Supplementary Appendix to "Asymptotic Power of Sphericity Tests for High-dimensional Data"

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June 22, 2012

Abstract

This note contains proofs of lemmas 4, 5, 6, 11, 12 and 13 in Onatski, Moreira and Hallin (2011), Asymptotic power of sphericity tests for highdimensional data, where we refer to for definitions and notation.

A Proof of Lemma 4

The original contour $\mathcal{K}$ is such that the singularities $z = \lambda_1, ..., z = \lambda_p$ of the integrand remain inside, whereas the singularity $z = \frac{1+h}{h}S$ remains outside the domain encircled by $\mathcal{K}$. Sufficient conditions for K to be similarly located with respect to the singularities of the integrand, and for $f(z)$ and $g(z)$ to be well-defined on K are

$$\min_{h\in(0,\bar{h}]} z_0(h) > \max\{b_p, \lambda_1\} \tag{A1}$$

and

$$\max_{h\in(0,\bar{h}]} \frac{h}{1+h}\frac{z_0(h)}{S} < 1. \tag{A2}$$

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Hence, to establish Lemma 4 it is enough to show that (A1) and (A2) hold with probability approaching one as $p, n \to \infty$ so that $c_p \to c$.

Let us fix a positive ε such that $\varepsilon < \left(\sqrt{c/\bar{h}} - \sqrt{\bar{h}}\right)^2$. Consider the event E that holds if and only if the following four inequalities simultaneously hold:

$$
\begin{aligned}
\min\nolimits_{h \in \left(0, \bar{h}\right]} \left(z_0(h) - b_p\right) &> \varepsilon, & \text{(A3)}\\
\left| b_p - \left(1 + \sqrt{c}\right)^2 \right| &< \varepsilon/4, & \text{(A4)}\\
\left| \lambda_1 - \left(1 + \sqrt{c}\right)^2 \right| &< \varepsilon/4, & \text{(A5)}\\
\min\nolimits_{h \in \left(0, \bar{h}\right]} \left(\frac{1+h}{h} S - z_0(h) \right) &> \varepsilon. & \text{(A6)}
\end{aligned}
$$

Clearly, E implies (A1) and (A2). On the other hand, $\Pr(E) \to 1$ as $n, p \to \infty$ so that $c_p \to c$. Indeed, by definition of $z_0(h)$ and b_p,

$$z_0(h) - b_p = \left(\sqrt{\frac{c_p}{h}} - \sqrt{h} \right)^2.$$

Therefore, as $c_p \to c$,

$$\min_{h \in \left(0, \bar{h}\right]} \left(z_0(h) - b_p\right) \to \min_{h \in \left(0, \bar{h}\right]} \left(\sqrt{\frac{c}{h}} - \sqrt{h} \right)^2 = \left(\sqrt{\frac{c}{\bar{h}}} - \sqrt{\bar{h}} \right)^2,$$

which is larger than ε by assumption. Hence, the probability of (A3) converges to one. Further, $b_p \to (1 + \sqrt{c})^2$ by definition, while $\lambda_1 \to (1 + \sqrt{c})^2$ almost surely under our null hypothesis, as shown, for example, in Geman (1980). Thus, the probabilities of (A4) and (A5) converge to one too. Finally, by definition of $z_0(h)$, $\frac{h}{1+h} z_0(h) = h + c_p$, so that

$$\min\nolimits_{h \in \left(0, \bar{h}\right]} \left(\frac{1+h}{h} S - z_0(h) \right) = \frac{1+\bar{h}}{\bar{h}} \left(S - \bar{h} - c_p \right).$$

But under our null hypothesis $S/p \to 1$ in probability, as $n, p \to \infty$ so that $c_p \to c$.

This follows, for example, from Theorem 1.1 of Bai and Silverstein (2004). Hence, the probability of (A6) also converges to one. It remains to note that $1-\Pr(E)$ equals the probability of the union of the events complementary to (A3)-(A6).

B Proof of Lemma 5

We have shown, in the proof of Lemma 4, that $\Pr(E) \to 1$. Therefore, it is sufficient to prove Lemma 5 under the assumption that E holds. Event E implies that $f(z)$ and $g(z)$ are analytic at $z_0(h)$ for any $h \in (0, \bar{h}]$. Furthermore, still under E,

$$f_1 \equiv \frac{\mathrm{d}}{\mathrm{d}z} f(z)|_{z=z_0(h)} = 0 \text{ and } f_2 \equiv \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} f(z)|_{z=z_0(h)} < 0.$$

Indeed, by definition, $z_0(h)$ is a critical point of $f(z)$ when $\bar{h} < \sqrt{c_p}$. But E implies $\bar{h} < \sqrt{c_p}$. Otherwise,

$$z_0(h) - b_p \equiv \left(\sqrt{\frac{c_p}{h}} - \sqrt{h}\right)^2 = 0 < \varepsilon$$

at $h = \sqrt{c_p} \leq \bar{h}$, which contradicts (A3). Further, a direct computation based on (3.3), (3.6), and (3.7)[1] shows that

$$f_2 = -\frac{1}{4}\frac{h^2}{(c_p - h^2)(1+h)^2} < 0. \tag{A7}$$

First, let us focus on the analysis of $\oint_{K_1} e^{-nf(z)} g(z)\mathrm{d}z$. Olver (1997) derives a useful representation for the part of $\oint_{K_1} e^{-nf(z)} g(z)\mathrm{d}z$ that corresponds to a portion of K_1 close to its boundary point, which in our case is $z_0(h)$. To make our exposition self-contained, we sketch Olver's derivation; for details, we refer the reader to pages

[1] Here and throughout this Supplement, numerical references are for equations in the main text.

121-124 of Olver's book.

Let us introduce new variables v and w by the equations

$$w^2 = v = f(z) - f_0, \tag{A8}$$

where the branch of w is determined by $\lim \{\arg(w)\} = 0$ as $z \to z_0(h)$ along K_1, and by continuity elsewhere.

Consider w as a function of z. Since $f_1 = 0$, there exists a small neighborhood of $z_0(h)$, where the indicated branch of $w(z)$ is an analytic function. Moreover, there exists a small number $\rho(h) > 0$ such that $w(z)$ maps the disk $|z - z_0(h)| < \rho(h)$ conformally on a domain Ω containing $w = 0$.

Let $z_1(h)$ be a point of K_1 chosen sufficiently close to $z_0(h)$ to insure that the disk $|w| \leq |f(z_1(h)) - f_0|^{1/2}$ is contained in Ω. Then the portion $[z_0, z_1] \equiv [z_0(h), z_1(h)]$ of contour K_1 can be deformed, without changing the value of the integral $\oint_{[z_0,z_1]} e^{-nf(z)} g(z)\mathrm{d}z$, to make its $w(z)$ map a straight line.

Transformation to the variable v gives

$$\oint_{[z_0,z_1]} e^{-nf(z)} g(z)\mathrm{d}z = e^{-nf_0} \oint_{[0,\tau(h)]} e^{-nv} \varphi(v)\mathrm{d}v, \tag{A9}$$

where

$$\tau(h) = f(z_1(h)) - f_0, \;\; \varphi(v) = \frac{g(z)}{f'(z)}, \tag{A10}$$

and the path for the integral on the right-hand side of (A9) is also a straight line.

For small $|v| \neq 0$, $\varphi(v)$ has a convergent expansion of the form

$$\varphi(v) = \sum_{s=0}^{\infty} a_s v^{(s-1)/2}, \tag{A11}$$

in which the coefficients a_s are related to f_s and g_s. The formulae for a_0, a_1, and

a_2 are given, for example, on p.86 of Olver (1997). We use them in the statement of Lemma 5.

Finally, define $\varphi_k(v)$, $k = 0, 1, 2, ...$ by the relations $\varphi_k(0) = a_k$ and

$$\varphi(v) = \sum_{s=0}^{k-1} a_s v^{(s-1)/2} + v^{(k-1)/2} \varphi_k(v) \text{ for } v \neq 0. \tag{A12}$$

Then the integral on the right-hand side of (A9) can be rearranged in the form

$$\oint_{[0,\tau(h)]} e^{-nv} \varphi(v) \mathrm{d}v = \sum_{s=0}^{k-1} \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{n^{(s+1)/2}} - \varepsilon_{k,1}(h) + \varepsilon_{k,2}(h), \tag{A13}$$

where

$$\varepsilon_{k,1}(h) = \sum_{s=0}^{k-1} \Gamma\left(\frac{s+1}{2}, \tau(h)n\right) \frac{a_s}{n^{(s+1)/2}}, \tag{A14}$$

$$\varepsilon_{k,2}(h) = \oint_{[0,\tau(h)]} e^{-nv} v^{(k-1)/2} \varphi_k(v) \mathrm{d}v, \tag{A15}$$

and

$$\Gamma(\alpha, x) = e^{-x} x^\alpha \int_0^\infty e^{-xt} (1+t)^{\alpha-1} \mathrm{d}t$$

is the incomplete Gamma function.

This completes our sketch of Olver's derivation. The remaining part of the proof of Lemma 5 is mostly concerned with two auxiliary lemmas establishing uniform asymptotic properties of $\varepsilon_{k,1}(h)$ and $\varepsilon_{k,2}(h)$. The first of these two lemmas provides explicit forms for $\rho(h)$, $z_1(h)$, and $\tau(h)$ allowing further analysis of their dependence on h.

Lemma A1. *Let $B(\alpha, R)$ and $\overline{B}(\alpha, R)$ denote, respectively, the open and closed balls in the complex plane with center at α and radius R. Further, let $r(h) = \min\left\{z_0(h) - \max\{b_p, \lambda_1\}, \frac{1+h}{h}S - z_0(h)\right\}$, $\rho(h) = \frac{1}{3\cdot 2^4} r(h)$, $z_1(h) = z_0(h) + \frac{i}{9\cdot 2^6} r(h)$, and $\tau(h) = f(z_1(h)) - f_0$. If event E holds, then,*

(i) *For any* ζ_1, ζ_2 *from* $\overline{B}\left(z_0(h), \rho(h)\right)$, *we have* $|w\left(\zeta_2\right) - w\left(\zeta_1\right)| > \frac{1}{2}\left|f_2^{1/2}\right|\left|\zeta_2 - \zeta_1\right|$;

(ii) *The function* $w(z)$ *is a one-to-one mapping of* $B\left(z_0(h), \rho(h)\right)$ *on an open set* Ω. *The inverse function* $z(w)$ *is analytic in* Ω;

(iii) *There exist positive constants* τ_1 and τ_2 *such that* $\operatorname{Re}\tau(h) > \tau_1$ *and* $\operatorname{Im}\tau\left(h\right) < \tau_2$ *for all* $h \in \left(0, \bar{h}\right]$;

(iv) $\overline{B}\left(0, 2\left|\tau(h)\right|^{1/2}\right)$ *is contained in* Ω.

Proof. Throughout this proof, we simplify the notation and write z_0, z_1, r, ρ, and τ instead of $z_0(h)$, $z_1(h)$, $r(h)$, $\rho(h)$, and $\tau(h)$, respectively. First, we show that $w(z)$ is analytic in $\bar{B}\left(z_0, \rho\right)$ and that $w'\left(z_0\right) = f_2^{1/2}$. Let $f^{(j)}(z)$ denote the j-th order derivative of $f(z)$. Consider the Taylor expansion of $f^{(j)}(z)$ at z_0 :

$$f^{(j)}\left(z\right) = \sum_{s=0}^{k} \frac{1}{s!} f^{(j+s)}\left(z_0\right)\left(z - z_0\right)^s + R_{j,k+1}.$$

In general, for any $z \in \overline{B}\left(z_0, R\right)$, the remainder $R_{j,k+1}$ satisfies

$$\left|R_{j,k+1}\right| \leq \frac{\left|z - z_0\right|^{k+1}}{(k+1)!} \max_{|t-z_0|\leq R}\left|f^{(j+k+1)}\left(t\right)\right|. \tag{A16}$$

From definition (3.3) of $f(z)$, we have

$$f^{(s)}\left(t\right) = \frac{c_p}{2}\left(-1\right)^{s-1}\left(s-1\right)! \int\left(t-\lambda\right)^{-s} \mathrm{d}\mathcal{F}_p\left(\lambda\right) \text{ for } s \geq 2. \tag{A17}$$

If $t \in B\left(z_0, \frac{1}{2}r\right)$, then $|t - \lambda| > \frac{1}{2}\left(z_0 - \lambda\right)$ for any λ in the support of $\mathcal{F}_p$. Therefore,

$$\left|t - \lambda\right|^{s+1} > \frac{1}{2^{s+1}}\left(z_0 - \lambda\right)^s r,$$

and using (A17) we get

$$\left|f^{(s+1)}(t)\right| < \frac{s2^{s+1}}{r}\left|f^{(s)}(z_0)\right| \text{ for } s \geq 2. \tag{A18}$$

Combining this with (A16), we obtain for $k+j \geq 2$ and $z \in B\left(z_0, \frac{k+1}{k+j}2^{-k-j-2}r\right)$,

$$|R_{j,k+1}| \leq \frac{|z-z_0|^k}{2k!}\left|f^{(k+j)}(z_0)\right|. \tag{A19}$$

Further, since

$$R_{j,k} = \frac{1}{k!}f^{(k+j)}(z_0)(z-z_0)^k + R_{j,k+1},$$

(A19) implies that, for $k+j \geq 2$ and $z \in B\left(z_0, \frac{k+1}{k+j}2^{-k-j-2}r\right)$,

$$\frac{1}{2k!}\left|f^{(k+j)}(z_0)\right||z-z_0|^k < |R_{j,k}| < \frac{3}{2k!}\left|f^{(k+j)}(z_0)\right||z-z_0|^k. \tag{A20}$$

Next, since $f^{(1)}(z_0) = 0$, inequalities (A20) imply that

$$|f(z)-f(z_0)| = |R_{0,2}| > \frac{1}{4}\left|f^{(2)}(z_0)\right||z-z_0|^2 \equiv \frac{1}{2}|f_2||z-z_0|^2 \tag{A21}$$

for any $z \in B\left(z_0, \frac{3}{2^5}r\right)$. Since $f_2 \neq 0$, inequality (A21) implies that $f(z)-f(z_0)$ does not have zeros in $B\left(z_0, \frac{3}{2^5}r\right)$ except a zero of the second order at $z = z_0$. Therefore,

$$\sqrt{\frac{f(z)-f(z_0)}{(z-z_0)^2}} = \frac{w(z)}{(z-z_0)}$$

is analytic inside $B\left(z_0, \frac{3}{2^5}r\right)$, which includes $\bar{B}(z_0, \rho)$, and converges to $f_2^{1/2}$ as $z \rightarrow z_0$. This implies that $w(z)$ is analytic in $\bar{B}(z_0, \rho)$ and $w'(z_0) = f_2^{1/2}$.

Now, let us show that, for any $z \in \overline{B}(z_0, \rho)$,

$$|w'(z) - w'(z_0)| < \frac{1}{2}|w'(z_0)|. \tag{A22}$$

Indeed, since

$$w'(z) = \frac{f'(z)}{2w(z)} = \frac{1}{2}(f(z) - f_0)^{-1/2} f'(z)$$

and $w'(z_0) = f_2^{1/2} \neq 0$,

$$\frac{w'(z)}{w'(z_0)} = \left(1 + \frac{R_{0,3}}{f_2(z - z_0)^2}\right)^{-\frac{1}{2}} \left(1 + \frac{R_{1,2}}{2f_2(z - z_0)}\right). \tag{A23}$$

Note that for any y_1 and y_2 such that $|y_2| < 1$,

$$\left|\frac{1 + y_1}{\sqrt{1 + y_2}} - 1\right| \leq \frac{|y_1| + |y_2|}{1 - |y_2|}, \tag{A24}$$

where the principal branch of the square root is used. This follows from the facts that, for $|y_2| < 1$, $\left|\sqrt{1 + y_2}\right| \geq 1 - |y_2|$ and $\left|1 + y_1 - \sqrt{1 + y_2}\right| \leq |y_1| + |y_2|$. Setting

$$y_1 = \frac{R_{1,2}}{2f_2(z - z_0)} \text{ and } y_2 = \frac{R_{0,3}}{f_2(z - z_0)^2}$$

and using (A23), (A20) and the fact that, for any $z \in \overline{B}(z_0, \rho)$,

$$\left|\frac{f^{(3)}(z_0)}{f^{(2)}(z_0)}\right| |z - z_0| < \frac{1}{3},$$

which follows from (A18), we get

$$\left|\frac{w'(z)}{w'(z_0)} - 1\right| < \frac{1}{2}.$$

Hence, (A22) holds.

Finally, let ζ_1 and ζ_2 be any two points in $\overline{B}(z_0, \rho)$, and let $\gamma(t) = (1 - t)\zeta_1 + t\zeta_2$, where $t \in [0, 1]$. We have

$$\int_0^1 (w'(\gamma(t)) - w'(z_0))\, \mathrm{d}t = \frac{w(\zeta_2) - w(\zeta_1)}{\zeta_2 - \zeta_1} - w'(z_0).$$

Therefore, using (A22), we obtain

$$\left|\frac{w(\zeta_2)-w(\zeta_1)}{\zeta_2-\zeta_1}-w'(z_0)\right|<\frac{1}{2}\left|w'(z_0)\right|.$$

This inequality and the fact that $w'(z_0)=f_2^{1/2}$ imply part (i) of the lemma.

Part (ii) of the lemma is a simple consequence of part (i) and of the analyticity of $w(z)$ in $\bar{B}(z_0,\rho)$, established above. Indeed, by the open mapping theorem, Ω is an open set. Next, by (i), $w(z)$ is one-to-one mapping of $B(z_0,\rho)$ on Ω and has a non-zero derivative in $B(z_0,\rho)$. Further, let $\psi(w)$ be defined on Ω by $\psi(w(z))=z$. Fix $\tilde{w}\in\Omega$. Then $\psi(\tilde{w})=\tilde{z}$ for a unique $\tilde{z}$ in $B(z_0,\rho)$. If $w\in\Omega$ and $\psi(w)=z$, we have

$$\frac{\psi(w)-\psi(\tilde{w})}{w-\tilde{w}}=\frac{z-\tilde{z}}{w(z)-w(\tilde{z})}.$$

By (i), $w\to\tilde{w}$ as $z\to\tilde{z}$, and the latter equality implies $\psi'(\tilde{w})=\frac{1}{w'(\tilde{z})}$. Therefore, $z(w)\equiv\psi(w)$ is an analytic inverse of $w(z)$ on Ω.

To see that part (iii) holds, note that

$$\operatorname{Re}\tau=\frac{c_p}{2}\int\ln\left|\frac{z_1-\lambda}{z_0-\lambda}\right|\mathrm{d}\mathcal{F}_p(\lambda), \tag{A25}$$

and for any λ such that $0\leq\lambda<z_0$, we have

$$\left|\frac{z_1-\lambda}{z_0-\lambda}\right|\geq\left|1+\frac{i}{9\cdot 2^6}\frac{r}{z_0}\right|.$$

When E holds, the latter expression is bounded from below by a fixed constant that is strictly larger than one for all $h\in\left(0,\bar{h}\right]$. Therefore, when E holds, (A25) implies that $\operatorname{Re}\tau>\tau_1>0$, for all $h\in\left(0,\bar{h}\right]$, where τ_1 is fixed.

Next, by definition of τ, we have

$$\operatorname{Im}\tau = -\frac{1}{2}\left(\frac{h}{1+h}\frac{r}{9\cdot 2^6} - c_p\int \arg\left(\frac{z_1-\lambda}{z_0-\lambda}\right)\right)\mathrm{d}\mathcal{F}_p(\lambda).$$

But

$$\frac{h}{1+h}r < \frac{h}{1+h}z_0 \equiv c_p + h,$$

which is smaller than a fixed positive number for all $h \in \left(0, \bar{h}\right]$ when E holds. Here the boundedness of h is obvious whereas the boundedness of c_p follows from (A4). Further,

$$\left|\arg\left(\frac{z_1-\lambda}{z_0-\lambda}\right)\right| < \frac{\pi}{2}$$

for all $h \in \left(0, \bar{h}\right]$ because $\operatorname{Re}\frac{z_1-\lambda}{z_0-\lambda} \equiv 1$. Hence, there exists τ_2 such that $|\operatorname{Im}\tau| < \tau_2$ for all $h \in \left(0, \bar{h}\right]$.

Finally, part (iv) of the lemma can be established as follows. Note that by part (i),

$$\left|w\left(z_0+\rho e^{i\theta}\right) - w\left(z_0\right)\right| > \frac{\rho}{2}\left|w'\left(z_0\right)\right|$$

for any $\theta \in [0, 2\pi]$. Therefore, for any w_1 such that $|w_1 - w(z_0)| \leq \frac{\rho}{4}\left|w'\left(z_0\right)\right|$, we have

$$\min_{\theta}\left|w_1 - w\left(z_0+\rho e^{i\theta}\right)\right| > \frac{\rho}{4}\left|w'\left(z_0\right)\right|.$$

By a corollary to the *maximum modulus theorem* (see Rudin (1987), p.212), the latter inequality implies that the function $w(z) - w_1$ has a zero in $B(z_0, \rho)$. Thus, region Ω includes $\overline{B}(0, \frac{\rho}{4}\left|w'\left(z_0\right)\right|)$. On the other hand,

$$2\left|\tau\right|^{1/2} < \frac{\rho}{4}\left|w'\left(z_0\right)\right|.$$

Indeed, consider the identity

$$\tau = f^{(1)}(z_0)(z_1 - z_0) + R_{0,2}.$$

Since $f^{(1)}(z_0) = 0$, (A20) together with (A7) imply

$$|\tau| < \frac{3}{2}|f_2||z_1 - z_0|^2.$$

Since $w'(z_0) = f_2^{1/2}$ and $|z_1 - z_0| = \frac{1}{9\cdot 2^6}r$, the latter inequality implies that

$$2|\tau|^{1/2} < \frac{\rho}{4}|w'(z_0)|.$$

Therefore, Ω includes $\overline{B}(0, 2|\tau|^{1/2})$.$\square$

Before proceeding with the proof of Lemma 5, we still need one more auxiliary lemma.

LEMMA A2. *Under the null hypothesis,* $\sup_{z\in\Theta_1}|g(z)| = O_p(1)$ *as* $n, p \to \infty$ *so that* $c_p \to c$, *where* $\Theta_1 = \left\{z : |\mathrm{Re}(z) - z_0(h)| < \frac{1}{2}r(h)\right\}$ *and* $O_p(1)$ *is uniform over* $h \in \left(0, \bar{h}\right]$.

PROOF. First, consider the case when $g(z) = \exp\left(-\frac{1}{2}\Delta_p(z)\right)$, where

$$\begin{aligned}\Delta_p(z) &\equiv \sum_{j=1}^{p}\ln(z - \lambda_j) - p\int \ln(z - \lambda)\,\mathrm{d}\mathcal{F}_p(\lambda)\\ &= \sum_{j=1}^{p}\ln\left(1 - \frac{\lambda_j}{z}\right) - p\int \ln\left(1 - \frac{\lambda}{z}\right)\mathrm{d}\mathcal{F}_p(\lambda).\end{aligned}$$

This statistic $\Delta_p(z)$ is a special form of a linear spectral statistic

$$\Delta_p(\varphi) \equiv \sum_{j=1}^{p}\varphi(\lambda_j) - p\int \varphi(\lambda)\,\mathrm{d}\mathcal{F}_p(\lambda)$$

studied by Bai and Silverstein (2004). According to their Theorem 1.1, if $\varphi(\cdot)$ is analytic on an open set containing interval $\mathcal{I}_c \equiv \left[0, (1+\sqrt{c})^2\right]$, then the sequence $\{\Delta_p(\varphi)\}$ is tight. That is, for any $\theta > 0$ there exists a bound B such that $\Pr(|\Delta_p(\varphi)| \leq B) > 1 - \theta$ for every $\Delta_p(\varphi)$ from the sequence.

A close inspection of Bai and Silverstein's (2004, pp.562-563) proof of tightness reveals that the bound B can be chosen so that it depends on $\varphi(\cdot)$ only through its supremum over an open area A that includes $\mathcal{I}_c$ and where $\varphi(\cdot)$ is analytic. In particular, if we denote by Φ a family of functions $\varphi(x)$, each of which is analytic in the area $A = \left\{x : \sup_{\lambda \in \mathcal{I}_c} |x - \lambda| < \varepsilon\right\}$, and if Φ is such that $\sup_{\varphi \in \Phi} \sup_{x \in A} |\varphi(x)| < \infty$, then $\left\{\sup_{\varphi \in \Phi} |\Delta_p(\varphi)|\right\}$ is tight.

Let $\Phi = \left\{\varphi(x) \equiv \ln\left(1 - \frac{x}{z}\right) : z \in \Theta_2\right\}$, where

$$\Theta_2 = \left\{z : \mathrm{Re}(z) > \left(1+\sqrt{c}\right)^2 + 2\varepsilon\right\}.$$

This family of functions satisfies the above requirements. Indeed,

$$\sup_{x \in A, z \in \Theta_2} \left|\frac{x}{z}\right| = \frac{(1+\sqrt{c})^2 + \varepsilon}{(1+\sqrt{c})^2 + 2\varepsilon} < 1$$

so that each of $\varphi(\cdot) \in \Phi$ is analytic in A. Moreover, since by definition

$$\ln\left(1 - \frac{x}{z}\right) = \ln\left|1 - \frac{x}{z}\right| + i \arg\left(1 - \frac{x}{z}\right),$$

we have

$$\sup_{\varphi \in \Phi} \sup_{x \in A} |\varphi(x)| < \ln|1 - R| + \frac{\pi}{2},$$

where

$$R \equiv \sup_{x \in A, z \in \Theta_2} \left|\frac{x}{z}\right| < 1.$$

Therefore, $\left\{\sup_{\varphi \in \Phi} |\Delta_p(\varphi)|\right\}$ is tight and $\sup_{z \in \Theta_2} |g(z)| = O_p(1)$, where $O_p(1)$ does not depend on h.

It remains to note that, as $p,n\to\infty$ so that $c_p\to c$,

$$\inf_{h\in\left(0,\bar{h}\right]}\left(z_0(h)-\frac{1}{2}r(h)\right)\to\frac{1}{2}\frac{\left(\bar{h}+1\right)\left(c+\bar{h}\right)}{\bar{h}}+\frac{1}{2}\left(1+\sqrt{c}\right)^2>\left(1+\sqrt{c}\right)^2$$

almost surely. Therefore, for a sufficiently small ε, $\Pr\left(\Theta_1\subseteq\Theta_2\right)\to 1$, and thus, $\sup_{z\in\Theta_1}|g(z)|=O_p(1)$, where $O_p(1)$ is uniform over $h\in\left(0,\bar{h}\right]$.

Now, consider the case when

$$g(z)=\exp\left\{-\frac{np-p+2}{2}\ln\left(1-\frac{h}{1+h}\frac{z}{S}\right)-\frac{n}{2}\frac{hz}{1+h}-\frac{\Delta_p\left(z\right)}{2}\right\}.$$

Since, as has just been shown, $\sup_{z\in\Theta_1}\left|\exp\left(-\frac{1}{2}\Delta_p(z)\right)\right|=O_p\left(1\right)$, we only need to prove that $\sup_{z\in\Theta_1}\tilde{g}\left(z\right)=O_p\left(1\right)$, where

$$\tilde{g}\left(z\right)=\exp\left\{-\frac{np-p+2}{2}\operatorname{Re}\ln\left(1-\frac{h}{1+h}\frac{z}{S}\right)-\frac{n}{2}\frac{h\operatorname{Re}z}{1+h}\right\}.$$

We have

$$\operatorname{Re}\ln\left(1-\frac{h}{1+h}\frac{z}{S}\right)=\ln\left|1-\frac{h}{1+h}\frac{z}{S}\right|>\ln\left(1-\frac{h}{1+h}\frac{\operatorname{Re}z}{S}\right).$$

Note that (A6) and the definition of Θ_1 imply that

$$\frac{h}{1+h}\frac{\operatorname{Re}z}{S}<1$$

for any $z\in\Theta_1$. In general, for any real x such that $0<x<1$, we have

$$\ln\left(1-x\right)>-\frac{x}{1-x}.$$

Therefore, for any $z \in \Theta_1$,

$$\ln\left(1-\frac{h}{1+h}\frac{\operatorname{Re} z}{S}\right) > -\left(S-\frac{h\operatorname{Re} z}{1+h}\right)^{-1}\frac{h\operatorname{Re} z}{1+h},$$

and we can write

$$\ln \tilde{g}(z) < \frac{p}{2c_p}\frac{h\operatorname{Re} z}{1+h}\left[\left(p-c_p+\frac{2}{n}\right)\left(S-\frac{h\operatorname{Re} z}{1+h}\right)^{-1}-1\right]. \tag{A26}$$

From the definition of Θ_1,

$$\left|\frac{h\operatorname{Re} z}{1+h}\right| < \frac{h}{1+h}\left(\frac{1}{2}r(h)+z_0(h)\right) < \frac{3}{2}\frac{hz_0(h)}{1+h} = \frac{3}{2}(h+c_p).$$

Further, $S-p \equiv \lambda_1+\ldots+\lambda_p-p = O_p(1)$ by Theorem 1.1 of Bai and Silverstein (2004). Combining these facts with (A26), we get $\sup_{z\in\Theta_1}\tilde{g}(z) = O_p(1)$ uniformly over $h \in \left(0,\bar{h}\right]$.□

Let us return to the proof of Lemma 5. Consider $\varphi(v)w$ as a function of w. According to (A8) and (A11), $\varphi(v)w$ has a convergent series representation

$$\varphi(v)w = \sum_{s=0}^{\infty} a_s w^s \tag{A27}$$

for sufficiently small $|w|$. Let us show that the series in (A27) converges for all $w \in \Omega$. Indeed, from (A10), we see that

$$\varphi(v)w = \left(2w'(z)\right)^{-1} g(z). \tag{A28}$$

By Lemma A1 (ii), z, viewed as the inverse of $w(z)$, is analytic in Ω. Further, $g(z)$

and $w'(z)$ are analytic in $z(\Omega) \equiv B(z_0(h), \rho(h))$. Finally,

$$|w'(z)| > \frac{1}{2}\left|f_2^{1/2}\right| \tag{A29}$$

for $z \in \overline{B}(z_0(h), \rho(h))$ by Lemma A1 (i), and $f_2^{1/2} \neq 0$ for $h \in \left(0, \bar{h}\right]$. Therefore, $\varphi(v)w$ must be analytic in Ω and the series (A27) must converge there.

Now, formula (A7) implies that $\inf_{h\in\left(0,\bar{h}\right]}\left\{\left|f_2^{1/2}\right|/h\right\} > 0$. Therefore, from Lemma A2 and (A29), we have

$$\sup_{w\in\Omega}|\varphi(v)w| \equiv \sup_{z\in\overline{B}(z_0(h),\rho(h))}\left|\frac{g(z)}{2w'(z)}\right| = h^{-1}O_p(1), \tag{A30}$$

where $O_p(1)$ is uniform in $h \in \left(0, \bar{h}\right]$.

By Lemma A1 (iii) and (iv), $|\tau(h)| > |\mathrm{Re}\,\tau(h)| > \tau_1$ and $B\left(0, |\tau_1|^{1/2}\right)$ is contained in Ω, where $\varphi(v)w$ is analytic. Using Cauchy's estimates for the derivatives of an analytic function (see Theorem 10.26 in Rudin (1987)), (A27) and (A30), we get

$$|a_s| \leq |\tau_1|^{-s/2} \sup_{w\in B\left(0,|\tau_1|^{1/2}\right)}|\varphi(v)w| = h^{-1}O_p(1). \tag{A31}$$

Next, Olver (1997, ch. 4, pp.109-110) shows that $\Gamma(\alpha,\zeta) = O\left(e^{-\zeta}\zeta^{\alpha-1}\right)$ as $|\zeta| \to \infty$, uniformly in the sector $|\arg(\zeta)| \leq \frac{\pi}{2} - \delta$ for an arbitrary positive δ. Let us take $\alpha = \frac{s+1}{2}$ and $\zeta = \tau(h)n$. Lemma A1 (iii) shows that

$$|\tau(h)n| > \tau_1 n \to \infty$$

and

$$|\arg(\tau(h)n)| = \left|\arctan\frac{\mathrm{Im}\,\tau(h)}{\mathrm{Re}\,\tau(h)}\right| < \arctan\frac{\tau_2}{\tau_1} < \frac{\pi}{2},$$

uniformly over $h \in \left(0, \bar{h}\right]$. Therefore,

$$\Gamma\left(\frac{s+1}{2}, \tau(h)\, n\right) = O\left(e^{-\tau(h)n}\left(\tau(h)\, n\right)^{\frac{s-1}{2}}\right) = O_p\left(e^{-\frac{1}{2}\tau_1 n}\right) \qquad \text{(A32)}$$

for any integer s, uniformly over $h \in \left(0, \bar{h}\right]$.

Equality (A32), the definition (A14) of $\varepsilon_{k,1}(h)$, and inequality (A31) imply that

$$\varepsilon_{k,1}(h) = h^{-1} O_p(e^{-\frac{1}{2}\tau_1 n}), \qquad \text{(A33)}$$

where $O_p(\cdot)$ is uniform over $h \in \left(0, \bar{h}\right]$.

Next, consider $w^k \varphi_k(v)$ as a function of w. Since, by definition,

$$w^k \varphi_k(v) = \varphi(v)\, w - \sum_{s=0}^{k-1} a_s w^s,$$

it can be interpreted as a remainder in the Taylor expansion of $\varphi(v)\, w$. As explained above, such an expansion is valid in Ω, which includes the ball $B\left(0, 2\left|\tau(h)\right|^{1/2}\right)$ by Lemma A1 (iv). By a general formula for remainders in Taylor expansions, for any $w \in B\left(0, \left|\tau(h)\right|^{1/2}\right)$,

$$\left|w^k \varphi_k(v)\right| \leq \frac{|w|^k}{k!} \max_{w \in B\left(0, |\tau(h)|^{1/2}\right)} \left|\frac{d^k}{dw^k}\left(w\varphi(v)\right)\right|. \qquad \text{(A34)}$$

Further, for any $w \in B\left(0, \left|\tau(h)\right|^{1/2}\right)$, a ball with radius $\left|\tau_1\right|^{1/2}$ centered in w is contained in the ball $B\left(0, 2\left|\tau(h)\right|^{1/2}\right) \subset \Omega$. Therefore, using (A30) and Cauchy's estimates for the derivatives of an analytic function (see Theorem 10.26 in Rudin (1987)), we get

$$\max_{w \in B\left(0, |\tau(h)|^{1/2}\right)} \left|\frac{d^k}{dw^k}\left(w\varphi(v)\right)\right| \leq k!\left|\tau_1\right|^{-k/2} \sup_{w \in \Omega}\left|w\varphi(v)\right| = h^{-1} O_p(1). \qquad \text{(A35)}$$

Combining (A34) and (A35), we have

$$\sup_{v\in(0,\tau(h)]} \left|\varphi_k\left(v\right)\right| = h^{-1}O_p(1).$$

This equality together with (A31) and the fact that, by definition, $\varphi_k\left(0\right) = a_k$ imply that

$$\max_{v\in[0,\tau(h)]} \left|\varphi_k\left(v\right)\right| = h^{-1}O_p(1), \tag{A36}$$

where $O_p(1)$ is uniform in $h \in \left(0,\bar{h}\right]$.

For $\varepsilon_{k,2}\left(h\right)$, the substitution of variable $v = \tau(h)\frac{x}{n}$ in the integral (A15) yields

$$\varepsilon_{k,2}\left(h\right) = n^{-(k+1)/2}\int_0^n e^{-\tau(h)x}x^{\frac{k-1}{2}}\tau(h)^{\frac{k+1}{2}}\varphi_k\left(v\right)\mathrm{d}x.$$

Therefore,

$$\begin{aligned}\left|\varepsilon_{k,2}\left(h\right)n^{(k+1)/2}\right| &< \max_{v\in[0,\tau(h)]}\left|\varphi_k\left(v\right)\right|\int_0^n e^{-\operatorname{Re}\tau(h)x}x^{\frac{k-1}{2}}\left|\tau(h)\right|^{\frac{k+1}{2}}\mathrm{d}x \\ &< \max_{v\in[0,\tau(h)]}\left|\varphi_k\left(v\right)\right|\int_0^\infty e^{-\frac{\operatorname{Re}\tau(h)}{\left|\tau(h)\right|}y}y^{\frac{k-1}{2}}\mathrm{d}y.\end{aligned} \tag{A37}$$

But by Lemma A1 (iii),

$$\frac{\operatorname{Re}\tau(h)}{\left|\tau\left(h\right)\right|} > \frac{\operatorname{Re}\tau(h)}{\left|\operatorname{Re}\tau(h)\right| + \left|\operatorname{Im}\tau\left(h\right)\right|} > \frac{\tau_1}{\tau_1+\tau_2}$$

for all $h \in \left(0,\bar{h}\right]$. Therefore, the integral in (A37) is bounded uniformly over $h \in \left(0,\bar{h}\right]$. Using (A36), we conclude that

$$\varepsilon_{k,2}\left(h\right) = h^{-1}O_p\left(n^{-(k+1)/2}\right). \tag{A38}$$

Combining (A9), (A13), (A33), and (A38), we get

$$\oint_{[z_0,z_1]} e^{-nf(z)}g(z)\mathrm{d}z = e^{-nf_0}\left(\sum_{s=0}^{k-1}\Gamma\left(\frac{s+1}{2}\right)\frac{a_s}{n^{(s+1)/2}} + \frac{O_p(1)}{hn^{(k+1)/2}}\right), \qquad \text{(A39)}$$

where $O_p(1)$ is uniform in $h \in \left(0, \bar{h}\right]$.

Let us now consider the contribution of $K_+ \backslash [z_0, z_1]$, that is, the part of contour K_+ excluding the segment $[z_0, z_1]$, to the contour integral $\oint_{K_+} e^{-nf(z)}g(z)\mathrm{d}z$. On K_1,

$$\operatorname{Re}\left(f(z) - f_0\right) = \frac{c_p}{2}\int \ln\left|1 + i\frac{\operatorname{Im} z}{z_0(h) - \lambda}\right| \mathrm{d}\mathcal{F}_p(\lambda)$$

is an increasing function of $\operatorname{Im}(z)$. Hence, on $K_1 \backslash [z_0, z_1]$,

$$\operatorname{Re}\left(f(z) - f_0\right) > \operatorname{Re}\tau \geq \tau_1.$$

Therefore,

$$\begin{aligned}\left|\oint_{K_1\backslash[z_0,z_1]} e^{-nf(z)}g(z)\mathrm{d}z\right| &\leq e^{-nf_0}e^{-n\tau_1}\oint_{K_1\backslash[z_0,z_1]}|g(z)\mathrm{d}z| \\ &= e^{-nf_0}e^{-n\tau_1}\left|3z_0(h)\right| O_p(1) \\ &= e^{-nf_0}e^{-n\tau_1}h^{-1}O_p(1). \qquad \text{(A40)}\end{aligned}$$

For the horizontal part K_2 of K_+, consider first the case when $g(z) = \exp\left\{-\frac{1}{2}\Delta_p(z)\right\}$. We have

$$\begin{aligned}\left|\oint_{K_2} e^{-nf(z)}g(z)dz\right| &= \left|\oint_{K_2} e^{\frac{n}{2}\frac{h}{1+h}z}\prod_{j=1}^{p}\left(z-\lambda_j\right)^{-\frac{1}{2}}\mathrm{d}z\right| \leq e^{-\frac{p}{2}\ln(3z_0(h))}\oint_{K_2}\left|e^{\frac{n}{2}\frac{h}{1+h}z}\mathrm{d}z\right| \\ &= \left(\frac{n}{2}\frac{h}{1+h}\right)^{-1}e^{-\frac{n}{2}\left(c_p\ln(3z_0(h)) - \frac{h}{1+h}z_0(h)\right)}. \qquad \text{(A41)}\end{aligned}$$

But $\frac{h}{1+h} z_0(h) \equiv h + c_p$, so that

$$c_p \ln\left(3 z_0\left(h\right)\right) - \frac{h}{1+h} z_0\left(h\right) > c_p \ln\left(z_0\left(h\right)\right) - h > 2 f_0 + c_p.$$

Combining such a lower bound with (A41), we get

$$\left| \oint\limits_{K_2} e^{-n f(z)} g(z) \mathrm{d}z \right| = e^{-n f_0} h^{-1} O\left(e^{-\frac{n}{2} c_p}\right) = e^{-n f_0} h^{-1} O_p\left(e^{-\frac{n}{4} c}\right), \tag{A42}$$

where $O_p\left(e^{-\frac{n}{4} c}\right)$ does not depend on h.

For the case when

$$g(z) = \exp\left\{ -\frac{np - p + 2}{2} \ln\left(1 - \frac{h}{1+h} \frac{z}{S}\right) - \frac{n}{2} \frac{hz}{1+h} - \frac{\Delta_p\left(z\right)}{2} \right\},$$

we have

$$\begin{aligned} \left| \oint\limits_{K_2} e^{-n f(z)} g(z) \mathrm{d}z \right| &= \left| \oint\limits_{K_2} \left(1 - \frac{h}{1+h} \frac{z}{S}\right)^{-\frac{np-p+2}{2}} \prod_{j=1}^{p} \left(z - \lambda_j\right)^{-\frac{1}{2}} \mathrm{d}z \right| \\ &\leq e^{-\frac{p}{2} \ln(3 z_0(h))} \oint\limits_{K_2} \left| \left(1 - \frac{h}{1+h} \frac{z}{S}\right)^{-\frac{np-p+2}{2}} \mathrm{d}z \right|. \end{aligned}$$

Further,

$$\begin{aligned} \oint\limits_{K_2} \left| \left(1 - \frac{h}{1+h} \frac{z}{S}\right)^{-\frac{np-p+2}{2}} \mathrm{d}z \right| &\leq \int_{-\infty}^{z_0(h)} \left(1 - \frac{h}{1+h} \frac{x}{S}\right)^{-\frac{np-p+2}{2}} \mathrm{d}x \\ &= \frac{2S}{np - p} \frac{1+h}{h} \left(1 - \frac{h}{1+h} \frac{z_0\left(h\right)}{S}\right)^{-\frac{np}{2} + \frac{p}{2}}. \end{aligned}$$

Hence, we can write

$$\left| \oint_{K_2} e^{-nf(z)} g(z) \mathrm{d}z \right| \leq \frac{2S}{np-p} \frac{1+h}{h} e^{-\frac{np-p}{2} \ln\left(1 - \frac{h}{1+h} \frac{z_0(h)}{S}\right) - \frac{p}{2} \ln(3z_0(h))}. \tag{A43}$$

Now, for any real x such that $0 < x < 1$, we have $\ln(1-x) > -\frac{x}{1-x}$. Hence,

$$-\frac{np-p}{2} \ln\left(1 - \frac{h}{1+h} \frac{z_0(h)}{S}\right) < (p - c_p)\left(S - \frac{hz_0(h)}{1+h}\right)^{-1} \frac{n}{2} \frac{hz_0(h)}{1+h}.$$

But

$$(p - c_p)\left(S - \frac{hz_0(h)}{1+h}\right)^{-1} = 1 + O_p\left(n^{-1}\right).$$

The $O_p(n^{-1})$ quantity here is uniform over $h \in (0, \bar{h}]$ in view of the facts that $S - p = O_p(1)$ by Theorem 1.1 of Bai and Silverstein (2004),

$$\left| \frac{hz_0(h)}{1+h} \right| = |h + c_p| \leq \left|\bar{h} + c_p\right|$$

for all $h \in (0, \bar{h}]$, and n and p diverge to infinity at the same rate. Therefore, (A43) implies

$$\left| \oint_{K_2} e^{-nf(z)} g(z) \mathrm{d}z \right| = \left(\frac{n}{2} \frac{h}{1+h}\right)^{-1} e^{-\frac{n}{2}\left(c_p \ln(3z_0(h)) - \frac{h}{1+h} z_0(h)\right)} O_p(1), \tag{A44}$$

which, similarly to (A41), implies (A42).

Combining (A39), (A40), and (A42), we get

$$\oint_{K_+} e^{-nf(z)} g(z) \mathrm{d}z = e^{-nf_0} \left(\sum_{s=0}^{k-1} \Gamma\left(\frac{s+1}{\mu}\right) \frac{a_s}{n^{(s+1)/2}} + \frac{O_p(1)}{hn^{(k+1)/2}} \right). \tag{A45}$$

Finally, note that

$$\oint_K e^{-nf(z)}g(z)\mathrm{d}z = \oint_{K_+} e^{-nf(z)}g(z)\mathrm{d}z - \oint_{\tilde{K}_-} e^{-nf(z)}g(z)\mathrm{d}z,$$

where $\tilde{K}_-$ is a contour that coincides with K_- but has the opposite orientation. As explained in Olver (1997, pp.121-122), a_s with odd s in the asymptotic expansion for $\oint_{\tilde{K}_-} e^{-nf(z)}g(z)\mathrm{d}z$ coincides with the corresponding a_s in the asymptotic expansion for $\oint_{K_+} e^{-nf(z)}g(z)\mathrm{d}z$. However, a_s with even s in the two expansions differ by the sign. Therefore, coefficients a_s with odd s cancel out, but those with even s double in the difference of the two expansions. Setting $k = 2m$, we have

$$\oint_K e^{-nf(z)}g(z)\mathrm{d}z = 2e^{-nf_0}\left(\sum_{s=0}^{m-1}\Gamma\left(s+\frac{1}{2}\right)\frac{a_{2s}}{n^{s+1/2}} + \frac{O_p(1)}{hn^{m+1/2}}\right),$$

which establishes Lemma 5.

C Proof of Lemma 6

Fix $0 < \varepsilon < \left(\sqrt{c/\tilde{h}} - \sqrt{\tilde{h}}\right)^2$, and consider the event E_1 that holds if and only if (A4) and (A5) hold,

$$z_0(\tilde{h}) - b_p > \varepsilon$$

and

$$\min_{h\in[\tilde{h},\infty)}\left(\frac{1+h}{h}S - z_0(\tilde{h})\right) > \varepsilon.$$

The fact that, with probability approaching 1, for all $h \in \left[\tilde{h}, \infty\right)$, the integrals in (2.9) and (2.10) do not change as $\mathcal{K}$ is deformed into $K(\tilde{h})$ can be established along the same lines as in the proof of Lemma 4 by replacing event E with event E_1.

Similarly, an equivalent, for $h \geq \tilde{h}$, of Lemma 2A, is easily proved along the same steps. Hence, since $\mathrm{Re}\left(f(z) - f\left(z_0\left(\tilde{h}\right)\right)\right)$ is an increasing function of $\mathrm{Im}\, z$

on $K_1\left(\tilde{h}\right)$,

$$\left|\oint_{K_1\left(\tilde{h}\right)} e^{-nf(z)}g(z)\mathrm{d}z\right| \leq e^{-nf\left(z_0\left(\tilde{h}\right)\right)}\oint_{K_1\left(\tilde{h}\right)}|g(z)\mathrm{d}z|$$
$$= e^{-nf\left(z_0\left(\tilde{h}\right)\right)}O_p\left(1\right). \tag{A46}$$

Further, as in (A41) and (A44), we have

$$\left|\oint_{K_2\left(\tilde{h}\right)} e^{-nf(z)}g(z)\mathrm{d}z\right| = \left(\frac{n}{2}\frac{h}{1+h}\right)^{-1} e^{-\frac{n}{2}\left(c_p\ln\left(3z_0\left(\tilde{h}\right)\right)-\frac{h}{1+h}z_0\left(\tilde{h}\right)\right)}O_p(1)$$
$$= e^{-nf\left(z_0\left(\tilde{h}\right)\right)}O_p(1). \tag{A47}$$

Combining (A46) and (A47), we get

$$\left|\oint_{K_+\left(\tilde{h}\right)} e^{-nf(z)}g(z)\mathrm{d}z\right| = e^{-nf\left(z_0\left(\tilde{h}\right)\right)}O_p(1).$$

Similarly,

$$\left|\oint_{K_-\left(\tilde{h}\right)} e^{-nf(z)}g(z)\mathrm{d}z\right| = e^{-nf\left(z_0\left(\tilde{h}\right)\right)}O_p(1).$$

Lemma 6 follows from the latter two equalities.

D Proof of Lemma 11

Consider

$$I\left(h\right) \equiv \int_{a_p}^{b_p} \ln\left(z_0(h)-\lambda\right)\psi_p\left(\lambda\right)\mathrm{d}\lambda,$$

where $\psi_p(\lambda)$ is defined in (3.2). Making the substitution $\lambda = 1 + c_p - 2\sqrt{c_p}\cos\theta$ and replacing $z_0(h)$ by the right-hand side of (3.7), we get

$$
\begin{aligned}
I(h) &= \frac{2}{\pi}\int_0^{\pi} \frac{\ln\left(h + h^{-1}c_p + 2\sqrt{c_p}\cos\theta\right)\sin^2\theta}{1 + c_p - 2\sqrt{c_p}\cos\theta}\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{\ln\left|\sqrt{c_p/h} + \sqrt{h}e^{i\theta}\right|^2 \sin^2\theta}{1 + c_p - 2\sqrt{c_p}\cos\theta}\mathrm{d}\theta.
\end{aligned}
$$

Further, changing the variable of integration from θ to $z = e^{i\theta}$, we get

$$
I(h) = \frac{-1}{2\pi i}\oint\limits_{|z|=1} \frac{\ln\left[\left(\sqrt{c_p/h} + \sqrt{h}z\right)\left(\sqrt{c_p/h} + \sqrt{h}z^{-1}\right)\right]\left(z - z^{-1}\right)^2}{2\left(\sqrt{c_p} - z\right)\left(z\sqrt{c_p} - 1\right)}\mathrm{d}z. \quad \text{(A48)}
$$

Representing the logarithm of a product as a sum of logarithms, splitting the integral into two parts corresponding to the summands, and changing the variable of integration in the second integral from z to z^{-1}, we get

$$
I(h) = \frac{-1}{2\pi i}\oint\limits_{|z|=1} \frac{\ln\left(\sqrt{c_p/h} + \sqrt{h}z\right)\left(z - z^{-1}\right)^2}{\left(\sqrt{c_p} - z\right)\left(z\sqrt{c_p} - 1\right)}\mathrm{d}z. \quad \text{(A49)}
$$

If $h < \sqrt{c_p}$, then function $\ln\left(\sqrt{c_p/h} + \sqrt{h}z\right)$ is analytic inside the ball $|z| \le 1$. Therefore, if $c_p < 1$, the integrand in (A49) has singularities only at zero and $\sqrt{c_p}$. If $c_p > 1$, the singularities are at zero and $\sqrt{1/c_p}$. If $c_p = 1$, the only singularity is at zero. Computing the residues of the integrand at the singularity points and using Cauchy's theorem, we get

$$
I(h) = \begin{cases} \frac{c_p - 1}{c_p}\ln(1 + h) + \frac{h}{c_p} + \ln\frac{c_p}{h} & \text{if } h < \sqrt{c_p} \text{ and } c_p < 1 \\ \frac{1 - c_p}{c_p}\ln\left(1 + \frac{h}{c_p}\right) + \frac{h}{c_p} + \frac{1}{c_p}\ln\frac{c_p}{h} & \text{if } h < \sqrt{c_p} \text{ and } c_p \ge 1 \end{cases}. \quad \text{(A50)}
$$

If $h > \sqrt{c_p}$, then represent the logarithm in (A48) in the form

$$\ln\left[\left(z\sqrt{c_p/h}+\sqrt{h}\right)\left(z^{-1}\sqrt{c_p/h}+\sqrt{h}\right)\right],$$

and proceed as above to get

$$I(h) = \begin{cases} \frac{c_p-1}{c_p}\ln(h+c_p)+\frac{1}{h}+\frac{1}{c_p}\ln h & \text{if } h>\sqrt{c_p} \text{ and } c_p<1 \\ \frac{1-c_p}{c_p}\ln(1+h)+\frac{1}{h}+\ln h & \text{if } h>\sqrt{c_p} \text{ and } c_p\geq 1 \end{cases}. \tag{A51}$$

Now, it is straightforward to verify that Lemma 11 follows from (A50), (A51), and from the facts that

$$f_0 = -\frac{1}{2}\left(\frac{h}{1+h}z_0(h) - c_p\int \ln\left(z_0(h)-\lambda\right)\mathrm{d}\mathcal{F}_p(\lambda)\right),$$

that $\frac{h}{1+h}z_0(h) = h + c_p$, and that the Marchenko-Pastur distribution has mass $\max\left(0, 1-c_p^{-1}\right)$ at zero.

E Proof of Lemma 12

Let $z_{0j} = \lim z_0(h_j)$ as $n, p \to \infty$. As follows from Bai and Silverstein (2004, p. 563),

$$\Delta_p\left(z_0(h_j)\right) = \oint_{\mathcal{C}} \ln\left(z_0(h_j)-z\right)M_p(z)\,\mathrm{d}z$$

and

$$\Delta_p\left(z_{0j}\right) = \oint_{\mathcal{C}} \ln\left(z_{0j}-z\right)M_p(z)\,\mathrm{d}z,$$

where $\mathcal{C}$ is a fixed contour of integration encircling the support of the Marchenko-Pastur distribution, but not $z_0(h_j)$ and z_{0j}, and

$$M_p(z) = \sum_{j=1}^{p} (\lambda_j - z)^{-1} - p \int (x-z)^{-1} \mathrm{d}\mathcal{F}_p(x).$$

Therefore,

$$\Delta_p(z_0(h_j)) - \Delta_p(z_{0j}) = \oint_{\mathcal{C}} \ln\left(\frac{z_0(h_j) - z}{z_{0j} - z}\right) M_p(z)\,\mathrm{d}z.$$

Further, as can be shown using arguments similar to those given on p.563 of Bai and Silverstein (2004),

$$\oint_{\mathcal{C}} \ln\left(\frac{z_0(h_j) - z}{z_{0j} - z}\right) M_p(z)\,\mathrm{d}z = \oint_{\mathcal{C}} \ln\left(\frac{z_0(h_j) - z}{z_{0j} - z}\right) \hat{M}_p(z)\,\mathrm{d}z + o_p(1),$$

where $\left\{\hat{M}_p(z), p = 1, 2, ...\right\}$ is a tight sequence of random continuous functions on $\mathcal{C}$. On the other hand, as $n, p \to \infty$,

$$\ln\left(\frac{z_0(h_j) - z}{z_{0j} - z}\right) \to 0$$

uniformly over $\mathcal{C}$. Hence,

$$\oint_{\mathcal{C}} \ln\left(\frac{z_0(h_j) - z}{z_{0j} - z}\right) \hat{M}_p(z)\,\mathrm{d}z = o_p(1),$$

and thus

$$\Delta_p(z_0(h_j)) - \Delta_p(z_{0j}) = o_p(1).$$

The latter equality implies that the vectors $(S - p, \Delta_p(z_0(h_1)), ..., \Delta_p(z_0(h_r)))$ and $(S - p, \Delta_p(z_{01}), ..., \Delta_p(z_{0r}))$ simultaneously diverge, or converge, in distribution, to the same limit.

Now, according to Theorem 1.1 of Bai and Silverstein (2004), $(S-p, \Delta_p(z_{01}), ...,$ $\Delta_p(z_{0r}))$ converges in distribution to a Gaussian vector $(\eta, \xi_1, ..., \xi_r)$ with means $\mathrm{E}\eta = 0$,

$$\mathrm{E}\xi_j = -\frac{1}{2\pi i}\oint \ln(z_{0j}-z)\frac{c\underline{m}^3(z)}{(1+\underline{m}(z))^3 - c\underline{m}^2(z)(1+\underline{m}(z))}\mathrm{d}z, \tag{A52}$$

covariances

$$\mathrm{Cov}(\xi_j, \xi_k) = -\frac{1}{2\pi^2}\oint\oint \frac{\ln(z_{0j}-z_1)\ln(z_{0k}-z_2)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}\frac{\mathrm{d}\underline{m}(z_1)}{\mathrm{d}z_1}\frac{\mathrm{d}\underline{m}(z_2)}{\mathrm{d}z_2}\mathrm{d}z_1\mathrm{d}z_2, \tag{A53}$$

$$\mathrm{Cov}(\xi_j, \eta) = -\frac{1}{2\pi^2}\oint\oint \frac{z_2\ln(z_{0j}-z_1)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}\frac{\mathrm{d}\underline{m}(z_1)}{\mathrm{d}z_1}\frac{\mathrm{d}\underline{m}(z_2)}{\mathrm{d}z_2}\mathrm{d}z_1\mathrm{d}z_2, \tag{A54}$$

and variance

$$\mathrm{Var}(\eta) = -\frac{1}{2\pi^2}\oint\oint \frac{z_1 z_2}{(\underline{m}(z_1)-\underline{m}(z_2))^2}\frac{\mathrm{d}\underline{m}(z_1)}{\mathrm{d}z_1}\frac{\mathrm{d}\underline{m}(z_2)}{\mathrm{d}z_2}\mathrm{d}z_1\mathrm{d}z_2, \tag{A55}$$

where

$$\underline{m}(z) = -(1-c)z^{-1} + cm(z)$$

with $m(z)$ given by (3.6) where c_p is replaced by c. That is,

$$\underline{m}(z) = \frac{-z+c-1+\sqrt{(z-c-1)^2-4c}}{2z}, \tag{A56}$$

where the branch of the square root is chosen so that the real and the imaginary parts of $\sqrt{(z-c-1)^2-4c}$ have the same signs as the real and the imaginary parts of $z-c-1$, respectively. The contours of integration in (A52)-(A55) are closed, oriented counterclockwise, enclose zero and the support of the Marchenko-Pastur distribution with parameter c, and do not enclose z_{0j} and z_{0k}.

The expressions for $\mathrm{E}\xi_j$, $\mathrm{Cov}\left(\xi_j, \xi_k\right)$, $\mathrm{Cov}\left(\xi_j, \eta\right)$ and $\mathrm{Var}\left(\eta\right)$ can be simplified along the same steps as in Bai and Silverstein (2004, pp.596-599). Exactly following the derivation of their formula 5.13, we get

$$\mathrm{E}\xi_j = \frac{\ln\left(\left(z_{0j}-a\right)\left(z_{0j}-b\right)\right)}{4} - \frac{1}{2\pi}\int_a^b \frac{\ln\left(z_{0j}-x\right)}{\sqrt{4c-\left(x-c-1\right)^2}}\mathrm{d}x, \tag{A57}$$

where $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$.

Making substitution $x = 1+c-2\sqrt{c}\cos\theta$ as in the above proof of Lemma 11, and using similar steps to those used in that proof, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_a^b \frac{\ln\left(z_{0j}-x\right)}{\sqrt{4c-\left(x-c-1\right)^2}}\mathrm{d}x = \frac{1}{2\pi}\int_0^\pi \ln\left|\sqrt{c/h_j}+\sqrt{h_j}e^{i\theta}\right|^2 \mathrm{d}\theta \\
= \; &\frac{1}{2\pi i}\int_{|z|=1} z^{-1}\ln\left(\sqrt{c/h_j}+\sqrt{h_j}z\right)\mathrm{d}z = \ln\sqrt{c/h_j}.
\end{aligned}$$

Using this in (A57), we get

$$\begin{aligned}
\mathrm{E}\xi_j &= \frac{1}{4}\ln\left(\left(\sqrt{c/h_j}+\sqrt{h_j}\right)^2\left(\sqrt{c/h_j}-\sqrt{h_j}\right)^2\right) - \ln\sqrt{c/h_j} \\
&= \frac{1}{2}\ln\left(1-c^{-1}h_j^2\right).
\end{aligned}$$

For the covariance $\mathrm{Cov}\left(\xi_j, \xi_k\right)$ we use formula 1.16 of Bai and Silverstein (2004), to get

$$\mathrm{Cov}\left(\xi_j, \xi_k\right) = -\frac{1}{2\pi^2}\oint\oint \frac{\ln\left(z_{0j}-z\left(m_1\right)\right)\ln\left(z_{0k}-z\left(m_2\right)\right)}{\left(m_1-m_2\right)^2}\mathrm{d}m_1\mathrm{d}m_2, \tag{A58}$$

where

$$z\left(m\right) = -\frac{1}{m}+\frac{c}{1+m}. \tag{A59}$$

Note that substituting $\underline{m}\left(z\right)$ as defined in (A56) in the right-hand side of (A59), we get z, so (A59) describes a function inverse to $\underline{m}(z)$.

Let us split the double integral in (A58) into three parts according to the decomposition

$$\operatorname{Cov}\left(\xi_j, \xi_k\right) = \frac{1}{2}\left[\operatorname{Var}(\xi_j) + \operatorname{Var}\left(\xi_k\right) - \operatorname{Var}\left(\xi_j - \xi_k\right)\right],$$

where

$$\operatorname{Var}(\xi_j) = -\frac{1}{2\pi^2}\oint\oint \frac{\ln\left(z_{0j} - z\left(m_1\right)\right)\ln\left(z_{0j} - z\left(m_2\right)\right)}{\left(m_1 - m_2\right)^2}\mathrm{d}m_1\mathrm{d}m_2, \tag{A60}$$

$$\operatorname{Var}(\xi_k) = -\frac{1}{2\pi^2}\oint\oint \frac{\ln\left(z_{0k} - z\left(m_1\right)\right)\ln\left(z_{0k} - z\left(m_2\right)\right)}{\left(m_1 - m_2\right)^2}\mathrm{d}m_1\mathrm{d}m_2, \tag{A61}$$

and

$$\operatorname{Var}\left(\xi_j - \xi_k\right) = -\frac{1}{2\pi^2}\oint\oint \frac{\ln\left(\frac{z_{0j} - z(m_1)}{z_{0k} - z(m_1)}\right)\ln\left(\frac{z_{0j} - z(m_2)}{z_{0k} - z(m_2)}\right)}{\left(m_1 - m_2\right)^2}\mathrm{d}m_1\mathrm{d}m_2. \tag{A62}$$

The contours of integration over m_1 and m_2 in (A60-A62) are obtained from the contours of integration over z_1 and z_2 in (A53) by transformation $\underline{m}(z)$. Recall that by assumption the contours over z_1 and z_2 intersect the real line to the left of zero and in between the upper boundary of the support of the Marchenko-Pastur distribution, $(1+\sqrt{c})^2$, and $\min\{z_{0j}, z_{0k}\}$. Therefore, as can be shown using the definition (A56) of $\underline{m}(z)$, the m_1-contour and m_2-contour are clockwise oriented and intersect the real line in between $-(1+\sqrt{c})^{-1}$ and $\min\{\underline{m}(z_{0j}), \underline{m}(z_{0k})\} = -\max\left\{h_j\left(h_j + c\right)^{-1}, h_k\left(h_k + c\right)^{-1}\right\}$ and to the right of zero. In particular, both contours enclose 0, $-h_j\left(h_j + c\right)^{-1}$ and $-h_k\left(h_k + c\right)^{-1}$, but not -1, $-\left(1 + h_j\right)^{-1}$ and $-\left(1 + h_k\right)^{-1}$.

Without loss of generality, assume that the m_2-contour encloses the m_1-contour.

For fixed m_2, we have

$$\oint \frac{\ln\left(z_{0j}-z\left(m_1\right)\right)}{\left(m_1-m_2\right)^2}\mathrm{d}m_1 = \oint \frac{-\frac{\mathrm{d}}{\mathrm{d}m_1}z\left(m_1\right)}{\left(z_{0j}-z\left(m_1\right)\right)\left(m_1-m_2\right)}\mathrm{d}m_1$$
$$= -\oint \frac{1/m_1^2 - c/\left(m_1+1\right)^2}{\left(z_{0j}+1/m_1-c/\left(m_1+1\right)\right)\left(m_1-m_2\right)}\mathrm{d}m_1, \tag{A63}$$

where the first equality follows from integration by parts and the fact that $\ln\left(z_{0j}-z\left(m_1\right)\right)$ is a single-valued function along the m_1-contour. To see this, note that

$$\ln\left(z_{0j}-z\left(m_1\right)\right) = \ln \frac{z_{0j}\left(m_1+\left(1+h_j\right)^{-1}\right)}{m_1+1} + \ln\left(m_1+\frac{h_j}{h_j+c}\right) - \ln m_1.$$

The first of the latter three terms is a single-valued function along the m_1-contour because it does not have singularities inside the contour. The second and the third terms are not single-valued, but their changes after passing once along the contour cancel each other.

Now, the integrand in (A63) has first-order poles at 0, $-h_j\left(h_j+c\right)^{-1}$, m_2, -1 and at $-\left(1+h_j\right)^{-1}$ and no other singularities. As explained above, only the first two of the above poles are enclosed by the m_1-contour. Using Cauchy's residue theorem, we get

$$\oint \frac{\ln\left(z_{0j}-z\left(m_1\right)\right)}{\left(m_1-m_2\right)^2}\mathrm{d}m_1 = 2\pi i\left(-\frac{1}{m_2}+\frac{1}{m_2+h_j\left(h_j+c\right)^{-1}}\right). \tag{A64}$$

Let us denote $-h_j(h_j+c)^{-1}$ as θ_j. Using (A64) and (A60), we get

$$\begin{aligned}
\mathrm{Var}(\xi_j) &= \frac{2\pi i}{2\pi^2}\oint \ln\left(z_{0j}-z\left(m_2\right)\right)\left(\frac{1}{m_2}-\frac{1}{m_2-\theta_j}\right)\mathrm{d}m_2 \\
&= \frac{2\pi i}{2\pi^2}\oint \ln\left(1-z_{0j}^{-1}z\left(m_2\right)\right)\left(\frac{1}{m_2}-\frac{1}{m_2-\theta_j}\right)\mathrm{d}m_2 \\
&= \frac{2\pi i}{2\pi^2}\oint \ln\left(\frac{m_2+\left(1+h_j\right)^{-1}}{m_2+1}\right)\left(\frac{1}{m_2}-\frac{1}{m_2-\theta_j}\right)\mathrm{d}m_2 \\
&\quad -\frac{2\pi i}{2\pi^2}\oint \ln\left(\frac{m_2-\theta_j}{m_2}\right)\left(\frac{1}{m_2}-\frac{1}{m_2-\theta_j}\right)\mathrm{d}m_2.
\end{aligned}$$

By Cauchy's residue theorem, the first term in the latter expression is equal to $-2\ln\left(1-c^{-1}h_j^2\right)$. The second term equals zero because the integrand has anti-derivative $-\frac{1}{2}\left[\ln\left(\frac{m_2-\theta_j}{m_2}\right)\right]^2$ which is a single-valued function along the contour.

Similarly, we can show that

$$\mathrm{Var}(\xi_k) = -2\ln\left(1-c^{-1}h_k^2\right)$$

and that

$$\mathrm{Var}(\xi_j-\xi_k) = 2\ln\frac{\left(1-c^{-1}h_jh_k\right)^2}{\left(1-c^{-1}h_j^2\right)\left(1-c^{-1}h_k^2\right)}.$$

Combining these results, we get

$$\begin{aligned}
\mathrm{Cov}\left(\xi_j,\xi_k\right) &= -\ln\left(1-c^{-1}h_j^2\right)-\ln\left(1-c^{-1}h_k^2\right) \\
&\quad -\ln\frac{\left(1-c^{-1}h_jh_k\right)^2}{\left(1-c^{-1}h_j^2\right)\left(1-c^{-1}h_k^2\right)} \\
&= -2\ln\left(1-c^{-1}h_jh_k\right).
\end{aligned}$$

For $\mathrm{Cov}\left(\xi_j,\eta\right)$ and $\mathrm{Var}\left(\eta\right)$, an analysis similar to but simpler than that leading to the above formula for $\mathrm{Cov}\left(\xi_j,\xi_k\right)$ shows that $\mathrm{Cov}\left(\xi_j,\eta\right)=-2h_j$ and $\mathrm{Var}\left(\eta\right)=2c$.

F Proof of Lemma 13

First, note that

$$CLR = \sum_{j=1}^{p} q(\lambda_j) - p \int q(x) \mathrm{d}\mathcal{F}_p(x),$$

where $q(x) = x - \ln x - 1$. Also, recall that, as shown in the proof of Lemma 12,

$$\Delta_p(z_0(h)) = \Delta_p(z_0) + o_p(1),$$

where $z_0 = \lim z_0(h)$ and

$$\Delta_p(z_0) = \sum_{j=1}^{p} s(\lambda_j) - p \int s(x) \mathrm{d}\mathcal{F}_p(x),$$

with $s(x) = \ln(z_0 - x)$. Therefore, in view of Theorem 1.1 of Bai and Silverstein (2004), CLR and $\Delta_p(z_0(h))$ jointly converge in distribution to a Gaussian vector with covariance

$$R = -\frac{1}{2\pi^2} \oint \oint \frac{s(z_1) q(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} \frac{\mathrm{d}\underline{m}(z_1)}{\mathrm{d}z_1} \frac{\mathrm{d}\underline{m}(z_2)}{\mathrm{d}z_2} \mathrm{d}z_1 \mathrm{d}z_2. \tag{A65}$$

Here $\underline{m}(z)$ is as defined in (A56), and the contours of integration are closed, oriented counterclockwise, enclose the support of the Marchenko-Pastur distribution with parameter $c < 1$, and do not enclose z_0. Further, we will choose such contours so that the z_1-contour encloses 0, but the z_2-contour does not.

Using Formula 1.16 of Bai and Silverstein (2004) we can simplify (A65) to get

$$R = -\frac{1}{2\pi^2} \oint \oint \frac{\ln(z_0 - z(m_1))(z(m_2) - \ln z(m_2) - 1)}{(m_1 - m_2)^2} \mathrm{d}m_1 \mathrm{d}m_2,$$

where

$$z(m) = -\frac{1}{m} + \frac{c}{1+m}$$

and the contours of integration over m_1 and m_2 are obtained from the contours of integration over z_1 and z_2 in (A65) by the transformation $\underline{m}(z)$. In particular, m_1-contour is oriented clockwise and encloses $-\frac{h}{h+c}$ and 0 but not -1 and $-\frac{1}{1+h}$, whereas m_2-contour is oriented counterclockwise and encloses $\frac{1}{c-1}$ and -1 but not $-\frac{h}{h+c}$ and 0.

Using (A64), we can write $R = R_1 + R_2 + R_3$, where

$$\begin{aligned} R_1 &= -\frac{i}{\pi} \oint \left(-\frac{1}{m_2} + \frac{1}{m_2 + h_j (h_j + c)^{-1}} \right) z(m_2)\, \mathrm{d}m_2, \\ R_2 &= \frac{i}{\pi} \oint \left(-\frac{1}{m_2} + \frac{1}{m_2 + h_j (h_j + c)^{-1}} \right) \ln z(m_2)\, \mathrm{d}m_2, \text{ and} \\ R_3 &= \frac{i}{\pi} \oint \left(-\frac{1}{m_2} + \frac{1}{m_2 + h_j (h_j + c)^{-1}} \right) \mathrm{d}m_2. \end{aligned}$$

Since $-\frac{1}{m_2} + \frac{1}{m_2 + h_j (h_j+c)^{-1}}$ is analytic in the area enclosed by the m_2-contour, $R_3 = 0$. Further, using Cauchy's theorem and the fact that

$$z(m_2) = -\frac{1}{m_2} + \frac{c}{1+m_2},$$

we get $R_1 = -2h$. Finally, integrating R_2 by parts, and using the fact that $\ln z(m_2)$ is a single-valued function on the m_2-contour, we get

$$R_2 = -\frac{i}{\pi} \oint \frac{\frac{1}{m_2^2} - \frac{c}{(1+m_2)^2}}{-\frac{1}{m_2} + \frac{c}{m_2+1}} \left(-\ln m_2 + \ln \left(m_2 + h_j (h_j + c)^{-1} \right) \right) \mathrm{d}m_2.$$

The integrand in the above integral has only two singularities in the area enclosed by the m_2-contour: a pole at $\frac{1}{c-1}$ and a pole at -1. Therefore, by Cauchy's residue theorem, we get $R_2 = 2\ln(1+h)$. To summarize, $R = R_1 + R_2 + R_3 = -2h + 2\ln(1+h)$, which establishes Lemma 13.

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