SUPPLEMENTAL ARTICLE: BAYESIAN SHRINKAGE METHODS FOR PARTIALLY OBSERVED DATA WITH MANY PREDICTORS

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APPENDIX A: ENUMERATION OF GIBBS STEPS

The complete-data log-likelihood, given in expression (4) in the manuscript, is

$$\begin{aligned}\ell_C = \ln[\boldsymbol{U}^{\mathrm{obs}}, \boldsymbol{U}^{\mathrm{mis}}|\boldsymbol{\phi}] &= \ln[\boldsymbol{y}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \beta_0, \boldsymbol{\beta}, \sigma^2] + \ln[\boldsymbol{w}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \psi, \nu, \tau^2] + \ln[\boldsymbol{x}_{\mathrm{A}}|\boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}] \\ &\quad + \ln[\boldsymbol{y}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \beta_0, \boldsymbol{\beta}, \sigma^2] + \ln[\boldsymbol{w}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \psi, \nu, \tau^2] + \ln[\boldsymbol{x}_{\mathrm{B}}|\boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}].\end{aligned}$$

For a choice of prior, $[\boldsymbol{\phi}]$, the partial conditional distributions implied by the product of the likelihood and the prior, $[\boldsymbol{U}^{\mathrm{obs}}, \boldsymbol{U}^{\mathrm{mis}}|\boldsymbol{\phi}] \times [\boldsymbol{\phi}]$, will yield the Gibbs steps. These are discussed in the order presented in the manuscript.

<u>VANILLA</u> The prior is given in expression (6) in the manuscript:

$$[\boldsymbol{\phi}] = [\beta_0, \boldsymbol{\beta}, \sigma^2, \psi, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}] \propto (\sigma^2\tau^2)^{-1}|\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}|^{(2p-1)/2} \exp\left\{-\tfrac{2p-1}{2}\mathrm{Tr}\,\left(\mathrm{diag}(\hat{\mathrm{Var}}[\boldsymbol{x}_{\mathrm{A}}])\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right)\right\},$$

where $\mathrm{diag}(\hat{\mathrm{Var}}[\boldsymbol{x}_{\mathrm{A}}])$ is the diagonal part of the empirical covariance of $\boldsymbol{x}_{\mathrm{A}}$. This is a Jeffreys prior on each parameter except $\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}$, and $\boldsymbol{\eta}$ (the vector of hyperparameters) is specified. Each Gibbs step is as follows:

$$
\begin{aligned}
&\left[\boldsymbol{x}_i | y_i, \boldsymbol{w}_i, \boldsymbol{\beta}, \beta_0, \sigma^2, \psi, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \left[y_i | \boldsymbol{x}_i, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{w}_i | \boldsymbol{x}_i, \psi, \nu, \tau^2\right] \left[\boldsymbol{x}_i | \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta_0 - \boldsymbol{x}_i^\top \boldsymbol{\beta})^2\right\} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{w}_i - \psi \mathbf{1}_p - \nu \boldsymbol{x}_i)^\top (\boldsymbol{w}_i - \psi \mathbf{1}_p - \nu \boldsymbol{x}_i)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}(\boldsymbol{x}_i - \boldsymbol{\mu}_{\boldsymbol{X}})^\top \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}_{\boldsymbol{X}})\right\} \\
&\quad \propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_i^\top [\boldsymbol{\beta}\boldsymbol{\beta}^\top/\sigma^2] \boldsymbol{x}_i - 2[(y_i - \beta_0)/\sigma^2] \boldsymbol{x}_i^\top \boldsymbol{\beta}\right.\right. \\
&\qquad\qquad \left.\left. +\ [\nu^2/\tau^2] \boldsymbol{x}_i^\top \boldsymbol{x}_i - 2[\nu/\tau^2] \boldsymbol{x}_i^\top [\boldsymbol{w}_i - \psi \mathbf{1}_p] + \boldsymbol{x}_i^\top \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} \boldsymbol{x}_i - 2\boldsymbol{x}_i^\top \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} \boldsymbol{\mu}_{\boldsymbol{X}}\right)\right\} \\
&\quad = N_p\left\{\Gamma\left([(y_i - \beta_0)/\sigma^2]\boldsymbol{\beta} + [\nu/\tau^2][\boldsymbol{w}_i - \psi \mathbf{1}_p] + \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} \boldsymbol{\mu}_{\boldsymbol{X}}\right), \Gamma\right\}, \\
&\Gamma = \left(\boldsymbol{\beta}\boldsymbol{\beta}^\top/\sigma^2 + \nu^2/\tau^2 + \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\boldsymbol{\beta} | \boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \beta_0, \sigma^2, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \left[\boldsymbol{y}_{\mathrm{A}} | \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{y}_{\mathrm{B}} | \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta}) - \frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\right\} \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}^\top \left[\boldsymbol{x}_{\mathrm{A}}^\top \boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top \boldsymbol{x}_{\mathrm{B}}\right] \boldsymbol{\beta} - 2\boldsymbol{\beta}^\top \left[\boldsymbol{x}_{\mathrm{A}}^\top (\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}}) + \boldsymbol{x}_{\mathrm{B}}^\top (\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}})\right]\right)\right\} \\
&\quad = N_p\left\{(\boldsymbol{x}_{\mathrm{A}}^\top \boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top \boldsymbol{x}_{\mathrm{B}})^{-1} (\boldsymbol{x}_{\mathrm{A}}^\top [\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}}] + \boldsymbol{x}_{\mathrm{B}}^\top [\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}}]), \sigma^2 (\boldsymbol{x}_{\mathrm{A}}^\top \boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top \boldsymbol{x}_{\mathrm{B}})^{-1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\beta_0 | \boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \sigma^2, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \left[\boldsymbol{y}_{\mathrm{A}} | \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{y}_{\mathrm{B}} | \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\right\} \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}\left([n_{\mathrm{A}} + n_{\mathrm{B}}]\beta_0^2 - 2\beta_0 (\boldsymbol{y}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top \mathbf{1}_{n_{\mathrm{A}}} - 2\beta_0 (\boldsymbol{y}_{\mathrm{B}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top \mathbf{1}_{n_{\mathrm{B}}}\right)\right\} \\
&\quad = N\left\{\frac{(\boldsymbol{y}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top \mathbf{1}_{n_{\mathrm{A}}} + (\boldsymbol{y}_{\mathrm{B}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top \mathbf{1}_{n_{\mathrm{B}}}}{n_{\mathrm{A}} + n_{\mathrm{B}}}, \frac{\sigma^2}{n_{\mathrm{A}} + n_{\mathrm{B}}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&[\sigma^2|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}] \\
&\quad \propto \left[\boldsymbol{y}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{y}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \\
&\quad \propto (\sigma^2)^{-n_{\mathrm{A}}/2} \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})\right\} \\
&\qquad \times (\sigma^2)^{-n_{\mathrm{B}}/2} \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\right\} (\sigma^2)^{-1} \\
&\quad = IG\Big\{\frac{1}{2}\left(n_{\mathrm{A}} + n_{\mathrm{B}}\right), \\
&\qquad\qquad \frac{1}{2}(\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{A}} - \beta_0 \mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta}) \\
&\qquad\qquad + \frac{1}{2}(\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top (\boldsymbol{y}_{\mathrm{B}} - \beta_0 \mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&[\psi|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}] \\
&\quad \propto \left[\boldsymbol{w}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \psi, \nu, \tau^2\right] \left[\boldsymbol{w}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \psi, \nu, \tau^2\right] \\
&\quad \propto \exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}}\right)^\top (\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}})\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}}\right)^\top (\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}})\right\} \\
&\quad \propto \exp\left\{-\frac{1}{2\tau^2}\left(\psi^2 \mathrm{Tr}\left[\mathbf{1}_p \mathbf{1}_{n_{\mathrm{A}}}^\top \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top + \mathbf{1}_p \mathbf{1}_{n_{\mathrm{B}}}^\top \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top\right] - 2\psi \mathrm{Tr}\left[\mathbf{1}_p \mathbf{1}_{n_{\mathrm{A}}}^\top(\boldsymbol{w}_{\mathrm{A}} - \nu\boldsymbol{x}_{\mathrm{A}}) + \mathbf{1}_p \mathbf{1}_{n_{\mathrm{B}}}^\top(\boldsymbol{w}_{\mathrm{B}} - \nu\boldsymbol{x}_{\mathrm{B}})\right]\right)\right\} \\
&\quad = N\left\{\frac{\mathbf{1}_{n_{\mathrm{A}}}^\top(\boldsymbol{w}_{\mathrm{A}} - \nu\boldsymbol{x}_{\mathrm{A}})\mathbf{1}_p + \mathbf{1}_{n_{\mathrm{B}}}^\top(\boldsymbol{w}_{\mathrm{B}} - \nu\boldsymbol{x}_{\mathrm{B}})\mathbf{1}_p}{(n_{\mathrm{A}} + n_{\mathrm{B}})p}, \frac{\tau^2}{(n_{\mathrm{A}} + n_{\mathrm{B}})p}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&[\nu|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2, \psi, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}] \\
&\quad \propto \left[\boldsymbol{w}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \psi, \nu, \tau^2\right] \left[\boldsymbol{w}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \psi, \nu, \tau^2\right] \\
&\quad \propto \exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}}\right)^\top (\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}})\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}}\right)^\top (\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}})\right\} \\
&\quad \propto \exp\left\{-\frac{1}{2\tau^2}\left(\nu^2 \mathrm{Tr}\left[\boldsymbol{x}_{\mathrm{A}}^\top \boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top \boldsymbol{x}_{\mathrm{B}}\right] - 2\nu \mathrm{Tr}\left[\boldsymbol{x}_{\mathrm{A}}^\top(\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top) + \boldsymbol{x}_{\mathrm{B}}^\top(\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top)\right]\right)\right\} \\
&\quad = N\left\{\frac{\mathrm{Tr}\left[\boldsymbol{x}_{\mathrm{A}}^\top(\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top) + \boldsymbol{x}_{\mathrm{B}}^\top(\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top)\right]}{\mathrm{Tr}\left[\boldsymbol{x}_{\mathrm{A}}^\top \boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top \boldsymbol{x}_{\mathrm{B}}\right]}, \frac{\tau^2}{\mathrm{Tr}\left[\boldsymbol{x}_{\mathrm{A}}^\top \boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top \boldsymbol{x}_{\mathrm{B}}\right]}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&[\tau^2|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2, \psi, \nu, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}] \\
&\quad \propto \left[\boldsymbol{w}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \psi, \nu, \tau^2\right] \left[\boldsymbol{w}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \psi, \nu, \tau^2\right] \\
&\quad \propto \left(\tau^2\right)^{-(n_{\mathrm{A}}+n_{\mathrm{B}})p/2} \exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}}\right)^\top (\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}})\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}}\right)^\top (\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}})\right\} (\tau^2)^{-1} \\
&\quad = IG\Big\{\frac{1}{2}(n_{\mathrm{A}} + n_{\mathrm{B}})p, \\
&\qquad\qquad \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}}\right)^\top (\boldsymbol{w}_{\mathrm{A}} - \psi \mathbf{1}_{n_{\mathrm{A}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{A}}) \\
&\qquad\qquad + \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}}\right)^\top (\boldsymbol{w}_{\mathrm{B}} - \psi \mathbf{1}_{n_{\mathrm{B}}}\mathbf{1}_p^\top - \nu \boldsymbol{x}_{\mathrm{B}})\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\boldsymbol{\mu}_{\boldsymbol{X}} | \boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2, \psi, \nu, \tau^2, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \left[\boldsymbol{x}_{\mathrm{A}} | \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \left[\boldsymbol{x}_{\mathrm{B}} | \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \exp\left\{-\frac{1}{2} \mathrm{Tr}\ (\boldsymbol{x}_{\mathrm{A}} - \mathbf{1}_{n_{\mathrm{A}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top) \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} (\boldsymbol{x}_{\mathrm{A}} - \mathbf{1}_{n_{\mathrm{A}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top)^\top - \frac{1}{2} \mathrm{Tr}\ (\boldsymbol{x}_{\mathrm{B}} - \mathbf{1}_{n_{\mathrm{B}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top) \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} (\boldsymbol{x}_{\mathrm{B}} - \mathbf{1}_{n_{\mathrm{B}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top)^\top \right\} \\
&\quad \propto \exp\left\{-\frac{1}{2} \left([n_{\mathrm{A}} + n_{\mathrm{B}}] \boldsymbol{\mu}_{\boldsymbol{X}}^\top \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} \boldsymbol{\mu}_{\boldsymbol{X}} - 2 \boldsymbol{\mu}_{\boldsymbol{X}}^\top \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} [\boldsymbol{x}_{\mathrm{A}}^\top \mathbf{1}_{n_{\mathrm{A}}} + \boldsymbol{x}_{\mathrm{B}}^\top \mathbf{1}_{n_{\mathrm{B}}}] \right) \right\} \\
&\quad = N_p \left\{ \frac{\boldsymbol{x}_{\mathrm{A}}^\top \mathbf{1}_{n_{\mathrm{A}}} + \boldsymbol{x}_{\mathrm{B}}^\top \mathbf{1}_{n_{\mathrm{B}}}}{n_{\mathrm{A}} + n_{\mathrm{B}}}, \frac{1}{n_{\mathrm{A}} + n_{\mathrm{B}}} \boldsymbol{\Sigma}_{\boldsymbol{X}} \right\} \\
&\left[\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} | \boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2, \psi, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}\right] \\
&\quad \propto \left[\boldsymbol{x}_{\mathrm{A}} | \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \left[\boldsymbol{x}_{\mathrm{B}} | \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \left[\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad = \left|\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right|^{(n_{\mathrm{A}} + n_{\mathrm{B}})/2} \exp\left\{-\frac{1}{2} \mathrm{Tr}\ \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1} \left[(\boldsymbol{x}_{\mathrm{A}} - \mathbf{1}_{n_{\mathrm{A}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top)^\top (\boldsymbol{x}_{\mathrm{A}} - \mathbf{1}_{n_{\mathrm{A}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top) + (\boldsymbol{x}_{\mathrm{B}} - \mathbf{1}_{n_{\mathrm{B}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top)^\top (\boldsymbol{x}_{\mathrm{B}} - \mathbf{1}_{n_{\mathrm{B}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top) \right] \right\} \\
&\qquad \times \left|\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right|^{(2p-1)/2} \exp\left\{-\tfrac{2p-1}{2} \mathrm{Tr}\ (\mathrm{diag}(\hat{\mathrm{Var}}[\boldsymbol{x}_{\mathrm{A}}]) \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}) \right\} \\
&\quad = W\Big\{ 3p + n_{\mathrm{A}} + n_{\mathrm{B}}, \\
&\qquad\qquad \Big((2p-1) \mathrm{diag}(\hat{\mathrm{Var}}[\boldsymbol{x}_{\mathrm{A}}]) \\
&\qquad\qquad + (\boldsymbol{x}_{\mathrm{A}} - \mathbf{1}_{n_{\mathrm{A}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top)^\top (\boldsymbol{x}_{\mathrm{A}} - \mathbf{1}_{n_{\mathrm{A}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top) + (\boldsymbol{x}_{\mathrm{B}} - \mathbf{1}_{n_{\mathrm{B}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top)^\top (\boldsymbol{x}_{\mathrm{B}} - \mathbf{1}_{n_{\mathrm{B}}} \boldsymbol{\mu}_{\boldsymbol{X}}^\top) \Big)^{-1} \Big\}
\end{aligned}
$$

The Inverse-Gamma distribution, $IG(a, b)$, is parametrized to have mean $\frac{b}{a-1}$ and the Wishart distribution, $W(d, S)$ is parametrized to have mean dS.

HIERBETAS, EBBETAS Replace the Jeffreys prior on $\boldsymbol{\beta}$ with

$$
[\boldsymbol{\beta} | \sigma^2, \lambda] \propto \left(\frac{\lambda}{\sigma^2}\right)^{p/2} \exp\left\{-\frac{1}{2} \frac{\lambda}{\sigma^2} \boldsymbol{\beta}^\top \boldsymbol{\beta}\right\},
$$

The following posterior steps are modified:

$$
\begin{aligned}
&\left[\boldsymbol{\beta}|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \beta_0, \sigma^2, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}, \lambda\right] \\
&\quad \propto \left[\boldsymbol{y}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{y}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{\beta}|\sigma^2, \lambda\right] \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top(\boldsymbol{y}_{\mathrm{A}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta}) - \frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{B}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top(\boldsymbol{y}_{\mathrm{B}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\right\} \exp\left\{-\frac{\lambda}{2\sigma^2}\boldsymbol{\beta}^\top\boldsymbol{\beta}\right\} \\
&\quad \propto \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{\beta}^\top\left[\boldsymbol{x}_{\mathrm{A}}^\top\boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top\boldsymbol{x}_{\mathrm{B}} + \lambda\boldsymbol{I}_p\right]\boldsymbol{\beta} - 2\boldsymbol{\beta}^\top\left[\boldsymbol{x}_{\mathrm{A}}^\top(\boldsymbol{y}_{\mathrm{A}} - \beta_0\mathbf{1}_{n_{\mathrm{A}}}) + \boldsymbol{x}_{\mathrm{B}}^\top(\boldsymbol{y}_{\mathrm{B}} - \beta_0\mathbf{1}_{n_{\mathrm{B}}})\right]\right)\right\} \\
&\quad = N_p\left\{(\boldsymbol{x}_{\mathrm{A}}^\top\boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top\boldsymbol{x}_{\mathrm{B}} + \lambda\boldsymbol{I}_p)^{-1}(\boldsymbol{x}_{\mathrm{A}}^\top[\boldsymbol{y}_{\mathrm{A}} - \beta_0\mathbf{1}_{n_{\mathrm{A}}}] + \boldsymbol{x}_{\mathrm{B}}^\top[\boldsymbol{y}_{\mathrm{B}} - \beta_0\mathbf{1}_{n_{\mathrm{B}}}]), \sigma^2(\boldsymbol{x}_{\mathrm{A}}^\top\boldsymbol{x}_{\mathrm{A}} + \boldsymbol{x}_{\mathrm{B}}^\top\boldsymbol{x}_{\mathrm{B}} + \lambda\boldsymbol{I}_p)^{-1}\right\} \\
&\left[\sigma^2|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}, \lambda\right] \\
&\quad \propto \left[\boldsymbol{y}_{\mathrm{A}}|\boldsymbol{x}_{\mathrm{A}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{y}_{\mathrm{B}}|\boldsymbol{x}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2\right] \left[\boldsymbol{\beta}|\sigma^2, \lambda\right] \left[\sigma^2\right] \\
&\quad \propto (\sigma^2)^{-n_{\mathrm{A}}/2} \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{A}} - \beta_0\mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top(\boldsymbol{y}_{\mathrm{A}} - \beta_0\mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})\right\} \\
&\qquad \times (\sigma^2)^{-n_{\mathrm{B}}/2} \exp\left\{-\frac{1}{2\sigma^2}(\boldsymbol{y}_{\mathrm{B}} - \beta_0\mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top(\boldsymbol{y}_{\mathrm{B}} - \beta_0\mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\right\} (\sigma^2)^{-1} \\
&\qquad \times (\sigma^2)^{-p/2} \exp\left\{-\frac{\lambda}{2\sigma^2}\boldsymbol{\beta}^\top\boldsymbol{\beta}\right\} \\
&\quad = IG\Big\{\frac{1}{2}(n_{\mathrm{A}} + n_{\mathrm{B}} + p), \\
&\qquad\quad \frac{1}{2}\lambda\boldsymbol{\beta}^\top\boldsymbol{\beta} + \frac{1}{2}(\boldsymbol{y}_{\mathrm{A}} - \beta_0\mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta})^\top(\boldsymbol{y}_{\mathrm{A}} - \beta_0\mathbf{1}_{n_{\mathrm{A}}} - \boldsymbol{x}_{\mathrm{A}}\boldsymbol{\beta}) \\
&\qquad\quad + \frac{1}{2}(\boldsymbol{y}_{\mathrm{B}} - \beta_0\mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})^\top(\boldsymbol{y}_{\mathrm{B}} - \beta_0\mathbf{1}_{n_{\mathrm{B}}} - \boldsymbol{x}_{\mathrm{B}}\boldsymbol{\beta})\Big\}
\end{aligned}
$$

The HIERBETAS update for λ is given as follows:

$$
\begin{aligned}
&\left[\lambda|\boldsymbol{y}_{\mathrm{A}}, \boldsymbol{x}_{\mathrm{A}}, \boldsymbol{w}_{\mathrm{A}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{x}_{\mathrm{B}}, \boldsymbol{w}_{\mathrm{B}}, \boldsymbol{\beta}, \beta_0, \sigma^2, \nu, \tau^2, \boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right] \\
&\quad \propto \left[\boldsymbol{\beta}|\sigma^2, \lambda\right] [\lambda] \\
&\quad \propto \lambda^{p/2} \exp\left\{-\frac{\lambda}{2\sigma^2}\boldsymbol{\beta}^\top\boldsymbol{\beta}\right\} \lambda^{-1} \\
&\quad = \mathcal{G}\left\{\frac{p}{2}, \frac{\boldsymbol{\beta}^\top\boldsymbol{\beta}}{2\sigma^2}\right\}
\end{aligned}
$$

To calculate the EBBETAS update for λ, observe that $\mathrm{E}_{\boldsymbol{\phi}|\boldsymbol{U}^{\mathrm{obs}},\lambda} \ln[\boldsymbol{\beta}|\sigma^2, \lambda] = (p/2)\ln\lambda - \lambda\mathrm{E}\left[\boldsymbol{\beta}^\top\boldsymbol{\beta}/\sigma^2\right]/2$. This is maximized with respect to λ when $\lambda = p/\mathrm{E}\left[\boldsymbol{\beta}^\top\boldsymbol{\beta}/\sigma^2\right]$.

EBSIGMAX, EBBOTH Leaving the inverse scale matrix $\boldsymbol{\Lambda}$ unspecified, the modified prior on $\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}$ is

$$
[\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}|\boldsymbol{\Lambda}] \propto |\boldsymbol{\Lambda}|^{3p/2}|\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}|^{(2p-1)/2} \exp\left\{-(1/2)\mathrm{Tr}\,(\boldsymbol{\Lambda}\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1})\right\}.
$$

$\mathbf{\Lambda}$ is the (unknown) positive-definite matrix of hyperparameters. The Gibbs step for $\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}$ becomes

$$\begin{aligned}
&\left[\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}|\boldsymbol{y}_{\mathrm{A}},\boldsymbol{x}_{\mathrm{A}},\boldsymbol{w}_{\mathrm{A}},\boldsymbol{y}_{\mathrm{B}},\boldsymbol{x}_{\mathrm{B}},\boldsymbol{w}_{\mathrm{B}},\boldsymbol{\beta},\beta_0,\sigma^2,\psi,\nu,\tau^2,\boldsymbol{\mu}_{\boldsymbol{X}}\right] \\
&\quad\propto \left[\boldsymbol{x}_{\mathrm{A}}|\boldsymbol{\mu}_{\boldsymbol{X}},\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}\right]\left[\boldsymbol{x}_{\mathrm{B}}|\boldsymbol{\mu}_{\boldsymbol{X}},\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}\right]\left[\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}|\mathbf{\Lambda}\right] \\
&\quad\propto \left|\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}\right|^{(n_{\mathrm{A}}+n_{\mathrm{B}})/2}\Big\{-\frac{1}{2}\mathrm{Tr}\ (\boldsymbol{x}_{\mathrm{A}}-\mathbf{1}_{n_{\mathrm{A}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}(\boldsymbol{x}_{\mathrm{A}}-\mathbf{1}_{n_{\mathrm{A}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})^{\top}\Big\} \\
&\qquad\times\exp\Big\{-\frac{1}{2}\mathrm{Tr}\ (\boldsymbol{x}_{\mathrm{B}}-\mathbf{1}_{n_{\mathrm{B}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}(\boldsymbol{x}_{\mathrm{B}}-\mathbf{1}_{n_{\mathrm{B}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})^{\top}\Big\} \\
&\qquad\times\left|\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}\right|^{(2p-1)/2}\exp\left\{-\frac{1}{2}\mathrm{Tr}\left(\mathbf{\Lambda}\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}\right)\right\} \\
&\quad= W\Big\{3p+n_{\mathrm{A}}+n_{\mathrm{B}}, \\
&\qquad\qquad\Big(\mathbf{\Lambda}+(\boldsymbol{x}_{\mathrm{A}}-\mathbf{1}_{n_{\mathrm{A}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})^{\top}(\boldsymbol{x}_{\mathrm{A}}-\mathbf{1}_{n_{\mathrm{A}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})+(\boldsymbol{x}_{\mathrm{B}}-\mathbf{1}_{n_{\mathrm{B}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})^{\top}(\boldsymbol{x}_{\mathrm{B}}-\mathbf{1}_{n_{\mathrm{B}}}\boldsymbol{\mu}_{\boldsymbol{X}}^{\top})\Big)^{-1}\Big\}
\end{aligned}$$

We now derive the Empirical Bayes update for the diagonal inverse-scale matrix $\mathbf{\Lambda} = \mathrm{diag}\{\Lambda_{11},\ldots,\Lambda_{pp}\}$. Observe that

$$\mathrm{E}_{\boldsymbol{\phi}|\boldsymbol{U}^{\mathrm{obs}},\mathbf{\Lambda}}\ln\left[\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}|\mathbf{\Lambda}\right]\propto p\ln\left|\mathbf{\Lambda}\right|-\mathrm{Tr}\ (\mathbf{\Lambda}\mathrm{E}[\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}])=3p\sum_{i=1}^{p}\log\Lambda_{ii}-\sum_{i=1}^{p}\Lambda_{ii}\mathrm{E}[\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}]_{ii}$$

Thus, each element Λ_{ii} may be optimized individually, yielding the Empirical Bayes update $\Lambda_{ii}\leftarrow 3p\mathrm{E}[\mathbf{\Sigma}_{\boldsymbol{X}}^{-1}]_{ii}^{-1}$

APPENDIX B: MODIFIED GIBBS STEPS FOR DATA ANALYSIS

Let $\boldsymbol{\psi}\equiv\{\psi_1,\ldots,\psi_p\}$, $\boldsymbol{\nu}\equiv\mathrm{diag}\{\nu_1,\ldots,\nu_p\}$ (that is, a diagonal matrix with components $\nu_1,\ldots,\nu_p$), and $\{\boldsymbol{e}_j\}$ the set of p-dimensional standard basic vectors. The conditional distributions with individual intercepts and slopes (using flat priors on each ψ_j and ν_j) are given by

$$
\begin{aligned}
&\left[\boldsymbol{\psi}|\boldsymbol{y}_{\rm A},\boldsymbol{x}_{\rm A},\boldsymbol{w}_{\rm A},\boldsymbol{y}_{\rm B},\boldsymbol{x}_{\rm B},\boldsymbol{w}_{\rm B},\boldsymbol{\beta},\beta_0,\sigma^2,\boldsymbol{\nu},\tau^2,\boldsymbol{\mu}_{\boldsymbol{X}},\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right]\\
&\quad\propto\left[\boldsymbol{w}_{\rm A}|\boldsymbol{x}_{\rm A},\boldsymbol{\psi},\boldsymbol{\nu},\tau^2\right]\left[\boldsymbol{w}_{\rm B}|\boldsymbol{x}_{\rm B},\boldsymbol{\psi},\boldsymbol{\nu},\tau^2\right]\\
&\quad\propto\exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\ (\boldsymbol{w}_{\rm A}-\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})^\top(\boldsymbol{w}_{\rm A}-\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})\right\}\\
&\qquad\times\exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\ (\boldsymbol{w}_{\rm B}-\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})^\top(\boldsymbol{w}_{\rm B}-\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})\right\}\\
&\quad\propto\exp\left\{-\frac{1}{2\tau^2}\left(\mathrm{Tr}\ [\boldsymbol{\psi}\mathbf{1}_{n_{\rm A}}^\top\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top+\boldsymbol{\psi}\mathbf{1}_{n_{\rm B}}^\top\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top]-2\mathrm{Tr}\ [\boldsymbol{\psi}\mathbf{1}_{n_{\rm A}}^\top(\boldsymbol{w}_{\rm A}-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})+\boldsymbol{\psi}\mathbf{1}_{n_{\rm B}}^\top(\boldsymbol{w}_{\rm B}-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})]\right)\right\}\\
&\quad=\exp\left\{-\frac{1}{2\tau^2}\left((n_{\rm A}+n_{\rm B})\sum_{j=1}^p\psi_j^2-\sum_{j=1}^p\psi_j\boldsymbol{e}_j^\top\left[(\boldsymbol{w}_{\rm A}-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})^\top\mathbf{1}_{n_{\rm A}}+(\boldsymbol{w}_{\rm B}-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})^\top\mathbf{1}_{n_{\rm B}}\right]\right)\right\}\\
&\left[\nu|\boldsymbol{y}_{\rm A},\boldsymbol{x}_{\rm A},\boldsymbol{w}_{\rm A},\boldsymbol{y}_{\rm B},\boldsymbol{x}_{\rm B},\boldsymbol{w}_{\rm B},\boldsymbol{\beta},\beta_0,\sigma^2,\psi,\tau^2,\boldsymbol{\mu}_{\boldsymbol{X}},\boldsymbol{\Sigma}_{\boldsymbol{X}}^{-1}\right]\\
&\quad\propto\left[\boldsymbol{w}_{\rm A}|\boldsymbol{x}_{\rm A},\psi,\nu,\tau^2\right]\left[\boldsymbol{w}_{\rm B}|\boldsymbol{x}_{\rm B},\psi,\nu,\tau^2\right]\\
&\quad\propto\exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\ (\boldsymbol{w}_{\rm A}-\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})^\top(\boldsymbol{w}_{\rm A}-\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})\right\}\\
&\qquad\times\exp\left\{-\frac{1}{2\tau^2}\mathrm{Tr}\ (\boldsymbol{w}_{\rm B}-\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})^\top(\boldsymbol{w}_{\rm B}-\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})\right\}\\
&\quad\propto\exp\left\{-\frac{1}{2\tau^2}\left(\mathrm{Tr}\ [\boldsymbol{\nu}^2\boldsymbol{x}_{\rm A}^\top\boldsymbol{x}_{\rm A}+\boldsymbol{\nu}^2\boldsymbol{x}_{\rm B}^\top\boldsymbol{x}_{\rm B}]-2\mathrm{Tr}\ [\boldsymbol{\nu}\boldsymbol{x}_{\rm A}^\top(\boldsymbol{w}_{\rm A}-\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top)+\boldsymbol{\nu}\boldsymbol{x}_{\rm B}^\top(\boldsymbol{w}_{\rm B}-\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top)]\right)\right\}
\end{aligned}
$$

From these, the modified Gibbs steps are

$$
\begin{aligned}
\psi_j&\leftarrow N\left\{\frac{\boldsymbol{e}_j^\top\left[(\boldsymbol{w}_{\rm A}-\boldsymbol{x}_{\rm A}\boldsymbol{\nu})^\top\mathbf{1}_{n_{\rm A}}+(\boldsymbol{w}_{\rm B}-\boldsymbol{x}_{\rm B}\boldsymbol{\nu})^\top\mathbf{1}_{n_{\rm B}}\right]}{(n_{\rm A}+n_{\rm B})},\frac{\tau^2}{(n_{\rm A}+n_{\rm B})}\right\}\\
\nu_j&\leftarrow N\left\{\frac{\boldsymbol{e}_j^\top\left[\boldsymbol{x}_{\rm A}^\top(\boldsymbol{w}_{\rm A}-\mathbf{1}_{n_{\rm A}}\boldsymbol{\psi}^\top)+\boldsymbol{x}_{\rm B}^\top(\boldsymbol{w}_{\rm B}-\mathbf{1}_{n_{\rm B}}\boldsymbol{\psi}^\top)\right]\boldsymbol{e}_j}{\boldsymbol{e}_j^\top\left[\boldsymbol{x}_{\rm A}^\top\boldsymbol{x}_{\rm A}+\boldsymbol{x}_{\rm B}^\top\boldsymbol{x}_{\rm B}\right]\boldsymbol{e}_j},\frac{\tau^2}{\boldsymbol{e}_j^\top[\boldsymbol{x}_{\rm A}^\top\boldsymbol{x}_{\rm A}+\boldsymbol{x}_{\rm B}^\top\boldsymbol{x}_{\rm B}]\boldsymbol{e}_j}\right\}
\end{aligned}
$$

independently for $j=1,\dots,p$.

APPENDIX C: DESCRIPTION OF PREPROCESSING STEPS FOR DATA ANALYSIS

The data were first normalized according to the prescription of Chen et al. (2011). 11 measurements in the qRT-PCR-only data, out of $47\times 91=4277$ total, or 0.26 percent, were missing; in order to use all observations, these values were imputed using chained equations (Su et al., 2011) and thereafter assumed known. Bayesian methods of imputation, like those discussed for imputing $\boldsymbol{x}_{\rm B}$, are a better approach to handle this missingness but, given the small percentage of missingness, would likely not affect the results. Additionally, four tumors, three in the Affymetrix-only sample and one in the validation sample, had event times less than one month after surgery; these were removed before analysis. Thus $n_{\rm A}=47$, $n_{\rm B}=389$, and the validation sample is size 100.

APPENDIX D: COMPUTATIONAL DETAILS

The Gibbs sampler code was written in the C language and called using `R` (R Core Team, 2012). The `R` environment was also used for all analyses. The simulation study in Section 4 of the manuscript was run on a cluster of 29 nodes with 392 cores, comprised of Intel Xeon X5660 and other Intel-based processors, each with 32GB of memory. With this configuration, using a burn-in period of 2500 iterations and 3500 total iterations, it took approximately 110 seconds to fit each Bayesian method to each simulated dataset. The data analysis in Section 5 of the manuscript was run on an Intel Core i7-3520M processor with 8GB of memory. With a burnin-period of 4000 iterations and 8000 total iterations, it took between 268–269 seconds to fit each Bayesian method.

Figures 2 and 3 in the manuscript were created with the `R` package `lattice` (Sarkar, 2008). The Scaled Integrated Brier Scores in Table 3 of the manuscript were calculated using code based on the `R` package `ipred` (Peters and Hothorn, 2012).

REFERENCES

Chen, G., Kim, S., Taylor, J. M. G., Wang, Z., Lee, O., Ramnath, N., Reddy, R. M., Lin, J., Chang, A. C., Orringer, M. B. and Beer, D. G. (2011). Development and validation of a qRT-PCR–classifier for lung cancer prognosis. *Journal of Thoracic Oncology* **6** 1481–1487.

Peters, A. and Hothorn, T. (2012). ipred: Improved Predictors R package version 0.9-1.

R Core Team, (2012). R: A Language and Environment for Statistical Computing, Vienna, Austria.

Sarkar, D. (2008). *Lattice: Multivariate Data Visualization with R.* Springer, New York.

Su, Y.-S., Gelman, A., Hill, J. and Yajima, M. (2011). Multiple Imputation with Diagnostics (mi) in R: Opening Windows into the Black Box. *Journal of Statistical Software* **45** 1–31.

$\{\rho, R^2\}$	$\{n_{\mathrm{B}}\}$	Method	MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{ppm}}$) $\tau = 0.01$	0.5	1.0	1.50	2.00	MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{pm}}$) 0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	700.2	691.4	694.4	697.8	692.7	700.2	691.4	694.4	697.8	692.7
		VANILLA	793.2	888.6	1074.8	1387.1	1499.2	793.2	919.6	1215.8	1566.4	1656.3
		HIERBETAS	**637.1**	**640.8**	**641.4**	**643.4**	**647.3**	**637.1**	**640.5**	**640.2**	**641.7**	**645.1**
		EBBETAS	643.8	645.7	645.4	648.1	648.4	643.8	645.4	644.4	646.7	646.8
		EBSIGMAX	793.2	878.5	1092.5	1432.2	1573.0	793.1	906.8	1238.2	1625.4	1744.6
		EBBOTH	644.1	645.8	645.3	648.1	648.8	644.1	645.5	644.3	646.8	647.1
0.75,0.1	400	RIDG	312.2	323.4	311.5	311.7	314.4	312.2	323.4	311.5	311.7	314.4
		VANILLA	347.6	341.9	361.3	381.8	399.9	347.6	348.8	386.2	429.6	462.7
		HIERBETAS	**281.3**	**282.6**	**283.3**	**284.6**	**287.3**	**281.3**	**282.6**	**283.2**	**284.4**	**287.1**
		EBBETAS	286.6	288.4	287.8	289.0	291.6	286.6	288.4	287.7	288.8	291.4
		EBSIGMAX	347.7	369.0	396.6	437.4	478.2	347.7	381.9	435.8	504.8	563.0
		EBBOTH	286.7	288.5	288.1	289.9	292.3	286.7	288.5	288.0	289.6	292.0
0.15,0.4	400	RIDG	124.7	125.2	125.2	125.7	127.4	124.7	125.2	125.2	125.7	127.4
		VANILLA	132.5	149.0	186.9	235.2	265.5	132.5	154.4	211.3	266.2	293.1
		HIERBETAS	111.2	112.3	113.9	115.2	117.4	111.2	112.2	113.7	114.9	117.0
		EBBETAS	**111.1**	112.0	113.5	**114.5**	**116.1**	**111.1**	112.0	**113.3**	**114.1**	**115.6**
		EBSIGMAX	132.5	147.3	188.9	246.7	283.5	132.5	152.2	214.6	278.8	311.5
		EBBOTH	111.1	**112.0**	**113.5**	114.5	116.3	111.1	**111.9**	113.3	114.1	115.8
0.75,0.4	400	RIDG	62.3	62.6	62.0	61.8	62.5	62.3	62.6	62.0	61.8	62.5
		VANILLA	58.1	57.6	61.2	68.0	73.1	58.1	58.7	65.4	77.2	85.3
		HIERBETAS	**49.1**	**49.7**	**50.2**	51.3	53.2	**49.1**	**49.7**	**50.2**	51.4	53.5
		EBBETAS	49.2	49.8	50.4	**51.2**	**53.1**	49.2	49.8	50.4	**51.3**	**53.3**
		EBSIGMAX	58.1	62.1	67.1	79.3	88.8	58.1	64.3	73.9	92.4	104.1
		EBBOTH	49.2	49.8	50.7	51.8	54.0	49.2	49.8	50.7	51.9	54.1
0.15,0.1	150	RIDG	694.4	712.1	691.8	699.3	701.2	694.4	712.1	691.8	699.3	701.2
		VANILLA	1245.3	1505.9	2321.9	2256.3	2222.5	1245.2	1612.4	2492.4	2412.2	2351.4
		HIERBETAS	**646.7**	**649.3**	**652.1**	**657.3**	669.3	**646.6**	**648.8**	**650.8**	**655.7**	667.5
		EBBETAS	650.0	654.9	654.9	658.2	662.6	650.0	654.5	653.9	656.8	660.8
		EBSIGMAX	1245.5	1483.5	2478.8	2369.8	2236.5	1245.4	1585.2	2636.2	2514.1	2374.6
		EBBOTH	650.4	654.5	654.8	658.6	**662.6**	650.4	654.1	653.8	657.1	**660.8**
0.75,0.1	150	RIDG	322.3	317.6	316.2	311.9	320.4	322.3	317.6	316.2	311.9	320.4
		VANILLA	539.6	550.4	656.1	689.5	713.6	539.5	584.4	725.3	767.2	789.4
		HIERBETAS	**287.2**	**289.0**	**289.5**	**291.8**	**294.8**	**287.2**	**288.9**	**289.2**	**291.6**	**294.9**
		EBBETAS	291.9	292.9	292.1	294.9	296.7	291.9	292.8	291.9	294.7	296.7
		EBSIGMAX	539.3	644.0	793.7	815.8	858.9	539.3	697.5	868.2	896.8	941.5
		EBBOTH	292.0	293.1	292.6	295.4	297.4	292.0	293.0	292.4	295.2	297.3
0.15,0.4	150	RIDG	127.4	127.8	128.4	127.3	126.8	127.4	127.8	128.4	127.3	126.8
		VANILLA	207.4	250.5	404.2	395.8	394.9	207.4	267.8	429.6	419.7	418.7
		HIERBETAS	115.6	116.2	118.1	120.3	124.3	115.6	116.1	117.8	120.0	124.0
		EBBETAS	**115.2**	115.7	116.9	**118.2**	**120.0**	**115.2**	115.5	**116.6**	**117.8**	**119.5**
		EBSIGMAX	207.5	247.2	416.7	404.7	405.3	207.5	264.3	443.2	429.6	431.3
		EBBOTH	115.2	**115.6**	**116.9**	118.3	120.2	115.2	**115.5**	116.6	117.9	119.8
0.75,0.4	150	RIDG	61.9	64.1	62.9	63.1	62.2	61.9	64.1	62.9	63.1	62.2
		VANILLA	90.8	91.9	110.5	126.3	133.2	90.8	97.5	121.9	140.1	147.3
		HIERBETAS	**51.7**	**52.2**	**53.1**	54.3	55.8	**51.7**	**52.2**	**53.1**	54.5	56.1
		EBBETAS	51.8	52.3	53.2	**54.1**	**55.2**	51.8	52.3	53.2	**54.2**	**55.4**
		EBSIGMAX	90.8	109.7	140.0	152.9	156.1	90.8	118.6	152.5	166.5	169.3
		EBBOTH	51.8	52.3	53.5	54.6	55.8	51.8	52.3	53.5	54.7	55.9

Table S1

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\{0.1\}_{k=1}^{k=8}, 1\}_{j=1}^{j=11}$ (a concentrated signal). When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

			MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{ppm}}$)					MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{pm}}$)				
$\{\rho, R^2\}$	$\{n_{\mathrm{B}}\}$	Method	$\tau = 0.01$	0.5	1.0	1.50	2.00	0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	74.09	73.55	73.40	74.01	74.20	74.09	73.55	73.40	74.01	74.20
		VANILLA	79.65	89.02	108.80	142.10	155.11	79.65	92.20	123.56	160.06	171.50
		HIERBETAS	**67.09**	**67.72**	**68.21**	**68.36**	**68.67**	**67.09**	**67.73**	**68.22**	**68.36**	**68.64**
		EBBETAS	67.99	68.42	68.66	68.73	68.77	67.99	68.42	68.66	68.73	68.76
		EBSIGMAX	79.66	88.02	113.97	147.55	164.04	79.65	90.89	129.24	167.26	180.22
		EBBOTH	68.00	68.43	68.67	68.74	68.77	68.00	68.44	68.67	68.73	68.76
0.75,0.1	400	RIDG	523.79	536.20	519.80	528.23	532.94	523.79	536.20	519.80	528.23	532.94
		VANILLA	589.99	574.68	605.71	635.76	680.29	589.97	586.67	647.97	714.70	785.98
		HIERBETAS	**471.93**	**470.84**	**472.84**	**474.42**	**480.87**	**471.92**	**470.74**	**472.56**	**473.88**	**480.37**
		EBBETAS	479.69	477.62	480.63	482.04	486.65	479.68	477.55	480.43	481.60	486.21
		EBSIGMAX	590.06	620.88	664.24	735.10	812.12	590.01	642.94	731.56	848.85	952.76
		EBBOTH	479.54	477.79	481.26	482.87	488.61	479.53	477.68	480.96	482.31	487.99
0.15,0.4	400	RIDG	17.87	17.54	17.65	17.46	17.74	17.87	17.54	17.65	17.46	17.74
		VANILLA	13.26	15.49	21.49	28.12	32.40	13.26	16.14	24.32	31.14	35.09
		HIERBETAS	**12.34**	**12.93**	**13.90**	**14.74**	**15.34**	**12.34**	**12.94**	**13.93**	**14.78**	**15.39**
		EBBETAS	13.44	13.55	14.41	15.14	15.74	13.43	13.56	14.44	15.17	15.77
		EBSIGMAX	13.26	15.27	21.91	30.26	34.17	13.26	15.85	24.87	33.37	36.86
		EBBOTH	13.48	13.54	14.43	15.16	15.75	13.48	13.55	14.45	15.18	15.78
0.75,0.4	400	RIDG	100.42	100.89	99.81	99.95	101.02	100.42	100.89	99.81	99.95	101.02
		VANILLA	97.91	97.28	102.18	110.06	120.13	97.90	99.31	109.61	124.62	139.82
		HIERBETAS	**81.01**	**81.69**	81.90	83.74	86.36	**81.01**	**81.68**	81.90	83.92	86.79
		EBBETAS	81.11	81.78	**81.73**	**83.65**	**86.02**	81.11	81.76	**81.72**	**83.76**	**86.27**
		EBSIGMAX	97.92	105.03	113.59	128.58	146.11	97.92	108.85	125.36	148.88	170.47
		EBBOTH	81.09	81.71	82.49	84.89	87.65	81.09	81.68	82.47	84.97	87.80
0.15,0.1	150	RIDG	73.44	73.80	74.22	74.13	73.53	73.44	73.80	74.22	74.13	73.53
		VANILLA	125.29	149.56	235.37	230.94	223.55	125.27	160.03	251.59	246.44	237.77
		HIERBETAS	**68.45**	**68.81**	**69.25**	69.50	69.31	**68.45**	**68.80**	**69.20**	69.42	69.26
		EBBETAS	68.77	69.07	69.30	**69.25**	**69.17**	68.77	69.07	69.28	**69.23**	**69.14**
		EBSIGMAX	125.21	150.29	239.49	240.20	233.70	125.21	160.60	256.19	254.68	248.14
		EBBOTH	68.78	69.07	69.30	69.27	69.19	68.78	69.06	69.29	69.25	69.16
0.75,0.1	150	RIDG	528.41	528.12	542.56	527.94	529.34	528.41	528.12	542.56	527.94	529.34
		VANILLA	929.85	916.60	1083.86	1175.04	1196.09	929.57	971.78	1195.87	1311.07	1330.01
		HIERBETAS	**483.01**	**483.69**	**487.68**	**491.58**	**493.62**	**483.00**	**483.50**	**487.21**	**491.14**	**493.77**
		EBBETAS	489.80	490.23	493.83	495.78	499.95	489.80	490.10	493.46	495.29	499.75
		EBSIGMAX	929.33	1093.62	1296.55	1401.16	1402.45	929.36	1181.01	1423.90	1532.15	1537.75
		EBBOTH	489.55	490.39	494.66	496.66	501.03	489.54	490.23	494.21	496.05	500.73
0.15,0.4	150	RIDG	17.81	17.73	17.74	17.65	17.67	17.81	17.73	17.74	17.65	17.67
		VANILLA	20.80	26.76	45.28	46.28	46.78	20.79	28.61	48.25	48.94	49.46
		HIERBETAS	**14.08**	**14.49**	**15.34**	16.16	16.81	**14.08**	**14.50**	**15.37**	16.19	16.84
		EBBETAS	14.72	14.94	15.71	**16.12**	16.46	14.72	14.94	15.72	**16.13**	16.47
		EBSIGMAX	20.79	26.41	46.27	47.74	48.51	20.79	28.22	49.19	50.48	51.00
		EBBOTH	14.71	14.95	15.70	16.13	**16.45**	14.71	14.95	15.72	16.14	**16.46**
0.75,0.4	150	RIDG	100.07	102.46	99.11	100.75	100.07	100.07	102.46	99.11	100.75	100.07
		VANILLA	150.97	155.73	183.32	212.93	225.54	150.95	164.84	203.10	234.98	246.53
		HIERBETAS	84.34	85.16	86.40	88.11	90.71	84.34	85.12	86.45	88.43	91.28
		EBBETAS	84.12	**85.02**	**86.03**	**87.64**	**89.92**	84.11	**84.98**	**86.03**	**87.82**	**90.30**
		EBSIGMAX	151.08	178.45	231.81	248.84	253.86	151.04	192.93	254.27	272.79	276.32
		EBBOTH	**84.11**	85.11	86.72	88.70	91.11	**84.11**	85.04	86.68	88.76	91.31

TABLE S2

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\frac{j}{100}\}_{j=-49}^{j=49}$ (a diffuse signal). When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

			MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{ppm}}$)					MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{pm}}$)				
$\{\rho, R^2\}$	$\{n_{\mathrm{B}}\}$	Method	$\tau = 0.01$	0.5	1.0	1.50	2.00	0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	694.3	702.3	695.4	690.8	695.2	694.3	702.3	695.4	690.8	695.2
		VANILLA	803.9	882.1	1086.9	1385.8	1455.6	803.9	912.9	1231.5	1564.9	1615.3
		HIERBETAS	**641.1**	**636.8**	**641.8**	**643.6**	**649.9**	**641.1**	**636.4**	**640.6**	**641.8**	**647.7**
		EBBETAS	646.7	640.1	643.9	646.7	650.7	646.7	639.8	642.9	645.3	648.8
		EBSIGMAX	804.1	872.2	1110.3	1481.1	1564.8	804.1	900.1	1259.3	1666.3	1724.4
		EBBOTH	646.4	640.1	643.9	647.1	651.0	646.4	639.8	642.9	645.7	649.1
0.75,0.1	400	RIDG	316.0	313.3	312.0	314.8	309.7	316.0	313.3	312.0	314.8	309.7
		VANILLA	347.8	341.2	362.2	383.9	404.3	347.8	348.2	386.5	430.8	463.5
		HIERBETAS	**282.3**	**281.9**	**284.4**	**284.8**	**286.9**	**282.3**	**281.8**	**284.3**	**284.5**	**286.5**
		EBBETAS	286.0	285.7	288.9	287.8	290.0	286.0	285.6	288.9	287.5	289.7
		EBSIGMAX	347.8	368.2	397.3	442.6	474.8	347.8	381.1	436.0	509.2	552.3
		EBBOTH	285.9	285.8	289.1	288.4	290.9	285.9	285.8	288.9	288.1	290.4
0.15,0.4	400	RIDG	125.9	127.6	127.1	127.2	126.6	125.9	127.6	127.1	127.2	126.6
		VANILLA	133.9	148.3	186.0	236.9	263.9	133.9	153.6	210.5	267.1	291.6
		HIERBETAS	**111.7**	**111.8**	114.1	114.8	117.2	**111.7**	111.8	113.9	114.5	116.9
		EBBETAS	111.8	111.9	113.7	**114.2**	**116.0**	111.8	111.8	113.4	**113.9**	**115.6**
		EBSIGMAX	133.9	146.5	190.4	248.2	274.5	133.9	151.3	216.1	280.2	303.9
		EBBOTH	111.8	111.8	**113.6**	114.4	116.1	111.8	**111.8**	**113.4**	114.1	115.7
0.75,0.4	400	RIDG	62.1	63.0	62.9	64.1	61.5	62.1	63.0	62.9	64.1	61.5
		VANILLA	58.3	58.2	62.3	69.1	72.6	58.3	59.4	66.8	77.8	83.8
		HIERBETAS	**49.3**	**49.9**	**50.6**	**51.8**	53.1	**49.3**	**49.9**	**50.6**	51.9	53.3
		EBBETAS	49.4	50.4	50.6	51.9	**53.1**	49.4	50.4	50.6	**51.9**	**53.2**
		EBSIGMAX	58.3	62.7	69.1	80.2	88.6	58.3	65.0	76.4	92.2	102.6
		EBBOTH	49.4	50.2	51.0	52.5	54.0	49.4	50.2	50.9	52.5	54.0
0.15,0.1	150	RIDG	686.3	690.7	691.4	700.9	689.7	686.3	690.7	691.4	700.9	689.7
		VANILLA	1264.1	1494.0	2337.3	2312.8	2169.8	1264.1	1597.8	2505.8	2442.4	2318.4
		HIERBETAS	**646.0**	**651.4**	**649.9**	**655.7**	658.5	**646.0**	**650.9**	**648.5**	**653.9**	656.2
		EBBETAS	649.8	654.3	653.1	656.3	**654.4**	649.8	653.9	652.1	655.0	**652.5**
		EBSIGMAX	1263.9	1490.5	2392.9	2287.6	2301.3	1263.9	1592.5	2555.5	2436.8	2438.5
		EBBOTH	650.3	654.2	653.6	656.4	654.8	650.3	653.7	652.6	655.1	652.9
0.75,0.1	150	RIDG	315.8	321.2	311.8	314.2	316.5	315.8	321.2	311.8	314.2	316.5
		VANILLA	546.8	540.1	635.4	676.1	707.2	546.8	573.7	701.9	753.7	787.1
		HIERBETAS	**286.9**	**288.1**	**291.0**	**295.2**	**295.3**	**286.9**	**288.0**	**290.8**	**294.8**	**295.4**
		EBBETAS	290.5	291.7	293.8	297.6	296.8	290.5	291.6	293.6	297.4	296.7
		EBSIGMAX	546.6	649.8	768.3	811.5	832.4	546.6	702.5	845.9	892.2	918.4
		EBBOTH	290.5	291.7	294.1	298.3	297.3	290.5	291.6	293.9	298.0	297.2
0.15,0.4	150	RIDG	127.9	125.2	124.7	124.8	124.1	127.9	125.2	124.7	124.8	124.1
		VANILLA	208.3	256.8	403.2	397.2	391.9	208.3	275.3	432.3	422.9	417.6
		HIERBETAS	115.3	115.6	117.2	119.4	124.5	115.3	115.5	116.9	119.0	124.0
		EBBETAS	**114.9**	115.1	**116.0**	**117.3**	120.3	**114.9**	114.9	**115.7**	**116.9**	119.6
		EBSIGMAX	208.2	253.7	414.9	428.2	407.8	208.2	271.7	443.1	449.7	433.0
		EBBOTH	114.9	**115.0**	116.1	117.5	**120.2**	114.9	**114.9**	115.8	117.0	**119.5**
0.75,0.4	150	RIDG	63.5	62.5	62.7	63.3	61.6	63.5	62.5	62.7	63.3	61.6
		VANILLA	90.6	91.0	111.7	127.7	132.5	90.6	96.6	123.4	140.3	146.1
		HIERBETAS	**51.5**	**52.2**	53.3	54.8	56.2	**51.5**	**52.1**	53.4	54.9	56.4
		EBBETAS	51.6	52.4	**53.3**	**54.4**	**55.8**	51.6	52.3	**53.3**	**54.5**	**55.9**
		EBSIGMAX	90.7	111.6	141.2	145.9	155.7	90.6	120.3	155.3	160.1	168.9
		EBBOTH	51.6	52.4	53.5	54.9	56.4	51.6	52.4	53.5	54.9	56.4

TABLE S3

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\{0.1\}_{k=1}^{k=8}, 1\}_{j=1}^{j=11}$ (a concentrated signal) when $\varepsilon + 1 \sim G(1,1)$. When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

$\{\rho, R^2\}$	$\{n_B\}$	Method	MSPE($\hat{\boldsymbol{\beta}}^{\text{ppm}}$) $\tau = 0.01$	0.5	1.0	1.50	2.00	MSPE($\hat{\boldsymbol{\beta}}^{\text{pm}}$) 0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	73.84	73.22	73.73	73.77	74.22	73.84	73.22	73.73	73.77	74.22
		VANILLA	79.61	88.78	109.85	139.23	147.96	79.61	91.95	124.40	157.32	164.91
		HIERBETAS	**67.27**	**67.35**	**68.41**	**68.36**	**68.85**	**67.27**	**67.35**	**68.40**	**68.35**	**68.82**
		EBBETAS	68.13	68.02	68.84	68.63	69.00	68.13	68.02	68.85	68.62	68.99
		EBSIGMAX	79.62	87.78	111.84	146.90	161.51	79.62	90.65	127.13	165.78	178.25
		EBBOTH	68.11	68.01	68.84	68.62	69.01	68.11	68.01	68.84	68.61	69.00
0.75,0.1	400	RIDG	525.97	519.50	529.63	529.43	539.07	525.97	519.50	529.63	529.43	539.07
		VANILLA	587.50	577.27	607.07	649.61	688.95	587.47	589.35	649.28	728.16	792.52
		HIERBETAS	**471.30**	**471.74**	**473.28**	**477.64**	**483.86**	**471.29**	**471.66**	**473.01**	**477.14**	**483.17**
		EBBETAS	479.13	478.80	481.39	484.81	490.66	479.13	478.74	481.19	484.38	490.16
		EBSIGMAX	587.59	624.21	665.89	737.95	824.74	587.58	646.43	731.43	847.48	955.73
		EBBOTH	478.90	478.88	481.92	485.81	492.33	478.89	478.79	481.64	485.27	491.71
0.15,0.4	400	RIDG	17.52	17.48	17.79	17.60	17.82	17.52	17.48	17.79	17.60	17.82
		VANILLA	13.25	15.31	21.20	29.35	31.43	13.25	15.95	23.96	32.21	34.35
		HIERBETAS	**12.31**	**12.86**	**13.81**	**14.80**	**15.48**	**12.31**	**12.87**	**13.84**	**14.84**	**15.54**
		EBBETAS	13.24	13.67	14.48	15.27	15.72	13.24	13.68	14.51	15.29	15.75
		EBSIGMAX	13.25	15.12	21.55	31.23	32.95	13.25	15.68	24.52	34.16	35.79
		EBBOTH	13.26	13.65	14.51	15.27	15.73	13.26	13.65	14.53	15.30	15.76
0.75,0.4	400	RIDG	100.36	99.95	102.19	101.45	100.97	100.36	99.95	102.19	101.45	100.97
		VANILLA	96.98	97.10	103.06	111.51	121.80	96.98	99.11	110.53	126.37	141.21
		HIERBETAS	**80.47**	**81.55**	**82.44**	83.77	87.21	**80.47**	**81.53**	82.45	83.93	87.60
		EBBETAS	80.76	81.81	82.44	**83.39**	**86.66**	80.76	81.79	**82.43**	**83.48**	**86.90**
		EBSIGMAX	97.01	104.86	113.87	129.09	150.33	97.01	108.63	125.65	150.12	174.52
		EBBOTH	80.76	81.82	83.08	84.63	88.16	80.75	81.79	83.06	84.67	88.21
0.15,0.1	150	RIDG	74.00	74.49	73.96	73.94	73.07	74.00	74.49	73.96	73.94	73.07
		VANILLA	124.35	152.68	231.58	235.34	229.21	124.34	163.65	248.06	249.57	244.11
		HIERBETAS	**68.69**	**68.37**	**69.11**	**69.25**	69.62	**68.68**	**68.36**	**69.07**	**69.20**	69.54
		EBBETAS	69.11	68.69	69.17	69.30	69.11	69.12	68.69	69.15	69.28	69.08
		EBSIGMAX	124.36	151.44	244.88	236.04	244.40	124.35	162.07	260.73	251.66	259.36
		EBBOTH	69.12	68.70	69.18	69.32	**69.11**	69.12	68.69	69.17	69.31	**69.07**
0.75,0.1	150	RIDG	534.02	527.23	531.07	523.91	526.81	534.02	527.23	531.07	523.91	526.81
		VANILLA	915.42	901.20	1074.68	1175.88	1133.98	915.29	957.72	1190.97	1305.25	1277.79
		HIERBETAS	**480.51**	**483.15**	**485.01**	**487.88**	**497.03**	**480.52**	**483.00**	**484.54**	**487.29**	**496.75**
		EBBETAS	486.68	489.21	491.33	492.71	499.40	486.69	489.09	490.93	492.08	498.76
		EBSIGMAX	915.69	1084.99	1325.31	1360.39	1423.70	915.61	1172.33	1463.78	1492.13	1567.57
		EBBOTH	486.58	489.73	491.92	493.80	500.29	486.59	489.60	491.45	493.14	499.71
0.15,0.4	150	RIDG	17.53	17.55	17.41	17.74	17.48	17.53	17.55	17.41	17.74	17.48
		VANILLA	20.63	26.37	45.47	46.16	47.40	20.63	28.27	48.29	48.86	49.99
		HIERBETAS	**14.11**	**14.63**	**15.35**	16.24	16.64	**14.11**	**14.64**	**15.38**	16.28	16.67
		EBBETAS	14.70	15.11	15.69	**16.18**	**16.35**	14.70	15.11	15.70	**16.19**	**16.34**
		EBSIGMAX	20.64	26.38	45.27	47.65	47.16	20.63	28.24	48.35	50.36	50.26
		EBBOTH	14.69	15.09	15.68	16.20	16.37	14.69	15.10	15.69	16.21	16.37
0.75,0.4	150	RIDG	100.53	103.75	100.36	100.05	101.42	100.53	103.75	100.36	100.05	101.42
		VANILLA	154.31	150.90	180.92	208.60	216.98	154.28	160.27	200.19	231.10	239.66
		HIERBETAS	**84.55**	85.61	86.85	88.51	90.85	**84.55**	85.59	86.84	88.79	91.42
		EBBETAS	84.66	**85.44**	**86.47**	**87.93**	**90.17**	84.65	85.41	**86.42**	**88.06**	**90.54**
		EBSIGMAX	154.36	184.58	232.36	239.26	252.38	154.34	199.06	253.12	263.95	277.45
		EBBOTH	84.65	85.45	86.88	88.92	91.44	84.64	**85.39**	86.77	88.98	91.64

TABLE S4

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\frac{j}{100}\}_{j=-49}^{j=49}$ (a diffuse signal) when $\varepsilon + 1 \sim G(1,1)$. When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

			MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{ppm}}$)					MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{pm}}$)				
$\{\rho, R^2\}$	$\{n_{\mathrm{B}}\}$	Method	$\tau = 0.01$	0.5	1.0	1.50	2.00	0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	689.0	693.5	703.6	696.7	710.2	689.0	693.5	703.6	696.7	710.2
		VANILLA	864.9	899.7	971.6	1093.5	1266.1	884.6	935.1	1054.5	1244.6	1452.6
		HIERBETAS	**638.0**	**638.6**	**637.4**	**641.1**	**644.5**	**637.7**	**638.3**	**636.7**	**639.9**	**643.2**
		EBBETAS	643.5	644.5	643.6	645.8	650.0	643.3	644.2	643.0	644.9	649.0
		EBSIGMAX	856.0	894.8	972.3	1120.2	1330.5	874.4	928.9	1053.9	1277.7	1511.8
		EBBOTH	643.6	644.1	643.4	645.5	649.8	643.5	643.9	642.9	644.6	648.9
0.75,0.1	400	RIDG	316.4	316.6	317.2	315.1	315.2	316.4	316.6	317.2	315.1	315.2
		VANILLA	348.8	348.2	350.0	358.4	372.5	354.7	356.1	364.1	382.7	410.9
		HIERBETAS	**282.9**	**283.3**	**282.8**	**282.6**	**284.5**	**282.9**	**283.3**	**282.8**	**282.5**	**284.2**
		EBBETAS	287.4	288.5	287.6	286.8	288.7	287.4	288.5	287.5	286.7	288.5
		EBSIGMAX	382.6	382.8	384.2	394.9	415.0	395.0	398.6	409.5	433.8	471.9
		EBBOTH	287.4	288.2	287.5	287.1	289.0	287.4	288.2	287.5	287.0	288.7
0.15,0.4	400	RIDG	125.6	126.4	126.4	126.4	126.5	125.6	126.4	126.4	126.4	126.5
		VANILLA	146.1	150.6	165.4	191.4	219.8	149.6	156.8	179.8	216.8	249.2
		HIERBETAS	112.2	111.5	113.3	114.0	114.1	112.2	111.5	113.2	113.8	113.8
		EBBETAS	112.0	111.3	113.0	113.6	113.5	112.0	111.3	112.8	113.4	**113.2**
		EBSIGMAX	144.6	149.6	166.0	193.2	233.9	147.8	155.4	180.3	220.0	265.6
		EBBOTH	**112.0**	**111.3**	**112.9**	**113.5**	**113.5**	**112.0**	**111.2**	**112.8**	**113.3**	113.2
0.75,0.4	400	RIDG	62.2	61.0	61.0	62.6	61.5	62.2	61.0	61.0	62.6	61.5
		VANILLA	59.3	58.8	59.6	61.8	65.1	60.3	60.1	62.0	66.3	72.3
		HIERBETAS	50.1	49.8	**50.0**	50.2	50.9	50.1	49.8	**50.0**	50.2	51.0
		EBBETAS	50.1	49.9	50.2	**50.1**	**50.9**	50.1	49.9	50.2	**50.1**	**50.9**
		EBSIGMAX	65.0	64.4	65.6	68.4	73.3	67.1	67.2	70.1	75.9	83.9
		EBBOTH	**49.8**	**49.6**	50.1	50.3	51.3	**49.8**	**49.6**	50.1	50.3	51.3
0.15,0.1	150	RIDG	700.8	692.6	696.9	695.6	685.1	700.8	692.6	696.9	695.6	685.1
		VANILLA	1448.2	1553.3	2123.5	2377.5	2272.9	1522.3	1671.9	2309.3	2541.8	2427.6
		HIERBETAS	**649.3**	**648.6**	**652.0**	**653.9**	**653.1**	**649.0**	**648.1**	**651.1**	**652.5**	**651.4**
		EBBETAS	655.4	653.2	656.8	658.2	656.5	655.1	652.7	656.2	657.1	655.3
		EBSIGMAX	1449.1	1546.1	2173.8	2370.7	2329.3	1525.3	1667.1	2368.1	2551.9	2492.8
		EBBOTH	655.4	653.0	656.8	657.4	656.6	655.1	652.6	656.2	656.4	655.4
0.75,0.1	150	RIDG	320.9	314.4	314.4	317.1	313.1	320.9	314.4	314.4	317.1	313.1
		VANILLA	572.4	570.4	601.9	647.3	660.5	600.2	607.2	656.6	718.3	741.9
		HIERBETAS	**289.5**	**288.6**	**288.7**	**290.7**	**291.3**	**289.4**	**288.5**	**288.6**	**290.5**	**291.2**
		EBBETAS	293.0	292.0	291.7	293.3	294.3	293.0	291.9	291.6	293.1	294.1
		EBSIGMAX	703.6	730.9	768.0	841.4	832.0	752.0	786.7	837.2	914.8	909.2
		EBBOTH	293.1	292.2	292.0	293.6	294.8	293.1	292.2	291.8	293.4	294.6
0.15,0.4	150	RIDG	126.1	125.8	126.4	125.8	124.4	126.1	125.8	126.4	125.8	124.4
		VANILLA	242.1	260.7	372.3	409.2	401.1	254.7	280.6	401.2	434.6	423.9
		HIERBETAS	115.3	115.9	116.4	117.3	119.1	115.3	115.7	116.1	117.0	118.7
		EBBETAS	114.9	115.3	115.5	116.3	**117.5**	114.8	115.1	115.3	**115.9**	**117.1**
		EBSIGMAX	242.4	261.4	376.0	415.3	408.5	255.3	281.7	405.8	442.2	433.9
		EBBOTH	**114.8**	**115.2**	**115.4**	**116.3**	117.5	**114.7**	**115.1**	**115.2**	116.0	117.1
0.75,0.4	150	RIDG	62.0	63.4	62.1	62.6	61.9	62.0	63.4	62.1	62.6	61.9
		VANILLA	98.4	100.0	106.0	116.0	125.1	103.2	106.4	115.5	127.6	138.0
		HIERBETAS	52.1	**52.4**	52.6	53.0	54.0	52.0	**52.4**	52.6	53.1	54.1
		EBBETAS	52.0	52.5	**52.6**	**52.9**	**53.7**	52.0	52.5	**52.5**	**52.9**	**53.7**
		EBSIGMAX	119.3	124.5	132.6	140.9	147.9	127.8	134.6	145.0	153.4	161.5
		EBBOTH	**52.0**	52.4	52.7	53.2	54.2	**52.0**	52.4	52.6	53.2	54.2

TABLE S5

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\{0.1\}_{k=1}^{k=8}, 1\}_{j=1}^{j=11}$ (a concentrated signal) when $\boldsymbol{W}|\boldsymbol{X} \sim N_p(\psi \mathbf{1}_p + \nu \boldsymbol{X}^2, \tau^2 \boldsymbol{I}_p)$, where $\boldsymbol{X}^2$ indicates the element-wise square of $\boldsymbol{X}$. When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

			MSPE($\hat{\boldsymbol{\beta}}^{\text{ppm}}$)					MSPE($\hat{\boldsymbol{\beta}}^{\text{pm}}$)				
$\{\rho, R^2\}$	$\{n_{\text{B}}\}$	Method	$\tau = 0.01$	0.5	1.0	1.50	2.00	0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	75.10	72.25	73.91	74.15	73.60	75.10	72.25	73.91	74.15	73.60
		VANILLA	87.54	89.92	98.18	112.78	129.54	89.67	93.61	106.70	128.09	147.15
		HIERBETAS	**67.43**	**67.29**	**67.72**	**68.04**	**68.43**	**67.43**	**67.31**	**67.73**	**68.06**	**68.42**
		EBBETAS	68.14	68.02	68.18	68.51	68.73	68.14	68.03	68.18	68.51	68.73
		EBSIGMAX	86.73	89.20	98.39	114.34	134.22	88.73	92.68	106.81	130.13	153.70
		EBBOTH	68.12	68.01	68.20	68.52	68.73	68.12	68.02	68.20	68.53	68.73
0.75,0.1	400	RIDG	524.29	528.26	531.41	541.77	529.11	524.29	528.26	531.41	541.77	529.11
		VANILLA	584.04	583.72	592.24	602.77	619.55	593.97	597.31	616.48	644.38	682.53
		HIERBETAS	**472.23**	**472.68**	**474.51**	**472.27**	**475.34**	**472.18**	**472.62**	**474.34**	**472.03**	**474.99**
		EBBETAS	479.90	481.08	480.97	479.17	482.72	479.86	481.03	480.84	478.97	482.45
		EBSIGMAX	639.99	641.80	650.82	661.56	693.07	660.47	668.71	694.08	728.55	787.97
		EBBOTH	479.45	481.21	481.50	479.55	483.35	479.40	481.14	481.30	479.26	482.99
0.15,0.4	400	RIDG	17.65	17.88	17.69	17.63	17.72	17.65	17.88	17.69	17.63	17.72
		VANILLA	15.01	15.63	17.74	21.95	26.74	15.42	16.36	19.50	24.75	29.87
		HIERBETAS	**12.82**	**12.96**	**13.41**	**13.97**	**14.48**	**12.84**	**12.97**	**13.43**	**14.00**	**14.52**
		EBBETAS	13.81	13.90	14.18	14.55	15.08	13.82	13.91	14.20	14.57	15.11
		EBSIGMAX	14.87	15.49	17.73	22.44	27.77	15.25	16.16	19.47	25.40	31.01
		EBBOTH	13.81	13.89	14.17	14.56	15.10	13.81	13.90	14.19	14.58	15.13
0.75,0.4	400	RIDG	101.25	101.28	101.29	100.41	102.31	101.25	101.28	101.29	100.41	102.31
		VANILLA	99.75	98.48	99.48	102.85	107.06	101.45	100.73	103.54	110.50	119.08
		HIERBETAS	82.30	81.98	82.12	**82.10**	**82.48**	82.28	81.97	82.09	**82.09**	**82.54**
		EBBETAS	82.21	82.08	82.01	82.12	82.61	82.19	82.06	81.98	82.09	82.62
		EBSIGMAX	109.15	108.12	109.00	113.94	120.77	112.76	112.74	116.36	126.31	138.51
		EBBOTH	**81.64**	**81.70**	**81.90**	82.54	83.35	**81.61**	**81.67**	**81.86**	82.50	83.36
0.15,0.1	150	RIDG	73.50	73.49	74.30	73.21	74.63	73.50	73.49	74.30	73.21	74.63
		VANILLA	145.41	156.82	219.88	247.75	228.21	153.00	168.59	238.20	263.22	244.01
		HIERBETAS	**68.84**	**68.67**	**68.82**	**69.22**	69.12	**68.84**	**68.66**	**68.80**	**69.18**	69.08
		EBBETAS	69.21	68.92	69.00	69.35	**69.07**	69.20	68.92	68.99	69.33	**69.05**
		EBSIGMAX	145.10	156.57	219.95	243.12	241.33	152.73	168.72	239.52	259.69	256.94
		EBBOTH	69.19	68.92	69.01	69.35	69.09	69.19	68.91	69.00	69.33	69.07
0.75,0.1	150	RIDG	528.96	529.17	530.14	528.99	530.43	528.96	529.17	530.14	528.99	530.43
		VANILLA	962.97	965.57	994.39	1063.71	1135.82	1010.05	1030.88	1083.29	1181.47	1260.04
		HIERBETAS	**483.81**	**483.32**	**483.71**	**486.35**	**484.78**	**483.71**	**483.16**	**483.49**	**486.05**	**484.40**
		EBBETAS	489.41	488.99	489.08	493.13	488.99	489.32	488.87	488.89	492.88	488.69
		EBSIGMAX	1154.89	1201.38	1249.35	1318.50	1328.89	1236.38	1304.62	1369.75	1452.56	1466.23
		EBBOTH	489.38	489.34	489.36	493.67	489.76	489.28	489.19	489.12	493.39	489.42
0.15,0.4	150	RIDG	17.43	17.61	17.64	17.62	17.57	17.43	17.61	17.64	17.62	17.57
		VANILLA	25.41	28.14	42.26	45.29	45.49	26.78	30.27	45.33	48.09	48.28
		HIERBETAS	**14.35**	**14.63**	**14.99**	**15.34**	**15.97**	**14.36**	**14.64**	**15.01**	**15.38**	16.01
		EBBETAS	14.94	15.19	15.40	15.62	15.99	14.94	15.20	15.41	15.64	16.01
		EBSIGMAX	25.42	28.51	41.43	46.08	46.27	26.83	30.71	44.80	49.03	49.09
		EBBOTH	14.94	15.20	15.42	15.61	15.99	14.94	15.21	15.43	15.63	**16.01**
0.75,0.4	150	RIDG	101.14	100.77	100.16	101.35	99.94	101.14	100.77	100.16	101.35	99.94
		VANILLA	163.37	160.10	176.08	187.37	201.27	171.37	170.75	192.54	206.59	222.86
		HIERBETAS	85.18	85.38	85.72	86.57	87.69	85.15	85.32	85.68	86.61	87.94
		EBBETAS	84.92	**85.07**	**85.50**	**86.53**	**86.99**	84.88	**85.01**	**85.44**	**86.52**	**87.13**
		EBSIGMAX	203.78	204.22	218.22	227.21	239.17	217.37	221.42	239.34	249.38	263.00
		EBBOTH	**84.86**	85.15	85.70	86.92	87.74	**84.81**	85.06	85.62	86.88	87.84

TABLE S6

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\frac{j}{100}\}_{j=-49}^{j=49}$ (a diffuse signal) when $\boldsymbol{W}|\boldsymbol{X} \sim N_p(\psi \mathbf{1}_p + \nu \boldsymbol{X}^2, \tau^2 \boldsymbol{I}_p)$, where $\boldsymbol{X}^2$ indicates the element-wise square of $\boldsymbol{X}$. When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

			MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{ppm}}$)					MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{pm}}$)				
$\{\rho, R^2\}$	$\{n_{\mathrm{B}}\}$	Method	$\tau = 0.01$	0.5	1.0	1.50	2.00	0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	684.4	691.3	688.6	688.9	682.7	684.4	691.3	688.6	688.9	682.7
		VANILLA	797.9	836.7	916.3	879.1	820.2	798.0	840.7	915.8	873.7	811.9
		HIERBETAS	637.6	732.2	863.1	845.8	797.4	637.6	731.7	861.2	842.7	793.4
		EBBETAS	**636.7**	731.8	863.5	845.5	797.4	636.7	731.3	861.7	842.7	793.8
		EBSIGMAX	798.0	878.1	1085.8	1386.5	1489.9	798.0	905.5	1223.6	1579.1	1672.9
		EBBOTH	636.7	**639.3**	**640.4**	**643.2**	**642.0**	**636.7**	**638.9**	**639.2**	**641.5**	**640.0**
0.75,0.1	400	RIDG	307.9	305.4	304.3	309.4	310.9	307.9	305.4	304.3	309.4	310.9
		VANILLA	350.3	410.8	486.4	451.2	404.6	350.3	408.7	477.3	438.7	390.1
		HIERBETAS	280.6	388.9	489.6	460.1	415.2	280.6	388.2	487.7	457.3	411.2
		EBBETAS	**280.2**	388.7	490.0	460.3	415.5	**280.2**	388.1	488.3	457.8	411.8
		EBSIGMAX	350.4	366.7	384.1	420.0	455.7	350.3	379.4	420.9	487.6	536.5
		EBBOTH	280.3	**280.6**	**283.6**	**287.8**	**292.3**	280.3	**280.4**	**283.2**	**287.1**	**291.3**
0.15,0.4	400	RIDG	121.2	122.2	120.7	121.5	120.3	121.2	122.2	120.7	121.5	120.3
		VANILLA	133.7	211.1	320.9	289.7	238.4	133.7	210.4	313.2	276.3	220.7
		HIERBETAS	111.6	200.4	326.9	303.0	254.2	111.6	199.8	324.3	299.0	248.7
		EBBETAS	**111.5**	200.2	327.0	303.4	254.5	**111.5**	199.6	324.5	299.6	249.2
		EBSIGMAX	133.7	146.9	186.2	238.6	265.3	133.7	151.6	210.8	271.2	295.2
		EBBOTH	111.5	**111.9**	**113.9**	**114.0**	**115.8**	111.5	**111.9**	**113.6**	**113.7**	**115.4**
0.75,0.4	400	RIDG	55.7	54.4	54.7	54.1	**55.0**	55.7	54.4	54.7	54.1	**55.0**
		VANILLA	57.9	145.7	234.1	200.5	149.5	57.9	143.1	225.9	188.6	134.4
		HIERBETAS	48.9	148.7	245.0	214.0	164.7	48.9	147.8	242.6	210.3	159.5
		EBBETAS	48.9	148.6	245.1	214.3	165.0	**48.9**	147.8	242.8	210.7	160.0
		EBSIGMAX	58.0	61.8	67.3	76.7	87.5	58.0	63.9	74.0	87.9	100.9
		EBBOTH	**48.9**	**49.6**	**51.1**	**53.5**	55.9	48.9	**49.6**	**51.1**	**53.4**	55.7
0.15,0.1	150	RIDG	683.7	677.1	682.0	686.0	683.8	683.7	677.1	682.0	686.0	683.8
		VANILLA	1251.9	1148.7	979.6	916.3	852.8	1251.9	1162.5	1006.0	947.1	889.4
		HIERBETAS	646.0	653.0	687.2	703.2	695.7	646.0	652.7	685.2	699.0	690.2
		EBBETAS	**643.8**	650.7	685.9	702.4	695.4	**643.8**	650.5	684.2	698.8	690.6
		EBSIGMAX	1251.8	1484.9	2344.1	2196.2	2188.1	1251.8	1586.4	2523.2	2349.6	2335.3
		EBBOTH	643.8	**645.0**	**644.0**	**645.7**	**649.5**	643.8	**644.4**	**642.7**	**644.0**	**647.6**
0.75,0.1	150	RIDG	305.1	312.5	298.8	304.3	308.0	305.1	312.5	298.8	304.3	308.0
		VANILLA	555.2	447.8	391.1	378.0	363.4	554.9	452.6	394.4	380.1	369.4
		HIERBETAS	288.5	294.6	326.1	340.2	330.5	288.5	294.4	324.1	335.8	325.0
		EBBETAS	**287.5**	293.7	325.9	340.2	330.6	**287.5**	293.5	324.1	336.2	325.5
		EBSIGMAX	555.9	625.2	752.6	746.4	804.2	555.8	675.6	830.0	833.3	888.0
		EBBOTH	287.5	**286.9**	**288.0**	**292.3**	**294.7**	287.5	**286.6**	**287.4**	**291.5**	**293.9**
0.15,0.4	150	RIDG	122.3	121.3	120.7	120.6	120.2	122.3	121.3	120.7	120.6	120.2
		VANILLA	209.2	199.8	202.0	192.3	178.9	209.1	201.7	201.1	186.9	175.3
		HIERBETAS	115.2	121.2	155.0	168.7	156.9	115.2	121.0	152.5	163.0	149.4
		EBBETAS	**114.7**	120.8	155.0	169.1	157.3	**114.7**	120.7	152.6	163.8	150.2
		EBSIGMAX	209.2	250.1	416.9	390.9	392.3	209.2	267.3	445.2	418.5	418.8
		EBBOTH	114.7	**115.1**	**115.8**	**118.0**	**118.7**	114.7	**114.9**	**115.5**	**117.6**	**118.3**
0.75,0.4	150	RIDG	54.5	55.4	54.2	55.6	**55.4**	54.5	55.4	54.2	55.6	**55.4**
		VANILLA	90.3	81.6	91.7	91.4	84.6	90.3	81.7	85.6	81.1	75.2
		HIERBETAS	51.5	58.2	88.3	97.2	86.4	51.5	58.0	85.5	91.3	79.2
		EBBETAS	51.4	58.1	88.5	97.6	86.9	51.4	57.9	85.9	92.0	79.9
		EBSIGMAX	90.3	108.8	133.5	143.3	151.5	90.3	117.5	146.5	157.2	165.7
		EBBOTH	**51.3**	**51.8**	**53.2**	**54.8**	56.4	**51.3**	**51.7**	**53.1**	**54.8**	56.3

Table S7

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\{0.1\}_{k=1}^{k=8}, 1\}_{j=1}^{j=11}$ (a concentrated signal) when $\boldsymbol{X}|Z \sim N_p\{1_{[Z=2]}(3 \times \mathbf{1}_p) - 1_{[Z=3]}(3 \times \mathbf{1}_p), \boldsymbol{\Sigma}_{\boldsymbol{X}}\}$, where $1_{[\cdot]}$ is the indicator function and $Z \overset{iid}{\sim}$ Unif$\{1,2,3\}$. When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

$\{\rho, R^2\}$	$\{n_{\mathrm{B}}\}$	Method	MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{ppm}}$) $\tau = 0.01$	0.5	1.0	1.50	2.00	MSPE($\hat{\boldsymbol{\beta}}^{\mathrm{pm}}$) 0.01	0.5	1.0	1.50	2.00
0.15,0.1	400	RIDG	73.46	73.31	73.75	74.23	74.42	73.46	73.31	73.75	74.23	74.42
		VANILLA	79.76	75.11	72.62	72.22	72.34	79.76	75.66	73.60	73.41	73.73
		HIERBETAS	**67.39**	**67.81**	**68.30**	**68.50**	**68.94**	**67.40**	**67.82**	**68.31**	**68.51**	**68.95**
		EBBETAS	68.37	68.55	68.63	68.68	69.08	68.37	68.55	68.63	68.69	69.08
		EBSIGMAX	79.77	88.60	109.87	142.72	152.09	79.77	91.55	124.17	161.67	171.01
		EBBOTH	68.37	68.54	68.72	68.74	69.11	68.37	68.54	68.71	68.73	69.11
0.75,0.1	400	RIDG	555.96	538.87	549.68	537.04	538.92	555.96	538.87	549.68	537.04	538.92
		VANILLA	588.83	522.00	501.32	498.25	496.15	588.80	525.02	505.62	503.76	503.82
		HIERBETAS	**473.82**	**473.39**	**475.77**	**481.26**	**484.05**	**473.81**	**473.41**	**475.84**	**481.38**	**484.26**
		EBBETAS	489.08	486.78	487.50	491.78	494.03	489.07	486.79	487.53	491.85	494.16
		EBSIGMAX	588.88	616.25	655.13	699.15	742.19	588.90	637.58	721.43	810.62	875.12
		EBBOTH	489.06	485.92	485.76	489.52	494.94	489.05	485.85	485.56	489.14	494.60
0.15,0.4	400	RIDG	17.72	17.95	17.74	17.64	17.98	17.72	17.95	17.74	17.64	17.98
		VANILLA	13.30	**13.10**	**13.93**	**14.73**	**15.22**	13.30	13.15	**13.97**	**14.76**	**15.25**
		HIERBETAS	**12.38**	13.14	14.78	15.93	16.42	**12.38**	**13.13**	14.78	15.94	16.43
		EBBETAS	13.49	14.07	15.39	16.25	16.57	13.49	14.06	15.38	16.26	16.59
		EBSIGMAX	13.30	15.25	21.20	28.61	31.23	13.30	15.82	24.05	31.96	34.42
		EBBOTH	13.49	13.84	14.71	15.38	15.77	13.49	13.85	14.72	15.40	15.79
0.75,0.4	400	RIDG	103.33	102.79	102.81	103.07	103.18	103.33	102.79	102.81	103.07	103.18
		VANILLA	97.90	89.02	87.13	86.54	**86.60**	97.89	89.38	87.28	86.66	**86.89**
		HIERBETAS	**81.34**	82.88	85.71	86.74	87.74	**81.34**	82.86	85.60	86.50	87.39
		EBBETAS	81.65	83.71	86.20	87.56	88.38	81.65	83.69	86.10	87.35	88.07
		EBSIGMAX	97.90	103.81	110.51	122.40	138.66	97.89	107.49	121.92	142.08	163.16
		EBBOTH	81.65	**82.09**	**82.75**	**84.90**	87.14	81.65	**82.06**	**82.71**	**84.96**	87.41
0.15,0.1	150	RIDG	74.64	74.37	73.49	73.54	74.09	74.64	74.37	73.49	73.54	74.09
		VANILLA	125.89	115.27	96.88	89.44	86.67	125.85	116.70	100.11	93.76	91.57
		HIERBETAS	**68.63**	**68.71**	**68.90**	**69.12**	**69.05**	**68.63**	**68.72**	**68.90**	**69.12**	**69.06**
		EBBETAS	68.83	68.94	69.07	69.19	69.10	68.83	68.94	69.07	69.19	69.10
		EBSIGMAX	125.91	152.16	241.47	238.10	228.09	125.91	162.64	258.98	252.56	242.79
		EBBOTH	68.83	69.00	69.16	69.30	69.30	68.82	68.99	69.16	69.28	69.28
0.75,0.1	150	RIDG	534.33	542.65	537.26	541.66	537.41	534.33	542.65	537.26	541.66	537.41
		VANILLA	915.80	732.87	612.52	578.99	567.38	915.80	741.96	632.87	606.28	599.87
		HIERBETAS	**484.74**	**485.32**	**488.42**	**490.18**	**496.19**	**484.70**	**485.30**	**488.46**	**490.36**	**496.46**
		EBBETAS	493.54	493.14	494.38	496.59	501.44	493.53	493.11	494.39	496.67	501.54
		EBSIGMAX	916.68	1039.01	1291.71	1302.14	1285.98	916.22	1122.00	1418.76	1436.86	1428.32
		EBBOTH	493.48	493.56	494.79	497.85	504.15	493.49	493.41	494.51	497.57	503.86
0.15,0.4	150	RIDG	17.80	17.82	17.63	17.69	17.71	17.80	17.82	17.63	17.69	17.71
		VANILLA	20.65	20.08	18.66	18.18	18.41	20.65	20.31	19.23	18.92	19.27
		HIERBETAS	**13.97**	**14.66**	**15.80**	16.53	16.87	**13.97**	**14.67**	**15.81**	16.54	16.87
		EBBETAS	14.81	15.22	16.16	16.69	16.99	14.81	15.22	16.17	16.70	17.00
		EBSIGMAX	20.65	26.69	45.81	45.73	46.57	20.65	28.58	48.66	48.73	49.29
		EBBOTH	14.81	15.18	15.80	**16.15**	**16.46**	14.81	15.19	15.81	**16.16**	**16.47**
0.75,0.4	150	RIDG	101.28	102.88	101.80	102.30	101.53	101.28	102.88	101.80	102.30	101.53
		VANILLA	153.45	123.36	103.77	99.22	97.75	153.40	124.85	107.05	103.61	102.75
		HIERBETAS	85.16	**85.43**	86.27	**88.20**	91.69	85.15	**85.43**	86.28	**88.19**	91.70
		EBBETAS	**85.13**	86.21	86.54	88.70	92.03	85.13	86.21	86.56	88.71	92.09
		EBSIGMAX	153.41	172.61	222.89	230.27	241.10	153.39	186.91	245.66	254.59	265.44
		EBBOTH	85.14	86.33	**86.02**	88.62	**89.86**	**85.13**	86.26	**85.98**	88.82	**90.34**

TABLE S8

MSPE($\hat{\boldsymbol{\beta}}^{ppm}$) and MSPE($\hat{\boldsymbol{\beta}}^{pm}$) under $\boldsymbol{\beta} = \{\frac{j}{100}\}_{j=-49}^{j=49}$ (a diffuse signal) when $\boldsymbol{X}|Z \sim N_p\{1_{[Z=2]}(3 \times \mathbf{1}_p) - 1_{[Z=3]}(3 \times \mathbf{1}_p), \boldsymbol{\Sigma}_{\boldsymbol{X}}\}$, where $1_{[\cdot]}$ is the indicator function and $Z \overset{iid}{\sim}$ Unif$\{1, 2, 3\}$. When $\rho = 0.15$, the error structure for $\boldsymbol{X}$ is compound symmetric, and when $\rho = 0.75$, the error structure is AR(1). The column-wise minima of each box are in **bold**

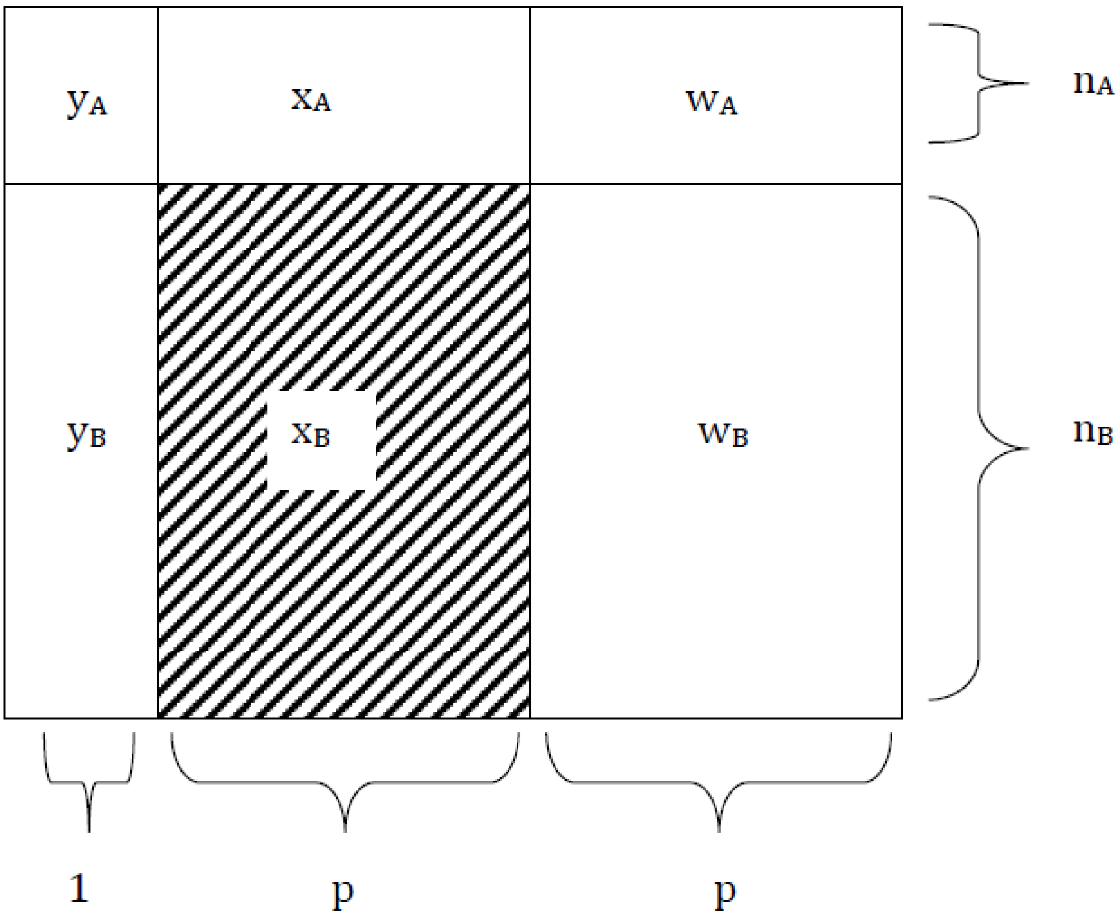

Fig S1. *Schematic representation of the prediction problem:* $(\boldsymbol{y}_A, \boldsymbol{x}_A, \boldsymbol{w}_A)$ *is measured on* n_A *subjects and* $(\boldsymbol{y}_B, \boldsymbol{w}_B)$ *is measured on* n_B *subjects.* $\boldsymbol{x}_B$ *is considered missing.*