of generalized $(r; g, s, m, \varphi)$-preinvex functions is introduced and some new integral inequalities for the left hand side of Gauss-Jacobi type quadrature formula involving generalized $(r; g, s, m, \varphi)$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals are established. These results not only extend the results appeared in the literature (see [1],[2]), but also provide new estimates on these types.

Key words: Hermite-Hadamard type inequality, Hölder's inequality, Minkowski's inequality, Cauchy's inequality, power mean inequality, Riemann-Liouville fractional integral, s-convex function in the second sense, m-invex, P-function.

AMS Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15.

1. Introduction and preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Fractional calculus (see [14]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. *Here* $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13],[14]).

Now, let us recall some definitions of various convex functions.

Definition 2 (see [4]). *A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0,+\infty)$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

Definition 3 (see [5]). *A function $f : [0,+\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2}$$

for all $x, y \geq 0$, $\lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $[0,+\infty)$ as usual. The s-convex functions in the second sense have been investigated in (see [5]).

Definition 4 (see [6]). *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details (see [6],[7]).

Definition 5 (see [8]). *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{3}$$

for certain $B_{m,k}$, γ_k and rest $R_m^{\star}|f|$ (see [9]). Recently, Liu (see [10]) obtained several integral inequalities for the left hand side of (3) under the Definition 2 of P-function. Also in (see [11]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section , the notion of generalized $(r; g, s, m, \varphi)$-preinvex function is introduced and some new integral inequalities for the left hand side of (3) involving generalized $(r; g, s, m, \varphi)$-preinvex functions are given. In Section , some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via fractional integrals are given. These general inequalities give us some new estimates for the left hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions

Definition 6 (see [3]). *A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1. In Definition 6, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$.

Definition 7 (see [12]). *A positive function f on the invex set K is said to be logarithmically preinvex, if*

$$f(u + t\eta(v,u)) \leq f^{1-t}(u) f^t(v)$$

for all $u, v \in K$ and $t \in [0,1]$.

Definition 8 (see [12]). *The function f on the invex set K is said to be r-preinvex with respect to η, if*

$$f(u+t\eta(v,u)) \leq M_r(f(u), f(v); t)$$

holds for all $u, v \in K$ and $t \in [0,1]$, where

$$M_r(x,y;t) = \begin{cases} \left[(1-t)x^r + ty^r\right]^{\frac{1}{r}}, & \text{if } r \neq 0 \\ x^{1-t}y^t, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers x and y.

We next give new definition, to be referred as generalized $(r; g, s, m, \varphi)$-preinvex function.

Definition 9. *Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $g : [0,1] \longrightarrow [0,1]$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0, +\infty)$ is said to be generalized $(r; g, s, m, \varphi)$-preinvex with respect to η, if*

$$f\left(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)\right) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, s; t) \tag{4}$$

holds for any fixed $s, m \in (0,1]$ and for all $x, y \in I, t \in [0,1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m, s; t) = \begin{cases} \left[m(1-g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y))\right]^{\frac{1}{r}}, & \text{if } r \neq 0 \\ \left[f(\varphi(x))\right]^{m(1-g(t))^s} \left[f(\varphi(y))\right]^{g^s(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2. In Definition 9, it is worthwhile to note that the class of generalized $(r; g, s, m, \varphi)$-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 3. Also, for $r = 1$, $g(t) = t$, $\forall t \in [0,1]$ and $\varphi(x) = x$, $\forall x \in I$, we get the notion of generalized (s, m)-preinvex function (see [3]).

Example 1. Let $f(x) = -|x|$, $g(t) = t$, $\varphi(x) = x$, $r = s = 1$ and

$$\eta(y,x,m) = \begin{cases} y - mx, & \text{if } x \geq 0,\ y \geq 0 \\ y - mx, & \text{if } x \leq 0,\ y \leq 0 \\ mx - y, & \text{if } x \geq 0,\ y \leq 0 \\ mx - y, & \text{if } x \leq 0,\ y \geq 0. \end{cases}$$

Then $f(x)$ is a generalized $(1; t, 1, m, x)$-preinvex function of with respect to $\eta : \mathbb{R} \times \mathbb{R} \times (0,1] \longrightarrow \mathbb{R}$ and any fixed $m \in (0,1]$. However, it is obvious that $f(x) = -|x|$ is not a convex function on $\mathbb{R}$.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized $(r;g,s,m,\varphi)$-preinvex functions, we need the following new interesting lemma:

Lemma 1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Then for any fixed $m \in (0,1]$ and $p,q > 0$, we have*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx \\
&\quad = \eta(\varphi(b),\varphi(a),m)^{p+q+1} \\
&\qquad \times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)].
\end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx \\
&\quad = \eta(\varphi(b),\varphi(a),m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m) - m\varphi(a))^p \\
&\qquad \times (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - m\varphi(a) - g(t)\eta(\varphi(b),\varphi(a),m))^q \\
&\qquad \times f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)] \\
&\quad = \eta(\varphi(b),\varphi(a),m)^{p+q+1} \\
&\qquad \times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)].
\end{aligned}$$

■

Theorem 2. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ is a continuous function on K° with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Let $k > 1$ and $0 < r \leq 1$. If $f^{\frac{k}{k-1}}$ is generalized $(r;g,s,m,\varphi)$-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1]$, then for any fixed $p,q > 0$,*

$$\begin{aligned}
(5)\qquad &\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) \\
&\qquad\qquad + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx \\
&\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \left(\frac{r}{s+r}\right)^{\frac{k-1}{k}} B^{\frac{1}{k}}(g(t);k,p,q)
\end{aligned}$$

$$\times\left[m\left((1-g(0))^{\frac{s}{r}+1}-(1-g(1))^{\frac{s}{r}+1}\right)^{r}f^{\frac{rk}{k-1}}(\varphi(a))\right.$$

$$\left.+\,(g^{\frac{s}{r}+1}(1)-g^{\frac{s}{r}+1}(0))^{r}f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}},$$

where $B(g(t);k,p,q)=\int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)]$.

Proof. Let $k>1$ and $0<r\leq 1$. Since $f^{\frac{k}{k-1}}$ is generalized $(r;g,s,m,\varphi)$-preinvex function on K, combining with Lemma 1, Hölder inequality and Minkowski inequality for all $t\in[0,1]$ and for any fixed $s,m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^{p}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{q}f(x)dx$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)]\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1\left|f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))\right|^{\frac{k}{k-1}}d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\int_0^1\left(m(1-g(t))^{s}f^{\frac{rk}{k-1}}(\varphi(a))+g^{s}(t)f^{\frac{rk}{k-1}}(\varphi(b))\right)^{\frac{1}{r}}d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}}(1-g(t))^{\frac{s}{r}}f^{\frac{k}{k-1}}(\varphi(a))d[g(t)]\right)^{r}\right.$$

$$\left.+\left(\int_0^1 g^{\frac{s}{r}}(t)f^{\frac{k}{k-1}}(\varphi(b))d[g(t)]\right)^{r}\right]^{\frac{k-1}{rk}}$$

$$=|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left(\frac{r}{s+r}\right)^{\frac{k-1}{k}}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[m\left((1-g(0))^{\frac{s}{r}+1}-(1-g(1))^{\frac{s}{r}+1}\right)^{r}f^{\frac{rk}{k-1}}(\varphi(a))\right.$$

$$\left.+\left(g^{\frac{s}{r}+1}(1)-g^{\frac{s}{r}+1}(0)\right)^{r}f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}}.$$

■

Corollary 1. *Under the same conditions as in Theorem 2 for $r = 1$ and $g(t) = t$, we get (see [1], Theorem 2.2).*

Theorem 3. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous function on K° with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $l \geq 1$ and $0 < r \leq 1$. If f^l is generalized $(r; g, s, m, \varphi)$-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$, then for any fixed $p, q > 0$,*

$$
\begin{aligned}
(6) \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} & (x - m\varphi(a))^p (m\varphi(a) \\
& + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} & B^{\frac{l-1}{l}}(g(t); p, q) \\
\times \Bigg[m f^{rl}(\varphi(a)) & B^r \left(g(t); p, q + \frac{s}{r} \right) + f^{rl}(\varphi(b)) B^r \left(g(t); p + \frac{s}{r}, q \right) \Bigg]^{\frac{1}{rl}},
\end{aligned}
$$

where $B(g(t); p, q) = \displaystyle\int_0^1 g^p(t)(1 - g(t))^q d[g(t)]$.

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is generalized $(r; g, s, m, \varphi)$-preinvex function on K, combining with Lemma 1, the well-known power mean inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$
\begin{aligned}
\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} & (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
&= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\
&\quad \times \int_0^1 \left[g^p(t)(1 - g(t))^q \right]^{\frac{l-1}{l}} \left[g^p(t)(1 - g(t))^q \right]^{\frac{1}{l}} \\
&\quad \times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
&\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 g^p(t)(1 - g(t))^q d[g(t)] \right]^{\frac{l-1}{l}} \\
&\quad \times \left[\int_0^1 g^p(t)(1 - g(t))^q \left| f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \right|^l d[g(t)] \right]^{\frac{1}{l}} \\
&\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t); p, q) \\
&\quad \times \Bigg[\int_0^1 g^p(t)(1 - g(t))^q \Big(m(1 - g(t))^s f^{rl}(\varphi(a))
\end{aligned}
$$

$$
\begin{aligned}
&+ g^s(t)f^{rl}(\varphi(b))\Big)^{\frac{1}{r}} d[g(t)]\Bigg]^{\frac{1}{l}} \\
&\le |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);p,q) \\
&\quad \times \Bigg[\left(\int_0^1 m^{\frac{1}{r}} g^p(t)(1-g(t))^{q+\frac{s}{r}} f^l(\varphi(a)) d[g(t)] \right)^r \\
&\quad + \left(\int_0^1 g^{p+\frac{s}{r}}(t)(1-g(t))^q f^l(\varphi(b)) d[g(t)] \right)^r \Bigg]^{\frac{1}{rl}} \\
&= |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);p,q) \\
&\quad \times \left[m f^{rl}(\varphi(a)) B^r\left(g(t);p,q+\frac{s}{r}\right) + f^{rl}(\varphi(b)) B^r\left(g(t);p+\frac{s}{r},q\right) \right]^{\frac{1}{rl}}.
\end{aligned}
$$

■

Corollary 2. *Under the same conditions as in Theorem 3 for $r = 1$ and $g(t) = t$, we get (see [1], Theorem 2.3).*

3. Hermite-Hadamard type fractional integral inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via fractional integrals.

Theorem 4. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ be generalized $(r; g, s, m, \varphi)$-preinvex function on an open m-invex set K°. Then for $\alpha > 0$ and $0 < r \le 1$, we have*

$$
\begin{aligned}
(7)\qquad &\frac{1}{\eta^\alpha(\varphi(b),\varphi(a),m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(b),\varphi(a),m)} (t - m\varphi(a))^{\alpha-1} f(t) dt \\
&\le \Bigg[m f^r(\varphi(a)) B^r\left(g(t);\alpha-1,\frac{s}{r}\right) \\
&\quad + f^r(\varphi(b)) \left(\frac{r}{\alpha r + s}\right)^r \left(g^{\frac{s}{r}+\alpha}(1) - g^{\frac{s}{r}+\alpha}(0)\right)^r \Bigg]^{\frac{1}{r}}.
\end{aligned}
$$

Proof. Let $0 < r \leq 1$. Since f is generalized $(r; g, s, m, \varphi)$-preinvex function on an open m-invex set K°, combining with Minkowski inequality for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\frac{1}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(b),\varphi(a),m)} (t - m\varphi(a))^{\alpha-1} f(t)dt$$
$$= \int_0^1 g^{\alpha-1}(t) f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$
$$\leq \int_0^1 g^{\alpha-1}(t)\Big[m(1-g(t))^s f^r(\varphi(a)) + g^s(t) f^r(\varphi(b))\Big]^{\frac{1}{r}} d[g(t)]$$
$$\leq \Bigg\{\bigg[\int_0^1 g^{\alpha-1+\frac{s}{r}}(t) f(\varphi(b))d[g(t)]\bigg]^r$$
$$+ \bigg[\int_0^1 m^{\frac{1}{r}} g^{\alpha-1}(t)(1-g(t))^{\frac{s}{r}} f(\varphi(a))d[g(t)]\bigg]^r\Bigg\}^{\frac{1}{r}}$$
$$= \Bigg[m f^r(\varphi(a)) B^r\left(g(t); \alpha - 1, \frac{s}{r}\right)$$
$$+ f^r(\varphi(b))\left(\frac{r}{\alpha r + s}\right)^r \left(g^{\frac{s}{r}+\alpha}(1) - g^{\frac{s}{r}+\alpha}(0)\right)^r\Bigg]^{\frac{1}{r}}.$$

■

Corollary 3. *Under the same conditions as in Theorem 4 for $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b),\varphi(a),m) = \eta(b,a)$ and $g(t) = t$, we get (see [2], Theorem 3.1).*

Theorem 5. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1)$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ are respectively generalized $(r; g, s, m, \varphi)$-preinvex function and generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K°. Then for $\alpha > 0$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have*

$$\frac{1}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(b),\varphi(a),m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt \tag{8}$$
$$\leq \frac{1}{2}\Bigg\{\bigg[m f^r(\varphi(a)) B^{\frac{r}{2}}\left(g(t); \frac{2(\alpha-1)}{r}, \frac{2s}{r}\right)$$

$$+ f^r(\varphi(b))\left(\frac{r}{2(\alpha-1+s)+r}\right)^{\frac{r}{2}}\left(g^{\frac{2(\alpha-1+s)}{r}+1}(1)-g^{\frac{2(\alpha-1+s)}{r}+1}(0)\right)^{\frac{r}{2}}\Bigg]^{\frac{2}{r}}$$

$$+\left[mh^l(\varphi(a))B^{\frac{l}{2}}\left(g(t);\frac{2(\alpha-1)}{l},\frac{2s}{l}\right)\right.$$

$$\left.+ h^l(\varphi(b))\left(\frac{l}{2(\alpha-1+s)+l}\right)^{\frac{l}{2}}\left(g^{\frac{2(\alpha-1+s)}{l}+1}(1)-g^{\frac{2(\alpha-1+s)}{l}+1}(0)\right)^{\frac{l}{2}}\right]^{\frac{2}{l}}\Bigg\}.$$

Proof. Let $r>1$ and $r^{-1}+l^{-1}=1$. Since f and h are respectively generalized $(r;g,s,m,\varphi)$-preinvex function and generalized $(l;g,s,m,\varphi)$-preinvex function on an open m-invex set K°, combining with Cauchy and Minkowski inequalities for all $t\in[0,1]$ and for any fixed $s,m\in(0,1]$, we get

$$\frac{1}{\eta^\alpha(\varphi(b),\varphi(a),m)}\int_{m\varphi(a)+g(0)\eta(\varphi(b),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(b),\varphi(a),m)}(t-m\varphi(a))^{\alpha-1}f(t)h(t)dt$$

$$=\int_0^1 g^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)}(t)f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))$$

$$\times h(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$

$$\le\int_0^1 g^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)}(t)\Big[m(1-g(t))^s f^r(\varphi(a))+g^s(t)f^r(\varphi(b))\Big]^{\frac{1}{r}}$$

$$\times\Big[m(1-g(t))^s h^l(\varphi(a))+g^s(t)h^l(\varphi(b))\Big]^{\frac{1}{l}}d[g(t)]$$

$$\le\frac{1}{2}\Bigg\{\int_0^1\Big[g^{\alpha-1+s}(t)f^r(\varphi(b))+mg^{\alpha-1}(t)(1-g(t))^s f^r(\varphi(a))\Big]^{\frac{2}{r}}d[g(t)]$$

$$+\int_0^1\Big[g^{\alpha-1+s}(t)h^l(\varphi(b))+mg^{\alpha-1}(t)(1-g(t))^s h^l(\varphi(a))\Big]^{\frac{2}{l}}d[g(t)]\Bigg\}$$

$$\le\frac{1}{2}\Bigg[\Bigg\{\left(\int_0^1 g^{\frac{2(\alpha-1+s)}{r}}(t)f^2(\varphi(b))d[g(t)]\right)^{\frac{r}{2}}$$

$$+\left(\int_0^1 m^{\frac{2}{r}}g^{\frac{2(\alpha-1)}{r}}(t)(1-g(t))^{\frac{2s}{r}}f^2(\varphi(a))d[g(t)]\right)^{\frac{r}{2}}\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{\left(\int_0^1 g^{\frac{2(\alpha-1+s)}{l}}(t)h^2(\varphi(b))d[g(t)]\right)^{\frac{l}{2}}$$

$$+\left(\int_0^1 m^{\frac{2}{l}}g^{\frac{2(\alpha-1)}{l}}(t)(1-g(t))^{\frac{2s}{l}}h^2(\varphi(a))d[g(t)]\right)^{\frac{l}{2}}\Bigg\}^{\frac{2}{l}}\Bigg]$$

$$= \frac{1}{2}\Bigg\{ \Bigg[mf^r(\varphi(a))B^{\frac{r}{2}}\left(g(t); \frac{2(\alpha-1)}{r}, \frac{2s}{r}\right)$$

$$+ f^{\,r}(\varphi(b))\left(\frac{r}{2(\alpha-1+s)+r}\right)^{\frac{r}{2}}\left(g^{\frac{2(\alpha-1+s)}{r}+1}(1) - g^{\frac{2(\alpha-1+s)}{r}+1}(0)\right)^{\frac{r}{2}}\Bigg]^{\frac{2}{r}}$$

$$+ \Bigg[mh^l(\varphi(a))B^{\frac{l}{2}}\left(g(t); \frac{2(\alpha-1)}{l}, \frac{2s}{l}\right)$$

$$+ h^l(\varphi(b))\left(\frac{l}{2(\alpha-1+s)+l}\right)^{\frac{l}{2}}\left(g^{\frac{2(\alpha-1+s)}{l}+1}(1) - g^{\frac{2(\alpha-1+s)}{l}+1}(0)\right)^{\frac{l}{2}}\Bigg]^{\frac{2}{l}}\Bigg\}.$$

■

Corollary 4. *Under the same conditions as in Theorem 5 for $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.3).*

Theorem 6. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ are respectively generalized $(r; g, s, m, \varphi)$-preinvex function and generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K°. Then for $\alpha > 0$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have*

$$(9)\quad \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(b),\varphi(a),m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt$$

$$\leq \left\{ \frac{f^r(\varphi(b))}{s+\alpha}\left(g^{s+\alpha}(1) - g^{s+\alpha}(0)\right) + mf^r(\varphi(a))B(g(t); \alpha-1, s) \right\}^{\frac{1}{r}}$$

$$+ \left\{ \frac{h^l(\varphi(b))}{s+\alpha}\left(g^{s+\alpha}(1) - g^{s+\alpha}(0)\right) + mh^l(\varphi(a))B(g(t); \alpha-1, s) \right\}^{\frac{1}{l}}.$$

Proof. Let $r > 1$ and $r^{-1}+l^{-1} = 1$. Since f and h are respectively generalized $(r; g, s, m, \varphi)$-preinvex function and generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K°, combining with Hölder inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(b),\varphi(a),m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt$$

$$
\begin{aligned}
&= \int_0^1 g^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)}(t) f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m)) \\
&\quad \times h(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m)) d[g(t)] \\
&\leq \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) f^r(\varphi(b)) + m g^{\alpha-1}(t)(1-g(t))^s f^r(\varphi(a)) \right]^{\frac{1}{r}} \right. \\
&\quad \left. \times \left[g^{\alpha-1+s}(t) h^l(\varphi(b)) + m g^{\alpha-1}(t)(1-g(t))^s h^l(\varphi(a)) \right]^{\frac{1}{l}} d[g(t)] \right\} \\
&\leq \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) f^r(\varphi(b)) + m g^{\alpha-1}(t)(1-g(t))^s f^r(\varphi(a)) \right] d[g(t)] \right\}^{\frac{1}{r}} \\
&\quad + \left\{ \int_0^1 \left[g^{\alpha-1+s}(t) h^l(\varphi(b)) + m g^{\alpha-1}(t)(1-g(t))^s h^l(\varphi(a)) \right] d[g(t)] \right\}^{\frac{1}{l}} \\
&= \left\{ \frac{f^r(\varphi(b))}{s+\alpha} \left(g^{s+\alpha}(1) - g^{s+\alpha}(0) \right) + m f^r(\varphi(a)) B(g(t); \alpha-1, s) \right\}^{\frac{1}{r}} \\
&\quad + \left\{ \frac{h^l(\varphi(b))}{s+\alpha} \left(g^{s+\alpha}(1) - g^{s+\alpha}(0) \right) + m h^l(\varphi(a)) B(g(t); \alpha-1, s) \right\}^{\frac{1}{l}}.
\end{aligned}
$$

■

Corollary 5. *Under the same conditions as in Theorem 6 for* $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ *and* $g(t) = t$, *we get* (*see [2], Theorem 3.9*).

Remark 3. For different choices of positive values $r, l = \frac{1}{2}, \frac{1}{3}, 2$, etc., for any fixed $s, m \in (0, 1]$, for a particular choices of a differentiable function $g(t) = e^{-t}, \ln(t+1), \sin\left(\frac{\pi t}{2}\right), \cos\left(\frac{\pi t}{2}\right)$, etc, and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, x^n for all $x > 0$ and for all $n \in \mathbb{N}$, etc, by Theorem 4, Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type fractional inequalities.

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